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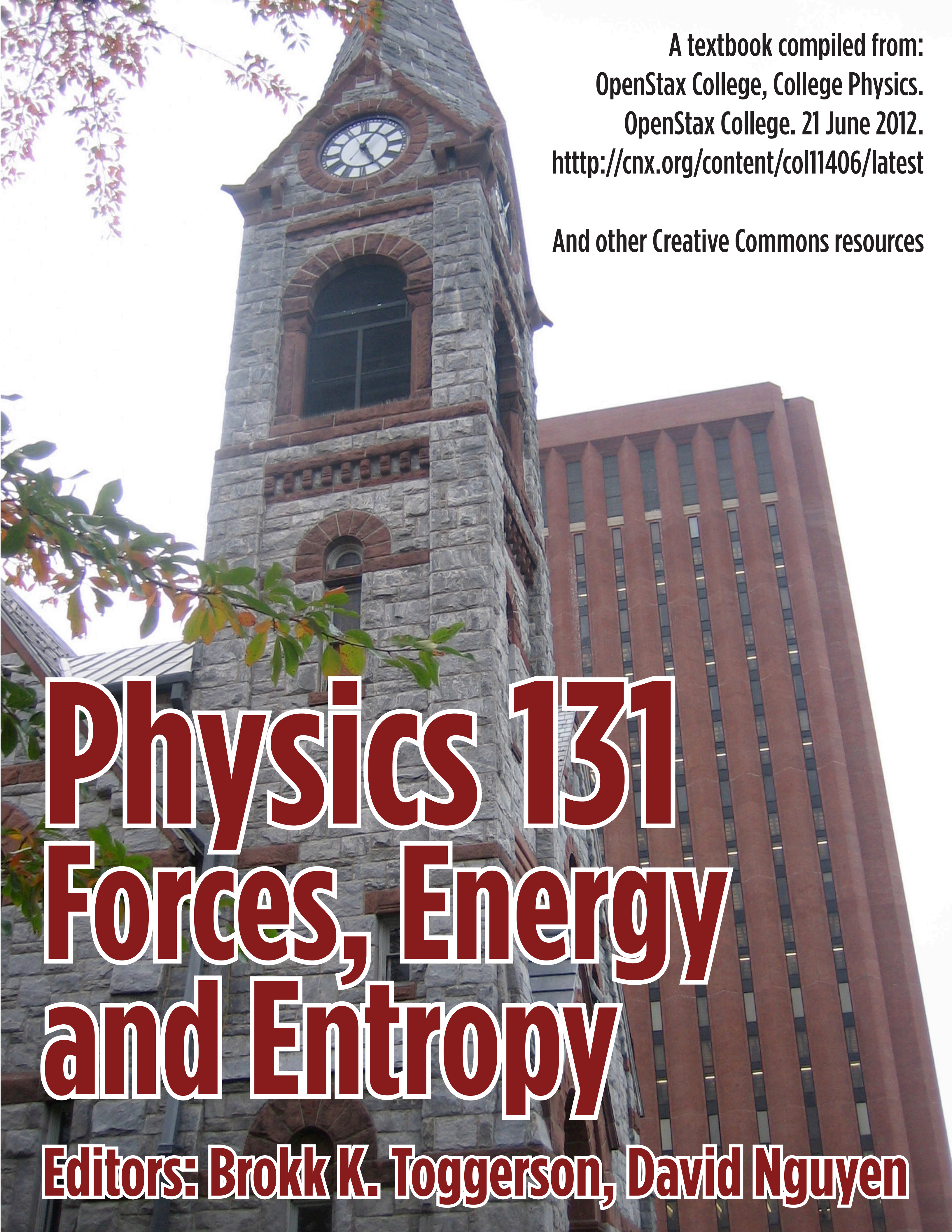
Physics 131: Forces, Energy and Entropy

Brokk Toggerson
University of Massachusetts Amherst, toggerson@physics.umass.edu

David Nguyen
University of Massachusetts Amherst, dnguyen@umass.edu



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OpenStax College, College Physics.
OpenStax College. 21 June 2012.
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Physics 131 Forces, Energy and Entropy

Editors: Brokk K. Toggerson, David Nguyen

About This Book

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Digital versions of the individual units can be found at

- *Unit I - Mathematical Tools and Foundational Concepts*: https://cnx.org/contents/8Gi8HetQ@2.1:G544Tn_j@3/Unit-1-Overview
- *Unit II - Forces:* *https://cnx.org/contents/Lkw2JNmw@1.1:w4pMFbR8@1/Unit-2-Overview*
- *Unit III - Forces and…*: https://cnx.org/contents/wT91GDZK@1.1:EuhP-FLm@1/Unit-3-Overview
- *Unit IV - Energy*: https://cnx.org/contents/bd13acOx@1.1:Pm3Sdcxq@1/Unit-4-Overview
- *Unit V - Entropy*: https://cnx.org/contents/8_JOW5TJ@1.2:8JZSZO77@2/Unit-5-Overview

Dear UMass Student,

Welcome to Physics 131! In this course, we will be exploring three of the biggest ideas in physics: forces, energy and entropy. These three ideas are absolutely fundamental to all of the other sciences including the life and chemical sciences. Moreover, as the frontier of knowledge continues to advance, the lines between biology, chemistry, and physics is becoming ever more blurred resulting in new fields like biochemistry, biophysics, and chemical physics. These interfaces between disciplines are where lots of new discoveries are being made.

In addition to the content, we hope that you learn a new way of thinking as well. Physics has a very particular way of thinking that, while it can seem strange at first glance, has been remarkably successful in developing our understanding of the world. The basic premise of physics thinking is to strip away all possible complications, figure out the basic rules of what is going on, and then add the complications back. Think about that process for a second. We have been studying physics a long time and we still find the fact that this approach works at all to be mind-boggling! By thinking this way, one gets to observe deep fundamental truths that explain the motions of atoms to stars expressed in beautifully simple-looking equations. Some might say the equations are almost deceptively simple looking. Note, however, we said “ideas represented in formulas” not “a list of equations.” Being cognizant of that distinction will serve you well: physics is about ideas, NOT equations.

To help you on your journey, we have compiled this book specifically for you - the University of Massachusetts, Amherst Physics 131 student. The selection of topics included in this book and the order in which they are presented will exactly match our class. While the structure of the course is detailed elsewhere (like your syllabus) the basic structure of the course is that it has five units:

Unit I - Mathematical Tools and Fundamental Concepts
Unit II - Forces
Unit III - Forces and… (Impulse, Torque, and Work)
Unit IV - Energy
Unit V - Entropy

This book follows this structure. There is a master table of contents at the front with links to each unit. Each unit has its own table of contents at the front. Pages re-start at 1 for each unit.

Since this edition is for you, we have been able to add Instructor’s Notes which look like the comment below. These can help you focus your reading and give you a heads up on what is important for your reading quizzes. Some sections written by us are also presented in video form and we provide links to the course YouTube page.

This textbook draws from many sources. We appreciate people putting so much great content on the web with a Creative Commons license for us to use. Most predominantly, we are of course, most indebted to the work of the OpenStax foundation whose College Physics textbook [1] provides the core of this book. We are also grateful to all of the work the University of Maryland NEXUS project did on their wikibook [2] which was the source of many additional pages along with the OpenStax Chemistry textbook [3]. Finally, thanks to John Denker for his permission to use his articles on introducing statistics that appear in Chapter 1 [4]. Any errors resulting from the modification of these materials is completely our own and should in no way reflect poorly on the great work of these teams and individuals.

This work would not have happened without the support of the UMass, Amherst Physics Department (http://physics.umass.edu) and the Open Education Initiative at the W.E.B. du Bois Library at UMass, Amherst (http://www.library.umass.edu/services/teaching-and-learning/oer/open-education-initiative/). Final thanks are due to the UMass Amherst Physics Education Group: Heath Hatch, Paul Bourgeois, Chris Ertl, and Chasya Church. Without you this book would just be a neat idea.

We hope you enjoy this book. If you have any feedback, please contact us at https://goo.gl/forms/BQQKirWgVL01iDpy2.

Brokk Toggerson, Ph.D.
David Nguyen

Editors

References

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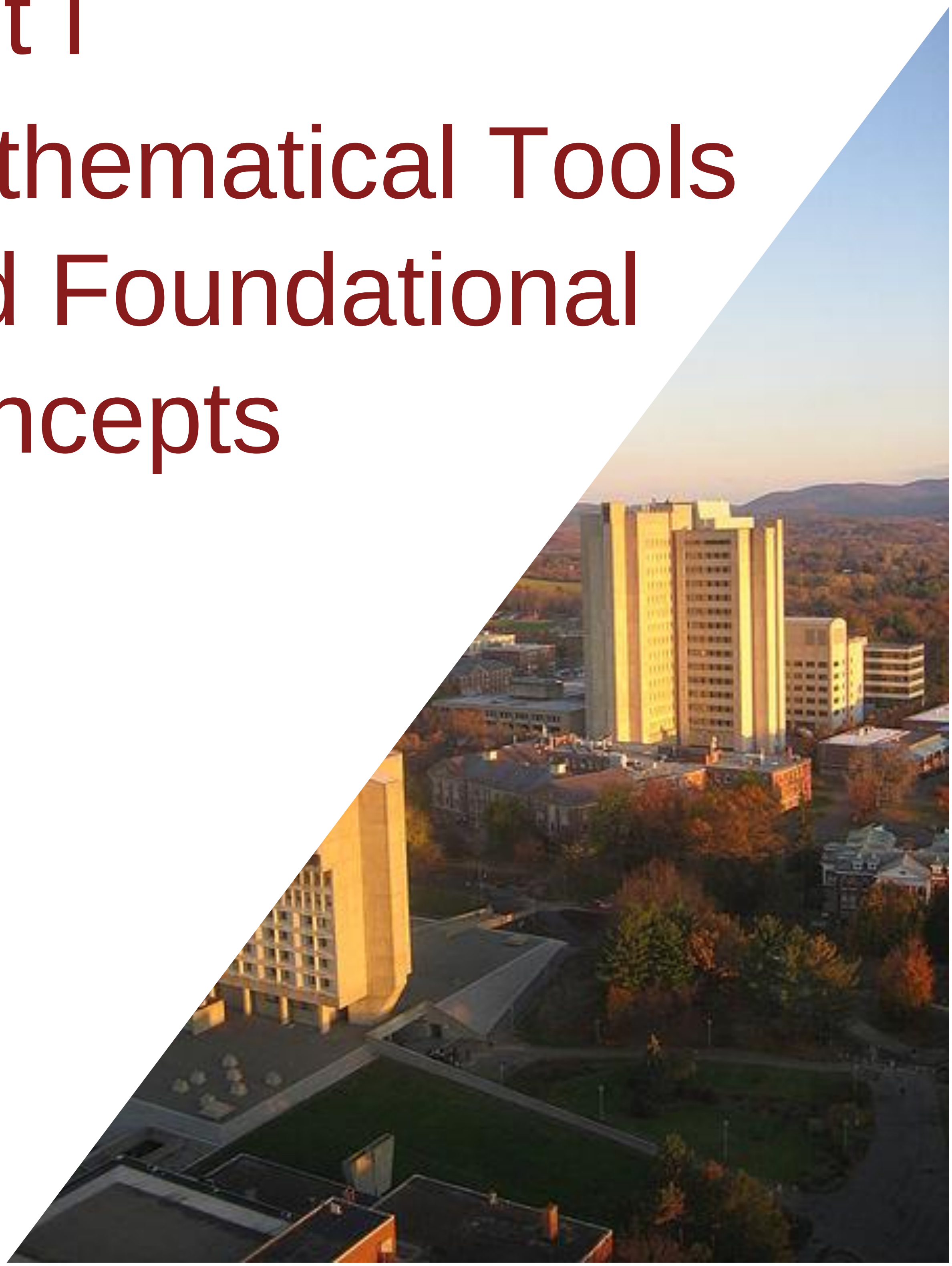
Unit I

Mathematical Tools and Foundational Concepts



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UNIT 1 OVERVIEW

UMASS AMHERST Instructor's Notes

Things to Consider as You Read:

This section provides an overview of the first unit on mathematical tools and foundational concepts for Physics 131; we will be using concepts from this unit throughout the class.

There are also a few hints and tips in the section on how to do the homework efficiently; I would suggest developing good homework habits early.

This overview is also available as a video here, or on Youtube at **https://www.youtube.com/watch?v=ikLsqv2dhY8 (https://www.youtube.com/watch?v=ikLsqv2dhY8)** .

In this unit, we will explore some of the fundamental mathematical tools and basic concepts that we will need throughout the rest of the course in our study of physics, including:

- An introduction to what physics is as a discipline and how that might be similar to or different from some of the other sciences you may have studied,
- A review of the basic idea of units
- My policy on significant figures
- Introduce the basic ideas of mean and standard deviation for use in the laboratory exercises within this course
- The definition of displacement, velocity, and acceleration; in particular, how velocity and acceleration are similar to and different from distance and speed
- How to use iterative methods to predict the motion of objects that move with non-uniform acceleration
- On a purely mathematical note, you will be exploring what a vector is and how they can be added and subtracted.

General Notes About Homework

The homework in this course is intended to provide you with some basic information. The material in the preparation will be the starting point for what we discuss in class. This helps to make sure that everyone with their varying backgrounds in physics is starting at the same point. We will then build upon this preparation in class, using in-class activities to get you ready for exams. This is somewhat different probably from your other courses where the purpose of the homework is to provide additional practice on in-class material to help you get ready for exams. **In this course, the homework gets you ready for class, and class is what gets you ready for the exams.**

How to be Successful

Each homework is divided up into sections. Within each section, the first question is your readings to do for that particular section, followed by a set of problems. The information you need to complete a set of problems will be in the readings at the beginning of that section. The readings are presented in terms of a checklist. This problem is not for a grade, it's just presented as a checklist to make sure that you get everything done. So, you may have various readings within the OpenStax textbook UMass edition which is on Perusall, and you may also have some videos. The videos are embedded directly within the online homework system, and you should be able to play them right there, but if you cannot, you can go and click this **link (https://www.youtube.com/channel/UCUMejMY1La0t8qi-jwFTyTw)** and it will take you to the course YouTube page, and you can watch the videos there. The transcripts for all of the videos are also included in the textbook themselves, so if you want to go and read the text because you prefer to read or you want to add some sort of Perusall comment to some of the content of the video, you can do that within the textbook in Perusall, so each video has an associated section in the textbook and in Perusall with the transcript of that video for you to comment.

Once you've completed the readings, you're now ready to move on to the actual homework problems. These problems are there to help you check that you understood what was in the various readings and videos, and to help you refine your understanding. Most of the individual parts of each problem are one-step. If you find yourself doing long chains of calculations, come get help in the consultation room. You're probably approaching the problem in a way that's not very efficient. When doing the homework, don't skip the readings and the videos. Your comments on the actual readings in Perusall are graded in accordance with the policy in the syllabus and form a part of your homework grade. We acknowledge that doing all of these readings and all of this homework is hard work, and we are here to help; we've provided quite a few resources to help you be successful in completing this assignment. Moreover, since it is so much work, the preparation is your entire homework for this course. There is no required end of chapter homework assignments; you only need to do this preparation. This is your big focus for your homework.

What to focus on in the Unit 1 Preparation Homework

I want you to focus on, while doing this homework, the definitions of the terms **position**, **velocity**, and **acceleration**, the few basic equations such as $\mathbf{v} = \frac{\Delta x}{\Delta t}$ and $\mathbf{a} = \frac{\Delta v}{\Delta t}$, including what all the symbols mean and when these equations can be applied. Many people in studying physics for the first time understand they need to know what the symbols mean, but they tend to skip over this second element, which is just as important, if not more so, because not every equation can be applied in every situation. I will also ask you to learn how to just “turn the crank” for various types of calculations, such as iterative calculations, and vector arithmetic. Don’t worry if you don’t really understand what you’re doing when you do these calculations. If conceptually it doesn’t make sense, that’s okay; we will spend time in class working with these ideas and getting an understanding of what you’re doing. I just want you to know how to do these calculations.

Finally, I would like to have a quick philosophical comment regarding motion with constant acceleration. If you have had any physics before, you may have seen the so-called kinematic equations, which are these two here:

$$d = v_0 t + \frac{1}{2} a t^2 \quad v^2 = {v_0}^2 + 2ad$$

We will NOT be using these equations in this class. We will be approaching the subject, and many others, in ways that may be different from how you may have seen them in a previous physics class. We believe that physics is not about memorizing equations and learning how to piece those equations together. We believe instead that physics is about fundamental ideas, and we will teach this course from this perspective. Occasionally, this will result in physics homework very different from what you may expect. A good example is the homework for the second unit, where you have some actual fill-in-the-blank type of questions. If you try to learn physics as a set of ideas instead of a set of equations to be pieced together, and start your analysis of situations from fundamental physical principles, then your physics experience will enrich and enhance your understanding of your other courses, as opposed to just being a course that you just “have to take for your major”.

1 INTRODUCTION: THE NATURE OF SCIENCE AND PHYSICS

Figure 1.1 Galaxies are as immense as atoms are small. Yet the same laws of physics describe both, and all the rest of nature—an indication of the underlying unity in the universe. The laws of physics are surprisingly few in number, implying an underlying simplicity to nature's apparent complexity. (credit: NASA, JPL-Caltech, P. Barmby, Harvard-Smithsonian Center for Astrophysics)

Chapter Outline

Chapter Overview

What is your first reaction when you hear the word "physics"? Did you imagine working through difficult equations or memorizing formulas that seem to have no real use in life outside the physics classroom? Many people come to the subject of physics with a bit of fear. But as you begin your exploration of this broad-ranging subject, you may soon come to realize that physics plays a much larger role in your life than you first thought, no matter your life goals or career choice.

For example, take a look at the image above. This image is of the Andromeda Galaxy, which contains billions of individual stars, huge clouds of gas, and dust. Two smaller galaxies are also visible as bright blue spots in the background. At a staggering 2.5 million light years from the Earth, this galaxy is the nearest one to our own galaxy (which is called the Milky Way). The stars and planets that make up Andromeda might seem to be the furthest thing from most people's regular, everyday lives. But Andromeda is a great starting point to think about the forces that hold together the universe. The forces that cause Andromeda to act as it does are the same forces we contend with here on Earth, whether we are planning to send a rocket into space or simply raise the walls for a new home. The same gravity that causes the stars of Andromeda to rotate and revolve also causes water to flow over hydroelectric dams here on Earth. Tonight, take a moment to look up at the stars. The forces out there are the same as the ones here on Earth. Through a study of physics, you may gain a greater understanding of the interconnectedness of everything we can see and know in this universe.

Think now about all of the technological devices that you use on a regular basis. Computers, smart phones, GPS systems, MP3 players, and satellite radio might come to mind. Next, think about the most exciting modern technologies that you have heard about in the news, such as trains that levitate above tracks, "invisibility cloaks" that bend light around them, and microscopic robots that fight cancer cells in our bodies. All of these groundbreaking advancements, commonplace or unbelievable, rely on the principles of physics. Aside from playing a significant role in technology, professionals such as engineers, pilots, physicians, physical therapists, electricians, and computer programmers apply physics concepts in their daily work. For example, a pilot must understand how wind forces affect a flight path and a physical therapist must understand how the muscles in the body experience forces as they move and bend. As you will learn in this text, physics principles are propelling new, exciting technologies, and these principles are applied in a wide range of careers.

In this text, you will begin to explore the history of the formal study of physics, beginning with natural philosophy and the ancient Greeks, and leading up through a review of Sir Isaac Newton and the laws of physics that bear his name. You will also be introduced to the standards scientists use when they study physical quantities and the interrelated system of measurements most of the scientific community uses to communicate in a single mathematical language. Finally, you will study the limits of our ability to be accurate and precise, and the reasons scientists go to painstaking lengths to be as clear as possible regarding their own limitations.

1.1 An Introduction to Physics

UMASS AMHERST Instructor's Notes

Things to Consider as You Read:

- What *is* science? Where does physics fit into this definition?
- How does the scientific process work?
- What is a scientific theory? What makes a theory credible?
- How will physics help you as a non-physics major?
- How are models used in science?
- What are some of the limits of classical physics?

Figure 1.2 The flight formations of migratory birds such as Canada geese are governed by the laws of physics. (credit: David Merrett)

The physical universe is enormously complex in its detail. Every day, each of us observes a great variety of objects and phenomena. Over the centuries, the curiosity of the human race has led us collectively to explore and catalog a tremendous wealth of information. From the flight of birds to the colors of flowers, from lightning to gravity, from quarks to clusters of galaxies, from the flow of time to the mystery of the creation of the universe, we have asked questions and assembled huge arrays of facts. In the face of all these details, we have discovered that a surprisingly small and unified set of physical laws can explain what we observe. As humans, we make generalizations and seek order. We have found that nature is remarkably cooperative—it exhibits the *underlying order and simplicity* we so value.

It is the underlying order of nature that makes science in general, and physics in particular, so enjoyable to study. For example, what do a bag of chips and a car battery have in common? Both contain energy that can be converted to other forms. The law of conservation of energy (which says that energy can change form but is never lost) ties together such topics as food calories, batteries, heat, light, and watch springs. Understanding this law makes it easier to learn about the various forms energy takes and how they relate to one another. Apparently unrelated topics are connected through broadly applicable physical laws, permitting an understanding beyond just the memorization of lists of facts.

The unifying aspect of physical laws and the basic simplicity of nature form the underlying themes of this text. In learning to apply these laws, you will, of course, study the most important topics in physics. More importantly, you will gain analytical abilities that will enable you to apply these laws far beyond the scope of what can be included in a single book. These analytical skills will help you to excel academically, and they will also help you to think critically in any professional career you choose to pursue. This module discusses the realm of physics (to define what physics is), some applications of physics (to illustrate its relevance to other disciplines), and more precisely what constitutes a physical law (to illuminate the importance of experimentation to theory).

Science and the Realm of Physics

Science consists of the theories and laws that are the general truths of nature as well as the body of knowledge they encompass. Scientists are continually trying to expand this body of knowledge and to perfect the expression of the laws that describe it. **Physics** is concerned with describing the interactions of energy, matter, space, and time, and it is especially interested in what fundamental mechanisms underlie every phenomenon. The concern for describing the basic phenomena in nature essentially defines the *realm of physics*.

Physics aims to describe the function of everything around us, from the movement of tiny charged particles to the motion of people, cars, and spaceships. In fact, almost everything around you can be described quite accurately by the laws of physics.

Consider a smart phone (**Figure 1.3**). Physics describes how electricity interacts with the various circuits inside the device. This knowledge helps engineers select the appropriate materials and circuit layout when building the smart phone. Next, consider a GPS system. Physics describes the relationship between the speed of an object, the distance over which it travels, and the time it takes to travel that distance. When you use a GPS device in a vehicle, it utilizes these physics equations to determine the travel time from one location to another.

Figure 1.3 The Apple “iPhone” is a common smart phone with a GPS function. Physics describes the way that electricity flows through the circuits of this device. Engineers use their knowledge of physics to construct an iPhone with features that consumers will enjoy. One specific feature of an iPhone is the GPS function. GPS uses physics equations to determine the driving time between two locations on a map. (credit: @gletham GIS, Social, Mobile Tech Images)

Applications of Physics

You need not be a scientist to use physics. On the contrary, knowledge of physics is useful in everyday situations as well as in nonscientific professions. It can help you understand how microwave ovens work, why metals should not be put into them, and why they might affect pacemakers. (See **Figure 1.4** and **Figure 1.5**.) Physics allows you to understand the hazards of radiation and rationally evaluate these hazards more easily. Physics also explains the reason why a black car radiator helps remove heat in a car engine, and it explains why a white roof helps keep the inside of a house cool. Similarly, the operation of a car’s ignition system as well as the transmission of electrical signals through our body’s nervous system are much easier to understand when you think about them in terms of basic physics.

Physics is the foundation of many important disciplines and contributes directly to others. Chemistry, for example—since it deals with the interactions of atoms and molecules—is rooted in atomic and molecular physics. Most branches of engineering are applied physics. In architecture, physics is at the heart of structural stability, and is involved in the acoustics, heating, lighting, and cooling of buildings. Parts of geology rely heavily on physics, such as radioactive dating of rocks, earthquake analysis, and heat transfer in the Earth. Some disciplines, such as biophysics and geophysics, are hybrids of physics and other disciplines.

Physics has many applications in the biological sciences. On the microscopic level, it helps describe the properties of cell walls and cell membranes (**Figure 1.6** and **Figure 1.7**). On the macroscopic level, it can explain the heat, work, and power associated with the human body. Physics is involved in medical diagnostics, such as x-rays, magnetic resonance imaging (MRI), and ultrasonic blood flow measurements. Medical therapy sometimes directly involves physics; for example, cancer radiotherapy uses ionizing radiation. Physics can also explain sensory phenomena, such as how musical instruments make sound, how the eye detects color, and how lasers can transmit information.

It is not necessary to formally study all applications of physics. What is most useful is knowledge of the basic laws of physics and a skill in the analytical methods for applying them. The study of physics also can improve your problem-solving skills. Furthermore, physics has retained the most basic aspects of science, so it is used by all of the sciences, and the study of physics makes other sciences easier to understand.

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While it might feel like you'll never need to use physics later on, knowing the physics will give you a deeper understanding, or at least a deeper appreciation, of what's happening. As we move forward in the course, try to connect the material in physics to your outside studies; it might be a lot easier than you expect!

Figure 1.4 The laws of physics help us understand how common appliances work. For example, the laws of physics can help explain how microwave ovens heat up food, and they also help us understand why it is dangerous to place metal objects in a microwave oven. (credit: MoneyBlogNewz)

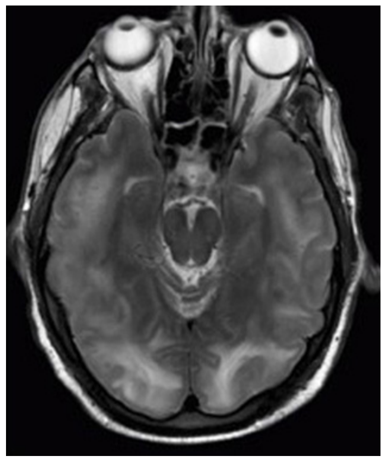

Figure 1.5 These two applications of physics have more in common than meets the eye. Microwave ovens use electromagnetic waves to heat food. Magnetic resonance imaging (MRI) also uses electromagnetic waves to yield an image of the brain, from which the exact location of tumors can be determined. (credit: Rashmi Chawla, Daniel Smith, and Paul E. Marik)

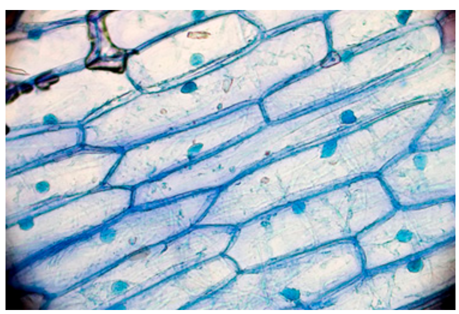

Figure 1.6 Physics, chemistry, and biology help describe the properties of cell walls in plant cells, such as the onion cells seen here. (credit: Umberto Salvagnin)

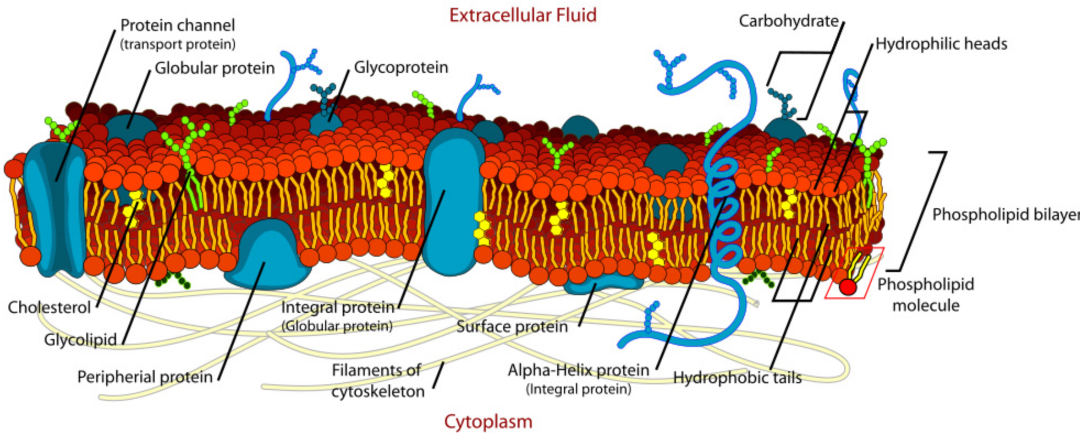

Figure 1.7 An artist's rendition of the the structure of a cell membrane. Membranes form the boundaries of animal cells and are complex in structure and function. Many of the most fundamental properties of life, such as the firing of nerve cells, are related to membranes. The disciplines of biology, chemistry, and physics all help us understand the membranes of animal cells. (credit: Mariana Ruiz)

Models, Theories, and Laws; The Role of Experimentation

The laws of nature are concise descriptions of the universe around us; they are human statements of the underlying laws or rules that all natural processes follow. Such laws are intrinsic to the universe; humans did not create them and so cannot change them. We can only discover and understand them. Their discovery is a very human endeavor, with all the elements of mystery, imagination, struggle, triumph, and disappointment inherent in any creative effort. (See **Figure 1.8** and **Figure 1.9**.) The cornerstone of discovering natural laws is observation; science must describe the universe as it is, not as we may imagine it to be.

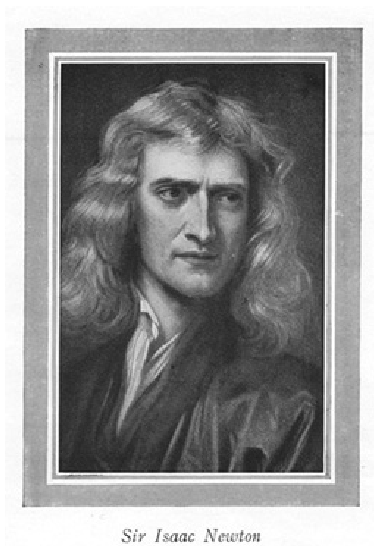

Figure 1.8 **Isaac Newton** (1642–1727) was very reluctant to publish his revolutionary work and had to be convinced to do so. In his later years, he stepped down from his academic post and became exchequer of the Royal Mint. He took this post seriously, inventing reeding (or creating ridges) on the edge of coins to prevent unscrupulous people from trimming the silver off of them before using them as currency. (credit: Arthur Shuster and Arthur E. Shipley: *Britain's Heritage of Science*. London, 1917.)

Figure 1.9 **Marie Curie** (1867–1934) sacrificed monetary assets to help finance her early research and damaged her physical well-being with radiation exposure. She is the only person to win Nobel prizes in both physics and chemistry. One of her daughters also won a Nobel Prize. (credit: Wikimedia Commons)

We all are curious to some extent. We look around, make generalizations, and try to understand what we see—for example, we look up and wonder whether one type of cloud signals an oncoming storm. As we become serious about exploring nature, we become more organized and formal in collecting and analyzing data. We attempt greater precision, perform controlled experiments (if we can), and write down ideas about how the data may be organized and unified. We then formulate models, theories, and laws based on the data we have collected and analyzed to generalize and communicate the results of these experiments.

A **model** is a representation of something that is often too difficult (or impossible) to display directly. While a model is justified with experimental proof, it is only accurate under limited situations. An example is the planetary model of the atom in which electrons are pictured as orbiting the nucleus, analogous to the way planets orbit the Sun. (See **Figure 1.10**.) We cannot observe electron orbits directly, but the mental image helps explain the observations we can make, such as the emission of light from hot gases (atomic spectra). Physicists use models for a variety of purposes. For example, models can help physicists analyze a scenario and perform a calculation, or they can be used to represent a situation in the form of a computer simulation. A **theory** is an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers. Some theories include models to help visualize phenomena, whereas others do not. Newton's theory of gravity, for example, does not require a model or mental image, because we can observe the objects directly with our own senses. The kinetic theory of gases, on the other hand, is a model in which a gas is viewed as being composed of atoms and molecules. Atoms and molecules are too small to be observed directly with our senses—thus, we picture them mentally to understand what our instruments tell us about the behavior of gases.

A **law** uses concise language to describe a generalized pattern in nature that is supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation. Laws and theories are similar in that they are both scientific statements that result from a tested hypothesis and are supported by scientific evidence. However, the designation *law* is reserved for a concise and very general statement that describes phenomena in nature, such as the law that energy is conserved during any process, or Newton's second law of motion, which relates force, mass, and acceleration by the simple equation $\mathbf{F} = m\mathbf{a}$. A theory, in contrast, is a less concise statement of observed phenomena. For example, the Theory of Evolution and the Theory of Relativity cannot be expressed concisely enough to be considered a law. The biggest difference between a law and a theory is that a theory is much more complex and dynamic. A law describes a single action, whereas a theory explains an entire group of related phenomena. And, whereas a law is a postulate that forms the foundation of the scientific method, a theory is the end result of that process.

Less broadly applicable statements are usually called principles (such as Pascal's principle, which is applicable only in fluids), but the distinction between laws and principles often is not carefully made.

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Thinking in terms of laws and principles will be an important aspect of this course. In terms of problem solving, one of the first steps in solving a problem will be to think about which laws and principles apply. Keep this in mind as we move forward; taking the time to stop and take this step will help you in the long run.

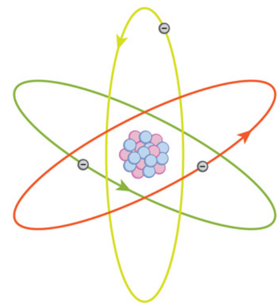


Figure 1.10 What is a model? This planetary model of the atom shows electrons orbiting the nucleus. It is a drawing that we use to form a mental image of the atom that we cannot see directly with our eyes because it is too small.

Models, Theories, and Laws

Models, theories, and laws are used to help scientists analyze the data they have already collected. However, often after a model, theory, or law has been developed, it points scientists toward new discoveries they would not otherwise have made.

The models, theories, and laws we devise sometimes *imply the existence of objects or phenomena as yet unobserved.* These predictions are remarkable triumphs and tributes to the power of science. It is the underlying order in the universe that enables scientists to make such spectacular predictions. However, if *experiment* does not verify our predictions, then the theory or law is wrong, no matter how elegant or convenient it is. Laws can never be known with absolute certainty because it is impossible to perform every imaginable experiment in order to confirm a law in every possible scenario. Physicists operate under the assumption that all scientific laws and theories are valid until a counterexample is observed. If a good-quality, verifiable experiment contradicts a well-established law, then the law must be modified or overthrown completely.

The study of science in general and physics in particular is an adventure much like the exploration of uncharted ocean. Discoveries are made; models, theories, and laws are formulated; and the beauty of the physical universe is made more sublime for the insights gained.

The Scientific Method

As scientists inquire and gather information about the world, they follow a process called the **scientific method**. This process typically begins with an observation and question that the scientist will research. Next, the scientist typically performs some research about the topic and then devises a hypothesis. Then, the scientist will test the hypothesis by performing an experiment. Finally, the scientist analyzes the results of the experiment and draws a conclusion. Note that the scientific method can be applied to many situations that are not limited to science, and this method can be modified to suit the situation.

Consider an example. Let us say that you try to turn on your car, but it will not start. You undoubtedly wonder: Why will the car not start? You can follow a scientific method to answer this question. First off, you may perform some research to determine a variety of reasons why the car will not start. Next, you will state a hypothesis. For example, you may believe that the car is not starting because it has no engine oil. To test this, you open the hood of the car and examine the oil level. You observe that the oil is at an acceptable level, and you thus conclude that the oil level is not contributing to your car issue. To troubleshoot the issue further, you may devise a new hypothesis to test and then repeat the process again.

The Evolution of Natural Philosophy into Modern Physics

Physics was not always a separate and distinct discipline. It remains connected to other sciences to this day. The word *physics* comes from Greek, meaning nature. The study of nature came to be called "natural philosophy." From ancient times through the Renaissance, natural philosophy encompassed many fields, including astronomy, biology, chemistry, physics, mathematics, and medicine. Over the last few centuries, the growth of knowledge has resulted in ever-increasing specialization and branching of natural philosophy into separate fields, with physics retaining the most basic facets. (See **Figure 1.11**, **Figure 1.12**, and **Figure 1.13**.) Physics as it developed from the Renaissance to the end of the 19th century is called **classical physics**. It was transformed into modern physics by revolutionary discoveries made starting at the beginning of the 20th century.

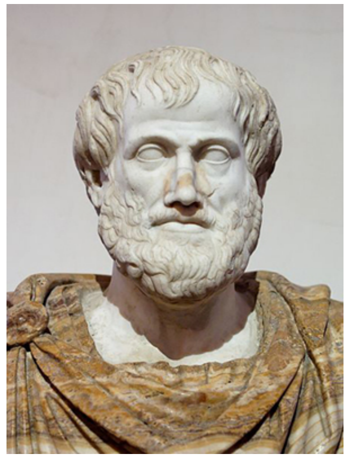


Figure 1.11 Over the centuries, natural philosophy has evolved into more specialized disciplines, as illustrated by the contributions of some of the greatest minds in history. The Greek philosopher **Aristotle** (384–322 B.C.) wrote on a broad range of topics including physics, animals, the soul, politics, and poetry. (credit: Jastrow (2006)/Ludovisi Collection)

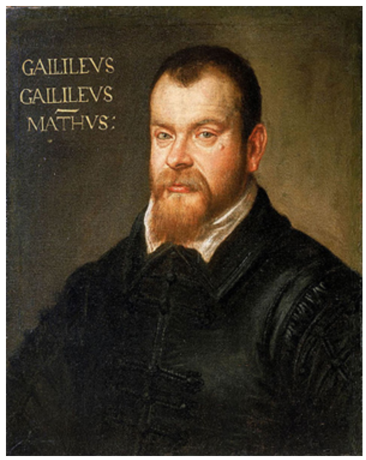


Figure 1.12 **Galileo Galilei** (1564–1642) laid the foundation of modern experimentation and made contributions in mathematics, physics, and astronomy. (credit: Domenico Tintoretto)

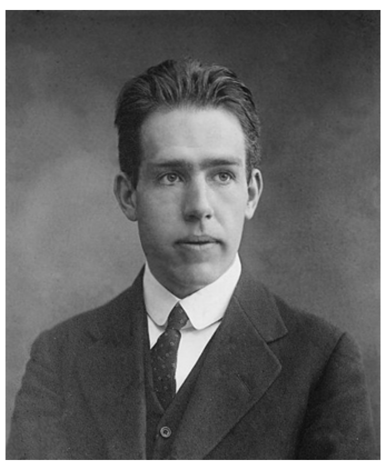


Figure 1.13 **Niels Bohr** (1885–1962) made fundamental contributions to the development of quantum mechanics, one part of modern physics. (credit: United States Library of Congress Prints and Photographs Division)

Classical physics is not an exact description of the universe, but it is an excellent approximation under the following conditions: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields, such as the field generated by the Earth, can be involved. Because humans live under such circumstances, classical physics seems intuitively reasonable, while many aspects of modern physics seem bizarre. This is why models are so useful in modern physics—they let us conceptualize phenomena we do not ordinarily experience. We can relate to models in human terms and visualize what happens when objects move at high speeds or imagine what objects too small to observe with our senses might be like. For example, we can understand an atom's properties because we can picture it in our minds, although we have never seen an atom with our eyes. New tools, of course, allow us to better picture phenomena we cannot see. In fact, new instrumentation has allowed us in recent years to actually “picture” the atom.

Limits on the Laws of Classical Physics

For the laws of classical physics to apply, the following criteria must be met: Matter must be moving at speeds less than about 1% of the speed of light, the objects dealt with must be large enough to be seen with a microscope, and only weak gravitational fields (such as the field generated by the Earth) can be involved.

UMASS AMHERST Instructor's Notes

This idea of creating models will be relevant in 132; we will be dealing with electrons and photons, which don't behave the way most things do. Being able to wrap your head around these ideas will be challenging, but being able to model this strangeness will help you over this hurdle.

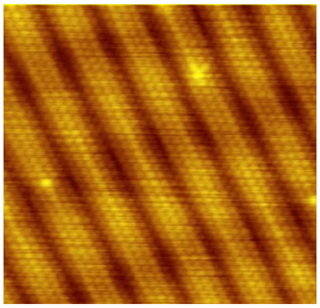


Figure 1.14 Using a scanning tunneling microscope (STM), scientists can see the individual atoms that compose this sheet of gold. (credit: Erwinrossen)

Some of the most spectacular advances in science have been made in modern physics. Many of the laws of classical physics have been modified or rejected, and revolutionary changes in technology, society, and our view of the universe have resulted. Like science fiction, modern physics is filled with fascinating objects beyond our normal experiences, but it has the advantage over science fiction of being very real. Why, then, is the majority of this text devoted to topics of classical physics? There are two main reasons: Classical physics gives an extremely accurate description of the universe under a wide range of everyday circumstances, and knowledge of classical physics is necessary to understand modern physics.

Modern physics itself consists of the two revolutionary theories, relativity and quantum mechanics. These theories deal with the very fast and the very small, respectively. **Relativity** must be used whenever an object is traveling at greater than about 1% of the speed of light or experiences a strong gravitational field such as that near the Sun. **Quantum mechanics** must be used for objects smaller than can be seen with a microscope. The combination of these two theories is *relativistic quantum mechanics,* and it describes the behavior of small objects traveling at high speeds or experiencing a strong gravitational field. Relativistic quantum mechanics is the best universally applicable theory we have. Because of its mathematical complexity, it is used only when necessary, and the other theories are used whenever they will produce sufficiently accurate results. We will find, however, that we can do a great deal of modern physics with the algebra and trigonometry used in this text.

Check Your Understanding

A friend tells you he has learned about a new law of nature. What can you know about the information even before your friend describes the law? How would the information be different if your friend told you he had learned about a scientific theory rather than a law?

Solution

Without knowing the details of the law, you can still infer that the information your friend has learned conforms to the requirements of all laws of nature: it will be a concise description of the universe around us; a statement of the underlying rules that all natural processes follow. If the information had been a theory, you would be able to infer that the information will be a large-scale, broadly applicable generalization.

PhET Explorations: Equation Grapher

Learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (e.g. $y = bx$) to see how they add to generate the polynomial curve.

PhET Interactive Simulation

Figure 1.15 Equation Grapher (http://legacy.cnx.org/content/m64001/1.37/equation-grapher_en.jar)

1.2 Physical Quantities and Units

UMASS AMHERST Instructor's Notes

This section should be a refresher for most of you, and so reading this section is optional. However, if this is unfamiliar territory for you, I would suggest reading through this section, and please come and see me if you are still uncomfortable with this material and we will work something out.

Things to Consider as You Read:

- Familiarizing yourself with the prefixes, such as nano-, milli-, centi-, will save you some trouble down the road.

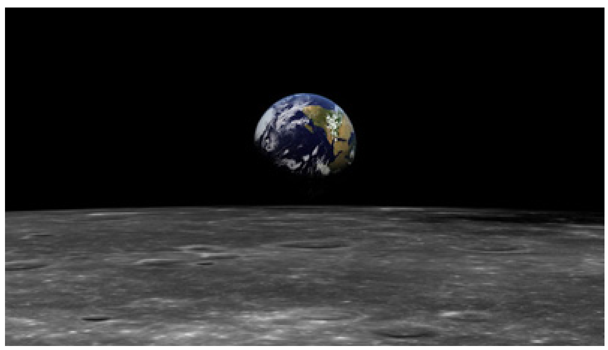


Figure 1.16 The distance from Earth to the Moon may seem immense, but it is just a tiny fraction of the distances from Earth to other celestial bodies. (credit: NASA)

The range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of the Earth, from the tiny sizes of sub-nuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than does qualitative description alone. To comprehend these vast ranges, we must also have accepted units in which to express them. And we shall find that (even in the potentially mundane discussion of meters, kilograms, and seconds) a profound simplicity of nature appears—all physical quantities can be expressed as combinations of only four fundamental physical quantities: length, mass, time, and electric current.

We define a **physical quantity** either by *specifying how it is measured* or by *stating how it is calculated* from other measurements. For example, we define distance and time by specifying methods for measuring them, whereas we define *average speed* by stating that it is calculated as distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way. (See **Figure 1.17**.)

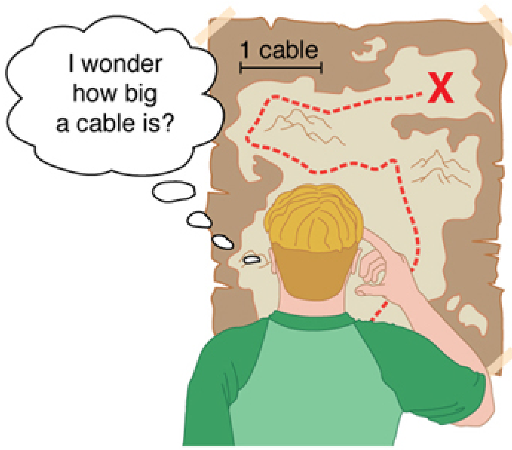


Figure 1.17 Distances given in unknown units are maddeningly useless.

There are two major systems of units used in the world: **SI units** (also known as the metric system) and **English units** (also known as the customary or imperial system). **English units** were historically used in nations once ruled by the British Empire and are still widely used in the United States. Virtually every other country in the world now uses SI units as the standard; the metric system is also the standard system agreed upon by scientists and mathematicians. The acronym "SI" is derived from the French *Système International*.

SI Units: Fundamental and Derived Units

Table 1.1 gives the fundamental SI units that are used throughout this textbook. This text uses non-SI units in a few applications where they are in very common use, such as the measurement of blood pressure in millimeters of mercury (mm Hg). Whenever non-SI units are discussed, they will be tied to SI units through conversions.

Table 1.1 Fundamental SI Units

Length	Mass	Time	Electric Current
meter (m)	kilogram (kg)	second (s)	ampere (A)

It is an intriguing fact that some physical quantities are more fundamental than others and that the most fundamental physical quantities can be defined *only* in terms of the procedure used to measure them. The units in which they are measured are thus called **fundamental units**. In this textbook, the fundamental physical quantities are taken to be length, mass, time, and electric current. (Note that electric current will not be introduced until much later in this text.) All other physical quantities, such as force and electric charge, can be expressed as algebraic combinations of length, mass, time, and current (for example, speed is length divided by time); these units are called **derived units**.

Units of Time, Length, and Mass: The Second, Meter, and Kilogram

The Second

The SI unit for time, the **second**(abbreviated s), has a long history. For many years it was defined as 1/86,400 of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a non-varying, or constant, physical phenomenon (because the solar day is getting longer due to very gradual slowing of the Earth's rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967 the second was redefined as the time required for 9,192,631,770 of these vibrations. (See **Figure 1.18**.) Accuracy in the fundamental units is essential, because all measurements are ultimately expressed in terms of fundamental units and can be no more accurate than are the fundamental units themselves.

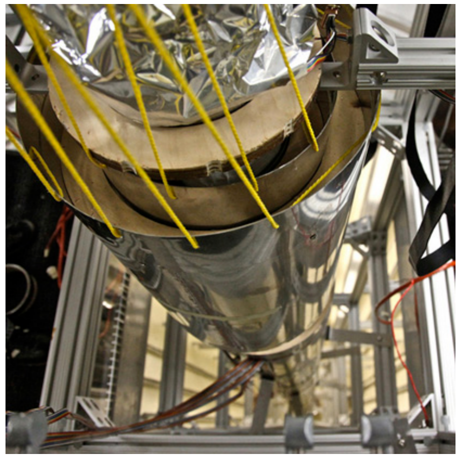


Figure 1.18 An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image is looking down from the top of an atomic fountain nearly 30 feet tall! (credit: Steve Jurvetson/Flickr)

The Meter

The SI unit for length is the **meter** (abbreviated m); its definition has also changed over time to become more accurate and precise. The meter was first defined in 1791 as 1/10,000,000 of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the distance between two engraved lines on a platinum-iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its present definition (partly for greater accuracy) as the distance light travels in a vacuum in 1/299,792,458 of a second. (See **Figure 1.19**.) This change defines the speed of light to be exactly 299,792,458 meters per second. The length of the meter will change if the speed of light is someday measured with greater accuracy.

The Kilogram

The SI unit for mass is the **kilogram** (abbreviated kg); it is defined to be the mass of a platinum-iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. Exact replicas of the standard kilogram are also kept at the United States' National Institute of Standards and Technology, or NIST, located in Gaithersburg, Maryland outside of Washington D.C., and at other locations around the world. The determination of all other masses can be ultimately traced to a comparison with the standard mass.

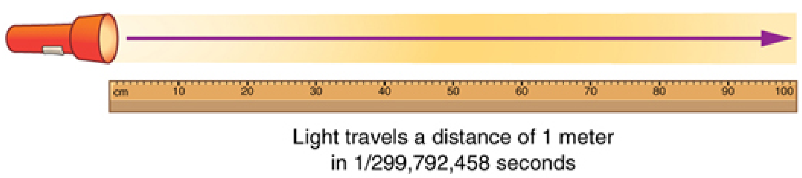


Figure 1.19 The meter is defined to be the distance light travels in 1/299,792,458 of a second in a vacuum. Distance traveled is speed multiplied by time.

Electric current and its accompanying unit, the ampere, will be introduced in **Introduction to Electric Current, Resistance, and Ohm's Law (https://legacy.cnx.org/content/m42339/latest/)** when electricity and magnetism are covered. The initial modules in this textbook are concerned with mechanics, fluids, heat, and waves. In these subjects all pertinent physical quantities can be expressed in terms of the fundamental units of length, mass, and time.

Metric Prefixes

SI units are part of the **metric system**. The metric system is convenient for scientific and engineering calculations because the units are categorized by factors of 10. **Table 1.2** gives metric prefixes and symbols used to denote various factors of 10.

Metric systems have the advantage that conversions of units involve only powers of 10. There are 100 centimeters in a meter, 1000 meters in a kilometer, and so on. In nonmetric systems, such as the system of U.S. customary units, the relationships are not as simple—there are 12 inches in a foot, 5280 feet in a mile, and so on. Another advantage of the metric system is that the same unit can be used over extremely large ranges of values simply by using an appropriate metric prefix. For example, distances in meters are suitable in construction, while distances in kilometers are appropriate for air travel, and the tiny measure of nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications.

The term **order of magnitude** refers to the scale of a value expressed in the metric system. Each power of 10 in the metric system represents a different order of magnitude. For example, $10^1, 10^2, 10^3$, and so forth are all different orders of magnitude. All quantities that can be expressed as a product of a specific power of 10 are said to be of the *same* order of magnitude. For example, the number 800 can be written as 8×10^2, and the number 450 can be written as 4.5×10^2. Thus, the numbers 800 and 450 are of the same order of magnitude: 10^2. Order of magnitude can be thought of as a ballpark estimate for the scale of a value. The diameter of an atom is on the order of $10^{-9}\ \text{m}$, while the diameter of the Sun is on the order of $10^9\ \text{m}$.

The Quest for Microscopic Standards for Basic Units

The fundamental units described in this chapter are those that produce the greatest accuracy and precision in measurement. There is a sense among physicists that, because there is an underlying microscopic substructure to matter, it would be most satisfying to base our standards of measurement on microscopic objects and fundamental physical phenomena such as the speed of light. A microscopic standard has been accomplished for the standard of time, which is based on the oscillations of the cesium atom.

The standard for length was once based on the wavelength of light (a small-scale length) emitted by a certain type of atom, but it has been supplanted by the more precise measurement of the speed of light. If it becomes possible to measure the mass of atoms or a particular arrangement of atoms such as a silicon sphere to greater precision than the kilogram standard, it may become possible to base mass measurements on the small scale. There are also possibilities that electrical phenomena on the small scale may someday allow us to base a unit of charge on the charge of electrons and protons, but at present current and charge are related to large-scale currents and forces between wires.

Table 1.2 Metric Prefixes for Powers of 10 and their Symbols

Prefix	Symbol	Value[1]	Example (some are approximate)			
exa	E	10^{18}	exameter	Em	$10^{18}\ \text{m}$	distance light travels in a century
peta	P	10^{15}	petasecond	Ps	$10^{15}\ \text{s}$	30 million years
tera	T	10^{12}	terawatt	TW	$10^{12}\ \text{W}$	powerful laser output
giga	G	10^{9}	gigahertz	GHz	$10^{9}\ \text{Hz}$	a microwave frequency
mega	M	10^{6}	megacurie	MCi	$10^{6}\ \text{Ci}$	high radioactivity
kilo	k	10^{3}	kilometer	km	$10^{3}\ \text{m}$	about 6/10 mile
hecto	h	10^{2}	hectoliter	hL	$10^{2}\ \text{L}$	26 gallons
deka	da	10^{1}	dekagram	dag	$10^{1}\ \text{g}$	teaspoon of butter
—	—	10^{0} (=1)				
deci	d	10^{-1}	deciliter	dL	$10^{-1}\ \text{L}$	less than half a soda
centi	c	10^{-2}	centimeter	cm	$10^{-2}\ \text{m}$	fingertip thickness
milli	m	10^{-3}	millimeter	mm	$10^{-3}\ \text{m}$	flea at its shoulders
micro	µ	10^{-6}	micrometer	µm	$10^{-6}\ \text{m}$	detail in microscope
nano	n	10^{-9}	nanogram	ng	$10^{-9}\ \text{g}$	small speck of dust
pico	p	10^{-12}	picofarad	pF	$10^{-12}\ \text{F}$	small capacitor in radio
femto	f	10^{-15}	femtometer	fm	$10^{-15}\ \text{m}$	size of a proton
atto	a	10^{-18}	attosecond	as	$10^{-18}\ \text{s}$	time light crosses an atom

Known Ranges of Length, Mass, and Time

The vastness of the universe and the breadth over which physics applies are illustrated by the wide range of examples of known lengths, masses, and times in **Table 1.3**. Examination of this table will give you some feeling for the range of possible topics and numerical values. (See **Figure 1.20** and **Figure 1.21**.)

1. See Appendix A (https://legacy.cnx.org/content/m42699/latest/) for a discussion of powers of 10.

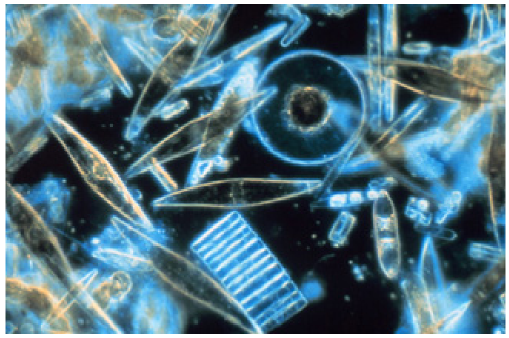


Figure 1.20 Tiny phytoplankton swims among crystals of ice in the Antarctic Sea. They range from a few micrometers to as much as 2 millimeters in length. (credit: Prof. Gordon T. Taylor, Stony Brook University; NOAA Corps Collections)

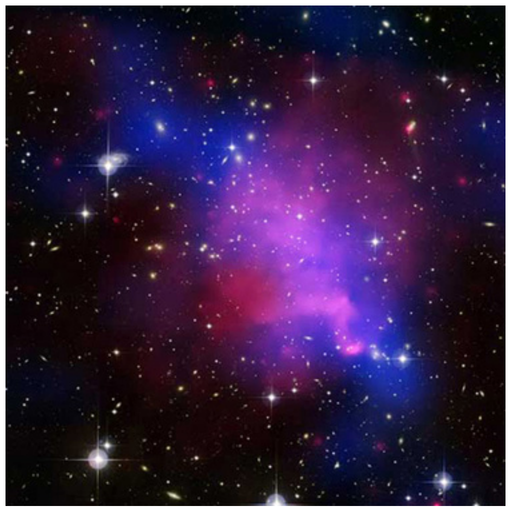


Figure 1.21 Galaxies collide 2.4 billion light years away from Earth. The tremendous range of observable phenomena in nature challenges the imagination. (credit: NASA/CXC/UVic./A. Mahdavi et al. Optical/lensing: CFHT/UVic./H. Hoekstra et al.)

Unit Conversion and Dimensional Analysis

It is often necessary to convert from one type of unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or, perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you will need to convert units of feet to miles.

Let us consider a simple example of how to convert units. Let us say that we want to convert 80 meters (m) to kilometers (km).

The first thing to do is to list the units that you have and the units that you want to convert to. In this case, we have units in *meters* and we want to convert to *kilometers*.

Next, we need to determine a **conversion factor** relating meters to kilometers. A conversion factor is a ratio expressing how many of one unit are equal to another unit. For example, there are 12 inches in 1 foot, 100 centimeters in 1 meter, 60 seconds in 1 minute, and so on. In this case, we know that there are 1,000 meters in 1 kilometer.

Now we can set up our unit conversion. We will write the units that we have and then multiply them by the conversion factor so that the units cancel out, as shown:

$$80\cancel{\text{m}}\times\frac{1\text{ km}}{1000\cancel{\text{m}}}=0.080\text{ km}. \tag{1.1}$$

Note that the unwanted m unit cancels, leaving only the desired km unit. You can use this method to convert between any types of unit.

Click **m42720 (https://legacy.cnx.org/content/m42720/latest/)** for a more complete list of conversion factors.

Table 1.3 Approximate Values of Length, Mass, and Time

Lengths in meters		Masses in kilograms (more precise values in parentheses)		Times in seconds (more precise values in parentheses)	
10^{-18}	Present experimental limit to smallest observable detail	10^{-30}	Mass of an electron $\left(9.11\times10^{-31}\text{ kg}\right)$	10^{-23}	Time for light to cross a proton
10^{-15}	Diameter of a proton	10^{-27}	Mass of a hydrogen atom $\left(1.67\times10^{-27}\text{ kg}\right)$	10^{-22}	Mean life of an extremely unstable nucleus
10^{-14}	Diameter of a uranium nucleus	10^{-15}	Mass of a bacterium	10^{-15}	Time for one oscillation of visible light
10^{-10}	Diameter of a hydrogen atom	10^{-5}	Mass of a mosquito	10^{-13}	Time for one vibration of an atom in a solid
10^{-8}	Thickness of membranes in cells of living organisms	10^{-2}	Mass of a hummingbird	10^{-8}	Time for one oscillation of an FM radio wave
10^{-6}	Wavelength of visible light	1	Mass of a liter of water (about a quart)	10^{-3}	Duration of a nerve impulse
10^{-3}	Size of a grain of sand	10^{2}	Mass of a person	1	Time for one heartbeat
1	Height of a 4-year-old child	10^{3}	Mass of a car	10^{5}	One day $\left(8.64\times10^{4}\text{ s}\right)$
10^{2}	Length of a football field	10^{8}	Mass of a large ship	10^{7}	One year (y) $\left(3.16\times10^{7}\text{ s}\right)$
10^{4}	Greatest ocean depth	10^{12}	Mass of a large iceberg	10^{9}	About half the life expectancy of a human
10^{7}	Diameter of the Earth	10^{15}	Mass of the nucleus of a comet	10^{11}	Recorded history
10^{11}	Distance from the Earth to the Sun	10^{23}	Mass of the Moon $\left(7.35\times10^{22}\text{ kg}\right)$	10^{17}	Age of the Earth
10^{16}	Distance traveled by light in 1 year (a light year)	10^{25}	Mass of the Earth $\left(5.97\times10^{24}\text{ kg}\right)$	10^{18}	Age of the universe
10^{21}	Diameter of the Milky Way galaxy	10^{30}	Mass of the Sun $\left(1.99\times10^{30}\text{ kg}\right)$		
10^{22}	Distance from the Earth to the nearest large galaxy (Andromeda)	10^{42}	Mass of the Milky Way galaxy (current upper limit)		
10^{26}	Distance from the Earth to the edges of the known universe	10^{53}	Mass of the known universe (current upper limit)		

Example 1.1 Unit Conversions: A Short Drive Home

Suppose that you drive the 10.0 km from your university to home in 20.0 min. Calculate your average speed (a) in kilometers per hour (km/h) and (b) in meters per second (m/s). (Note: Average speed is distance traveled divided by time of travel.)

Strategy

First we calculate the average speed using the given units. Then we can get the average speed into the desired units by picking the correct conversion factor and multiplying by it. The correct conversion factor is the one that cancels the unwanted unit and leaves the desired unit in its place.

Solution for (a)

(1) Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now—average speed and other motion concepts will be covered in a later module.) In equation form,

$$\text{average speed} = \frac{\text{distance}}{\text{time}}. \tag{1.2}$$

(2) Substitute the given values for distance and time.

$$\text{average speed} = \frac{10.0\ \text{km}}{20.0\ \text{min}} = 0.500\frac{\text{km}}{\text{min}}. \tag{1.3}$$

(3) Convert km/min to km/h: multiply by the conversion factor that will cancel minutes and leave hours. That conversion factor is $60\ \text{min/hr}$. Thus,

$$\text{average speed} = 0.500\frac{\text{km}}{\text{min}} \times \frac{60\ \text{min}}{1\ \text{h}} = 30.0\frac{\text{km}}{\text{h}}. \tag{1.4}$$

Discussion for (a)

To check your answer, consider the following:

(1) Be sure that you have properly cancelled the units in the unit conversion. If you have written the unit conversion factor upside down, the units will not cancel properly in the equation. If you accidentally get the ratio upside down, then the units will not cancel; rather, they will give you the wrong units as follows:

$$\frac{\text{km}}{\text{min}} \times \frac{1\ \text{hr}}{60\ \text{min}} = \frac{1}{60}\frac{\text{km} \cdot \text{hr}}{\text{min}^2}, \tag{1.5}$$

which are obviously not the desired units of km/h.

(2) Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of km/h and we have indeed obtained these units.

(3) Check the significant figures. Because each of the values given in the problem has three significant figures, the answer should also have three significant figures. The answer 30.0 km/hr does indeed have three significant figures, so this is appropriate. Note that the significant figures in the conversion factor are not relevant because an hour is *defined* to be 60 minutes, so the precision of the conversion factor is perfect.

(4) Next, check whether the answer is reasonable. Let us consider some information from the problem—if you travel 10 km in a third of an hour (20 min), you would travel three times that far in an hour. The answer does seem reasonable.

Solution for (b)

There are several ways to convert the average speed into meters per second.

(1) Start with the answer to (a) and convert km/h to m/s. Two conversion factors are needed—one to convert hours to seconds, and another to convert kilometers to meters.

(2) Multiplying by these yields

$$\text{Average speed} = 30.0\frac{\text{km}}{\text{h}} \times \frac{1\ \text{h}}{3{,}600\ \text{s}} \times \frac{1{,}000\ \text{m}}{1\ \text{km}}, \tag{1.6}$$

$$\text{Average speed} = 8.33\frac{\text{m}}{\text{s}}. \tag{1.7}$$

Discussion for (b)

If we had started with 0.500 km/min, we would have needed different conversion factors, but the answer would have been the same: 8.33 m/s.

You may have noted that the answers in the worked example just covered were given to three digits. Why? When do you need to be concerned about the number of digits in something you calculate? Why not write down all the digits your calculator produces? The module **Accuracy, Precision, and Significant Figures (https://legacy.cnx.org/content/m42120/latest/)** will help you answer these questions.

Nonstandard Units

While there are numerous types of units that we are all familiar with, there are others that are much more obscure. For example, a **firkin** is a unit of volume that was once used to measure beer. One firkin equals about 34 liters. To learn more about nonstandard units, use a dictionary or encyclopedia to research different “weights and measures.” Take note of any unusual units, such as a barleycorn, that are not listed in the text. Think about how the unit is defined and state its relationship to SI units.

Check Your Understanding

Some hummingbirds beat their wings more than 50 times per second. A scientist is measuring the time it takes for a hummingbird to beat its wings once. Which fundamental unit should the scientist use to describe the measurement? Which factor of 10 is the scientist likely to use to describe the motion precisely? Identify the metric prefix that corresponds to this factor of 10.

Solution
The scientist will measure the time between each movement using the fundamental unit of seconds. Because the wings beat

so fast, the scientist will probably need to measure in milliseconds, or 10^{-3} seconds. (50 beats per second corresponds to 20 milliseconds per beat.)

Check Your Understanding

One cubic centimeter is equal to one milliliter. What does this tell you about the different units in the SI metric system?

Solution
The fundamental unit of length (meter) is probably used to create the derived unit of volume (liter). The measure of a milliliter is dependent on the measure of a centimeter.

1.3 Writing Numbers

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Estimating the number of appropriate digits for any calculation. One big thing is that we will use properties of data to predict/determine the number of decimal points that are appropriate.

Other Things to Consider as You Read:

- While we won't be using sig-fig rules, there are some guidelines on the appropriate amount of digits to include in a result.

In general, three digits after the decimal point is what we will go with.

The following section is based upon, with permission:
Denker, J. Uncertainty as Applied to Measurements and Calculations. Uncertainty as Applied to Measurments and Calculations (2011). Available at: **http://www.av8n.com/physics/uncertainty.htm (http://www.av8n.com/physics/uncertainty.htm)** . (Accessed: 26th August 2016)

How Many Digits To Include

Here are some simple rules that apply whenever you are writing down a number:

1. Use many enough digits to avoid unintended loss of information.

2. Use few enough digits to be reasonably convenient.

Important note: The previous two sentences tell you everything you need to know for most purposes, including real-life situations as well as academic situations at every level from primary school up to and including introductory college level.

3. When using a calculator, it is good practice to leave intermediate results in the machine. This is simultaneously more accurate and more convenient than writing them down and then keying them in again.

Seriously: The primary rule is to use plenty of digits. You hardly even need to think about it. Too many is vastly better than too few. To say the same thing the other way: If you ever have more digits than you need *and* they are causing major inconvenience, then you can think about reducing the number of digits.

When to Write Down Uncertainty

In many cases, when you write down a number, you need not *and should not* associate it with any notion of uncertainty.

- One way this can happen is if you have a number with zero uncertainty. If you roll a pair of dice and observe five spots, the number of spots is 5. This is a raw data point, with no uncertainty whatsoever. So just write down the number. Similarly, the number of centimeters per inch is 2.54, by definition, with no uncertainty whatsoever. Again: just write down the number.
- Another possibility is that there is a cooked data blob, which in principle must have “some” uncertainty, but the uncertainty is too small to be interesting. It is insignificant. It is unimportant. It is immaterial. There are plenty of situations a moderately rough approximation is sufficient. There are even some situations where an *extremely* rough approximation is called for.

Along the same lines, here is a less-extreme example that arises in the introductory chemistry class. Suppose the assignment is to balance the equation for the combustion of gasoline, namely

$a\ C_8H_{18} + b\ O_2 \rightarrow x\ CO_2 + y\ H_2O$

by finding numerical values for the coefficients a, b, x, and y. The conventional answer is (a, b, x, y) = (2, 25, 16, 18). The outcome of the real reaction must have “some” uncertainty, because there will generally be some non-idealities, including the presence of other molecules such as CO or C_{60}, not to mention NO_2 or whatever. However, my point is that we don’t necessarily care about these non-idealities. We can perfectly well find the idealized solution to the idealized equation and postpone worrying about the non-idealities and uncertainties until much, much later.

As another example, suppose you use a digital stopwatch to measure some event, and the reading is 1.234 seconds. We call this number the *indicated* time, and we distinguish it from the *true* time of the event. In principle, there is no chance that the indicated time will be exactly equal to the true time; true time is a *continuous variable*, which means that it can take on an infinite amount of values, such as 1.234, 1.2341, 1.23406, 1.2360000009, and so on, whereas the indicated time is *quantized*, which means that it can only take on certain values, such as, if you have a stopwatch with only three decimal places, you can have 1.234 and 1.235, but no values in between, such as 1.2347. However, in many cases you may decide that it is close enough, in which case you should just write down the indicated reading and not worry about the quantization error.

Let us continue with the stopwatch example. Suppose we make two observations. The first reading is 1.234 seconds, and the second reading is just the same, namely 1.234 seconds. Meanwhile, however, you may believe that if you repeated the experiment many times, the resulting set of readings would have some amount of scatter, namely ± 0.01 seconds. The two observations that we actually have don’t show any scatter at all, so your estimate of the uncertainty remains hypothetical and theoretical. Theoretical information is still information, and should be written down in the lab book, plain and simple. For example, you might write a sentence that says “Intuition suggests the timing data is reproducible ± 0.01 seconds.” It would be even better to include some explanation of why you think so. The principle is simple: Write down what you know. Say what you mean, and mean what you say. The same principle applies to the indicated values. The recommend practice is to write down each indicated value, as-is, plain and simple.

You are not trying write down the true values. You don’t know the true values (except insofar as the indicated values represent them, indirectly). You don’t need to know the true values, so don’t worry about it. The rule is: *Write down what you know.* So write down the indicated value. Also: You are not obliged to attribute any uncertainty to the numbers you write down. Normal lab-book entries do not express an uncertainty using $A \pm B$ notation or otherwise, and they do not “imply” an uncertainty using sig figs or otherwise. We are always uncertain about the true value, but we aren’t writing down the true value, so that’s not a concern.

Some people say there must be some uncertainty “associated” with the number you write down, and of course there is, indirectly, in the sense that the indicated value is “associated” with some range of true values. We are always uncertain about the true value, but that does not mean we are uncertain about the indicated value. These things are “associated” ... but they are not the same thing.

In a well-designed experiment, things like readability and quantization error usually do not make a large contribution to the overall uncertainty anyway. Please do not confuse such things with “the” uncertainty.

It is usually a good practice to keep all the original data. When reading an instrument, read it as precisely as the instrument permits, and write down the reading “as is” ... without any conversions, any roundoff, or anything else.

Why We’re Not Using Significant Figures

No matter what you are trying to do, significant figures are the wrong way to do it.

When writing, do not use the number of digits to imply anything about the uncertainty. If you want to describe a distribution, describe it explicitly, perhaps using expressions such as 1.234 ± 0.055.

When reading, do not assume the number of digits tells you anything about the overall uncertainty, accuracy, precision, tolerance, or anything else, unless you are absolutely sure that’s what the writer intended ... and even then, beware that the meaning is very unclear.

People who care about their data don’t use sig figs.

Significant-digit dogma destroys your data and messes up your thinking in many ways, including:

- Given data that can be described by an expression such as $A \pm B$, such as 1.234 ± 0.055, converting it to sig figs gives you an excessively crude and erratic representation of the uncertainty, B.
- Converting to sig figs can cause excessive roundoff error in the nominal value, A. This is a big problem.
- Sig figs cause people to misunderstand the distinction between roundoff error and uncertainty.
- Sig figs cause people to misunderstand the distinction between uncertainty and significance. Sig figs cause people to misunderstand the distinction between the *indicated value* and the corresponding range of *true values*.
- Sig figs cause people to misunderstand the distinction between distributions and numbers. Distributions have width, whereas numbers don’t. Uncertainty is necessarily associated with some distribution, not with any particular point that might have been drawn from the distribution.
- As a consequence, sig figs make people hesitate to write down numbers. They think they need to know the amount of supposedly “associated” uncertainty before they can write the number, when in fact they don’t. Very commonly, there simply isn’t any “associated” uncertainty anyway.
- Sig figs weaken people’s understanding of the axioms of the decimal numeral system.
- Sig figs provide no guidance as to the appropriate decimal representation for repeating decimals such as 80 ÷ 81, or irrational numbers such as $\sqrt{2}$ or π.

UMASS AMHERST Instructor's Notes

For homework problems online, however, answers will be judged as correct if they are within ***two percent*** of the correct value, so my suggestion is to just put in plenty of digits. You may get a message that says something about incorrect number of significant figures if you do this.. However, don't worry about it, the problem has been graded correctly. Mastering physics is just telling you that you did your significant figures wrong. However, since I don't care about significant figures, it's not really a big deal; you will get full credit either way.

In regards to how to present data in the labs, section 1.4 has all the information on how people present data, and so you should follow these guidelines for the labs as well..

1.4 Introduction to Statistics

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Mean and Standard Deviation: Know the definition of both of these as well as how to calculate them for a given data set.
- Using mean and standard deviation in calculations.

This section focuses on some basic statistics facts that we will need for this class. This section is also available as a video. *Link to the video:* **https://www.youtube.com/watch?v=sGjq350QY7c (https://www.youtube.com/watch?v=sGjq350QY7c)**

How to present data for laboratory exerciese

How do people actually present data? A common method is $\mu \pm \sigma$, where μ is the **mean**, and σ is the **standard deviation**. Before we talk about mean and standard deviation, we have to discuss a little bit about measurement. Most objects have some variation. People come in a variety of heights, for example, and even manufactured objects like, say pencils, will have a variety of lengths. If you can measure them precisely enough, this variation may be very small, for objects made by machines for example, but it will still be there. Another example that's not just lengths or heights is the number of blood cells passing through a capillary per second. This quantity will, of course, vary from second to second. Another example might be that if you have a spring based launcher of a ball, the ball will travel slightly different distances each time, if for no other reason than the spring coils slightly different ways on the molecular level with each launch. These types of variation are intrinsic, and result in variation in your measurement. However, sometimes measuring something directly is tough, and you need to use indirect methods, like we do for our library lab. One way to get a feel for the precision of your method is to make the measurement with a few different methods, with what we expect to have similar levels of precision. The variation in the results of the different methods can give a sense of the precision of your measurements; this is how we will evaluate our methods for the library lab.

Mean and Standard Deviation

To talk about the ideas of mean and standard deviation, it's helpful to have an example. Say we took the height of 20 men from the United States and presented the data in the table below.

Table 1.4

Person	Height [cm]
1	177.7
2	181.4
3	179.4
4	164.9
5	180.4
6	174.0
7	178.6
8	176.1
9	181.9
10	179.7
11	175.8
12	175.0
13	180.9
14	181.9
15	181.0
16	180.5
17	169.2
18	173.3
19	171.8
20	176.5

Note that there are no uncertainties listed in this table. Yes, the ruler that we're using has some limit of precision, which is apparently .1 cm according to this table, but the variation between measurements is much larger than this, so the precision of the ruler won't be too important. In a well-designed experiment the precision of your instruments should be much less than the intrinsic variation that you are trying to measure.

Mean

The most complete way to report this data would be to report the entire table as we've done here. However, this becomes impractical as more and more data are collected. Moreover, it becomes very difficult to see trends when you just got big lists of numbers. Therefore, we need ways to characterize our data. If you're looking for a way to characterize the data, the first thing that you might think of to do would be to take the average. Now in mathematics, the word average is replaced with the word mean; they're synonyms. There are many different symbols for mean and each discipline seems to have their own one, so I'm going to present you with all of them. I wish we could agree on which symbol to use, but we can't, so I'm just going to show you all of what's out there. The Greek letter μ here is a very general symbol for mean. A general tip I would have is that you learn your Greek alphabet. Another way to represent mean is, let's say we're using the variable H to represent the height of a man, then you might see $\bar{H}$ or $\langle H \rangle$ to represent the mean. The formula for the mean is given by

$$\mu = \frac{1}{N}\sum_i x_i$$

Many of the equations that you might see in this section can get pretty ugly looking, but they are manageable if you stop and parse them down and read what the equation is trying to tell you. A good tip for questions is to read actually right to left. So, let's give that a shot with this equation. The letter i represents an index over the measurements. Here we have 20 measurements, so i is an integer that runs from 1 to 20. x_i is one specific measurement, so x_5 is the height of the fifth person which, according to our table, is 180.4 cm. To calculate the mean, we add up all the measurements, and then divide by the number of measurements. Let's calculate the mean. For these data, when we add up all of the measurements, we get a sum of 3540 cm. We divide by the number of measurements; take $\frac{3540cm}{20cm}$, which gives us a result of 177 cm. This is our average or mean.

Standard Deviation

The mean provides a great starting point for characterizing data, but it's insufficient because it's missing a key feature. Just representing the mean gives us no clue on how spread out these data are. Phrased differently, we don't have any information on what is the average distance for a random data point to the mean. So if we're looking at this question, let's try to translate this question into mathematics. Well, the distance from a given data point x_i to the mean would be $\langle H \rangle - x_i$, and the average distance would be, well, take all of these different distances, $\mu - x_i$, add them all up and divide by the number of measurements.

However, this idea has a problem. Some distances are lower than the mean. For example, person 6 is slightly shorter than our average. So, his distance to the mean will be positive, while some people are taller than the average, for example person 2, so their distance to the mean will be negative. If I add up positive numbers and negative numbers, I'll probably get a result that's very close to zero due to the cancellation. So how can we get around this problem? Well, you might think absolute values, but for calculus reasons, absolute values have some problems, so a better way to get around having negative numbers is to look at squaring them, because no matter what, when I take a number and square it, the result is positive. So, let's look at the average squared distance from the mean. Mathematically, the average squared distance from the mean would be, take the distance from the data point to the mean, just mean minus data point, square it, add them all up and divide by the number of measurements.

$$\sigma^2 = \frac{1}{n}\sum_{i}^{n} (x_i - \mu)^2$$

Now all the numbers being added are positive, so there's no cancellation. This quantity is called the variance, and we will label it with the variable σ^2 , for reasons that will become apparent in a moment. Let's calculate the variance for this data. Again, variance is kind of an ugly formula, so you really got to slow down and take it one piece at a time. So let's take an entry *i*=1 and what do we do, take the entry and subtract it from the mean. This for *i*=1 is -0.7 cm. We repeat this for all of our data. We get these results. Next in variance, we see we should square so for *i*=1 the result is .49 cm^2. I want to point out that we've now moved from cm to cm^2, because when you square a number with units, you got to square the units too, and when we repeat it for all of our data and get these results to calculate variance, we take all of these numbers and add them up, which gives us 403.5 cm2. To get the variance divided by the number of measurements, which in this case is 20 giving us a variance of 20.2 cm^2.

Now variance has different units than mean, as we've already seen. The mean for this data set is 177.0 cm while the variance is 20.2 cm ^{2}. It's very difficult to compare numbers with different units. To deal with this we, instead of looking at the variance, look at the standard deviation, which we represent by σ . So, the standard deviation is the square root of the variance. This is why we represent variance with a σ^2 . In this example to get the standard deviation, we take square root of the variance so the square root of 20.2 cm^2, to give us 4.49 cm. Now we have two quantities that are both in cm, and allows us to compare them.

So how do we report numbers in the laboratory exercises? In this class, most of the labs in this course will have multiple measurements. We can use these different trials to calculate a mean and a standard deviation, and we can use this standard deviation as an uncertainty and use it to tell us how many decimals we should record. In our height example, we had a mean of 177 cm and a standard deviation a 4.49 cm. An appropriate way to represent this result would be 177 plus or minus 4.5 cm. This representation has a lot of advantages; it represents the average 177, and we have the standard deviation, which gives the person reading the number a sense of the spread of the data, and we have a reasonable number of digits. I've gone with one digit past the decimal point, and I did this based upon the standard deviation. While our standard deviation is officially 4.49, I rounded it to 4.5, because .01 is very small relative to our standard deviation, so I can't really trust that .01. So while I removed some certainty of nice hard sig-fig rules, this is how numbers are actually reported in research, and this is how we'll do it in class. Part of the point of the laboratory exercises to get you some experience with this sort of usage.

The following section is based upon:
Denker, J. Uncertainty as Applied to Measurements and Calculations. Uncertainty as Applied to Measurments and Calculations (2011). Available at: **http://www.av8n.com/physics/uncertainty.htm (http://www.av8n.com/physics/uncertainty.htm)** . (Accessed: 26th August 2016)

Incorporating Mean and Standard Deviation into Calculations - Crank Three Times

Here's a simple yet powerful way of estimating the uncertainty of a result, given the uncertainty of the thing(s) it depends on.

Here's the procedure, in the simple case when there is only one input variable with appreciable uncertainty:

a. Set up the calculation. Do it once in the usual way, using the nominal, best-estimate values for all the input variables.
b. Then re-do the calculation with the uncertain variable at the end of its upper error bar.
c. Then re-do the calculation with the uncertain variable at the end of its lower error bar.

I call this the *Crank Three Times* method. Here is an example:

Table 1.5

x	$\frac{1}{x}$
2.02 (high case)	.495
2 (nominal case)	.5
1.98	.505

This table tells us that if *x* is distributed according to *x*=2±.02 then $\frac{1}{x}$ is distributed according to $\frac{1}{x}$=.5±.005. The Crank Three Times method is by no means an exact error analysis. It is an approximation. The nice thing is that you can understand the nature of the approximation. One of the glories of the Crank Three Times method is that in cases where it doesn't work, it will tell you it isn't working, provided you listen to what it's trying to tell you. If you get asymmetrical error bars, you need to investigate further. Something bad is happening, and you need to check closely to see whether it is a little bit bad or very, very bad.

As far as I can tell, for every flaw that this method has, the sig-figs method has the same flaw plus others ... which means Crank Three Times is therefore superior.

Crank Three Times shouldn't require more than a few minutes of labor. Once a problem is set up, turning the crank should take only a couple of minutes; if it takes longer than that you should have been doing it on a spreadsheet all along. And if you are using a spreadsheet, Crank Three Times is super-easy and super-quick.

If you have *N* variables that are (or might be) making a significant contribution to the uncertainty of the result, the Crank Three Times method could more precisely be called the Crank 2*N*+1 Times method. Here's the procedure: Set up the spreadsheet and wiggle each variable in turn, and see what happens. Wiggle them *one* at a time, leaving the other *N*−1 at their original, nominal values.

For example, let's say you're looking for the area of a rectangle, and the length and width of the rectangle are measured to be 5±.6 and 2±.4, respectively. Using $A = lw$, the nominal crank is 10 cm^2. Wiggling the length results in 10±1.2 cm^2 and wiggling the width results in 10±2 cm^2. Note that even though the width has a lower uncertainty associated with it, its uncertainty it creates in the result is higher than that of the length.

If you are worried about what happens when two of the input variables are simultaneously at the ends of their error bars, you can check that case if you want. However, beware that if there are many variables, checking all the possibilities is exponentially laborious. Furthermore, it is improbable that many variables would simultaneously take on extreme values, and checking extreme cases can lead you to overestimate the uncertainty. For these reasons, and others, if you have numerous variables and need to study the system properly, at some point you need to give up on the Crank Three Timesmethod and do something more sophisticated called a Monte Carlo analysis which we will not discuss in this class. The Crank Three Times method can be considered an ultra-simplified variation of the Monte Carlo method, suitable for introductory reconnaissance.

In the *rare* situation where you want a worst-case analysis, you can move each variable to whichever end of its error bar makes a positive contribution to the final answer, and then flip them all so that each one makes a negative contribution. In most cases, however, a worst-case analysis is wildly over-pessimistic, especially when there are more than a few uncertain variables.

Remember: there are many cases, especially when there are multiple uncertain variables and/or correlations among the variables and/or nonlinearities for which you will need to be more sophisticated.. The Crank Three Times method can be considered an ultra-simplified variation of the Monte Carlo method, suitable for introductory reconnaissance.

Here is another example, which is more interesting because it exhibits nonlinearity:

Table 1.6

x	$\frac{1}{x}$
2.9 (high case)	.35
2 (nominal case)	.5
1.1 (low case)	.91

Here we see that if *x* is distributed according to *x*=2±.9 then $\frac{1}{x}$ is distributed according to $\frac{1}{x}=5^{-0.16}_{+0.42}$. Even though the error bars on *x* are symmetric, the error bars on $\frac{1}{x}$ are markedly lopsided.

Lopsided error bars are fairly common in practice. Sometimes they are merely a symptom of a harmless nonlinearity, but sometimes they are a symptom of something much worse. As an example, let's say you had a calculation that was $\frac{1}{\text{x-2}}$, and the value of x was found to be 3±2.When you do the crank three times method, the nominal crank is 1, the upper crank is $\frac{1}{3}$, and the lower crank is -1. Both the upper and lower cranks give values less than the nominal; the result is $1^{-2/3}_{-2}$, which doesn't make much sense. The absurdity arises because at x=2, the function $\frac{1}{\text{x-2}}$ is equal to $\frac{1}{0}$ and the function is undefined, i.e. has a ***singularity*****.** Here's a graph of the function and the data, which is the value of x and its uncertainty:

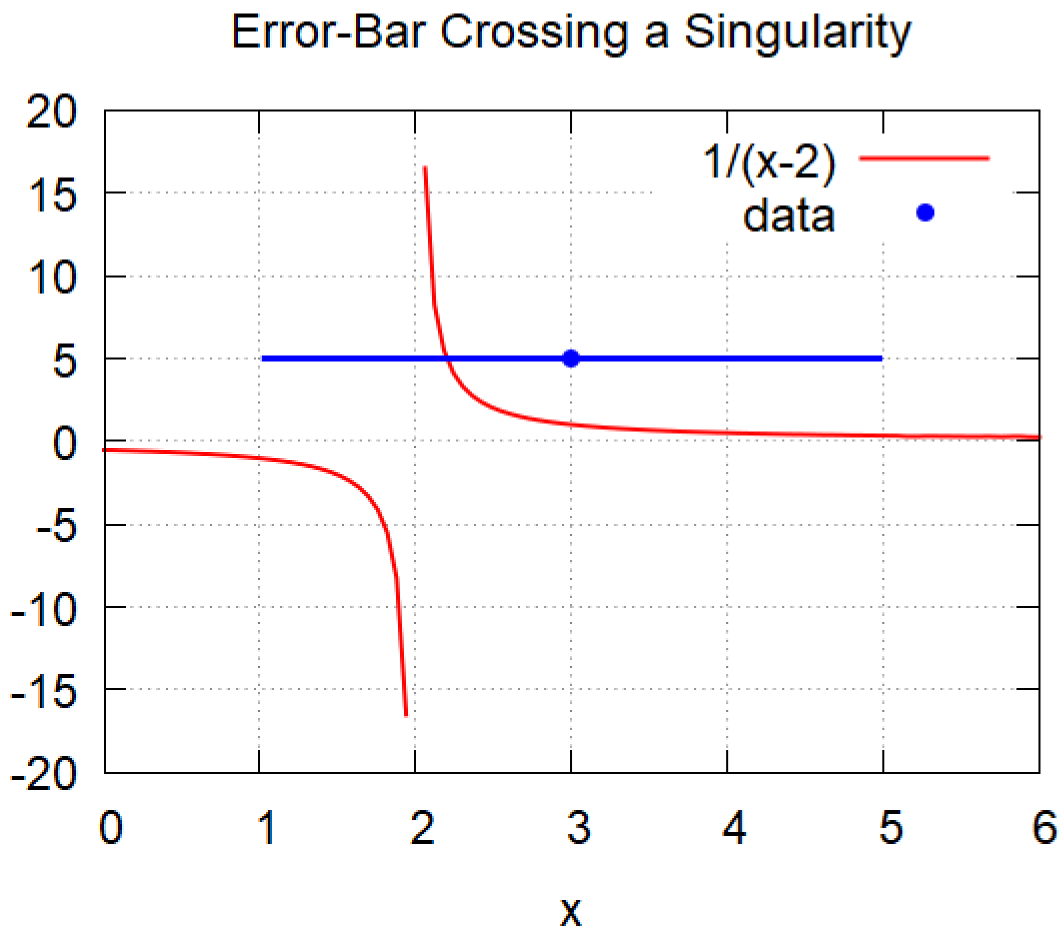


Figure 1.22

Notice that for all the values above the nominal value of x (indicated by the point), the function behaves normally, but for the values below, the function has a ‘break’ in it at x=2, where the function becomes a division by zero. Notice as well that the function spikes up around the point x=2 as well. If we were to continue the function as it approaches closer and closer to 2, we would see that the function would go up to infinity and to negative infinity, and infinite values tend to break uncertainty calculations. What the nonsense result is trying to tell you is that your error bars contain a problem point, such as the one above. Results such as these are the ones you should be wary of.

1.5 Accuracy and Precision

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Intepreting data and results. This ties into understanding how mean and standard deviation relate to uncertainty.

Figure 1.23 A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The “known masses” are typically metal cylinders of standard mass such as 1 gram, 10 grams, and 100 grams. (credit: Serge Melki)

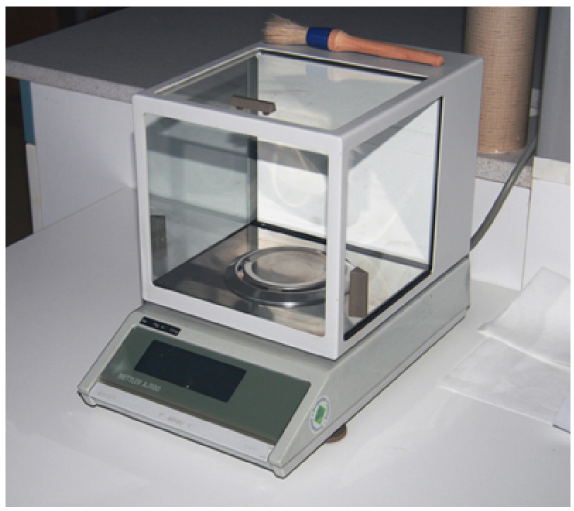


Figure 1.24 Many mechanical balances, such as double-pan balances, have been replaced by digital scales, which can typically measure the mass of an object more precisely. Whereas a mechanical balance may only read the mass of an object to the nearest tenth of a gram, many digital scales can measure the mass of an object up to the nearest thousandth of a gram. (credit: Karel Jakubec)

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the correct value for that measurement. For example, let us say that you are measuring the length of standard computer paper. The packaging in which you purchased the paper states that it is 11.0 inches long. You measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the correct value of 11.0 inches. In contrast, if you had obtained a measurement of 12 inches, your measurement would not be very accurate.

The **precision** of a measurement system is refers to how close the agreement is between repeated measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements would be to determine the range, or difference, between the lowest and the highest measured values. In that case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by at most 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9, 11.1, and 11.9, then the measurements would not be very precise because there would be significant variation from one measurement to another.

The measurements in the paper example are both accurate and precise, but in some cases, measurements are accurate but not precise, or they are precise but not accurate. Let us consider an example of a GPS system that is attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the center of a bull’s-eye target, and think of each GPS attempt to locate the restaurant as a black dot. In **Figure 1.25**, you can see that the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the center of the target. This indicates a low precision, high accuracy measuring system. However, in **Figure 1.26**, the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high precision, low accuracy measuring system.

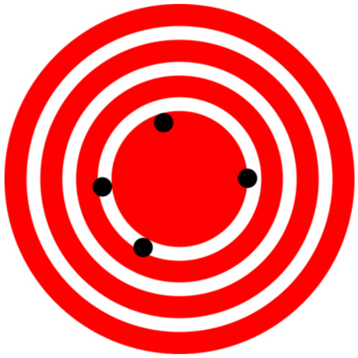


Figure 1.25 A GPS system attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy. (credit: Dark Evil)

Figure 1.26 In this figure, the dots are concentrated rather closely to one another, indicating high precision, but they are rather far away from the actual location of the restaurant, indicating low accuracy. (credit: Dark Evil)

Accuracy, Precision, and Uncertainty

The degree of accuracy and precision of a measuring system are related to the **uncertainty** in the measurements. Uncertainty is a quantitative measure of how much your measured values deviate from a standard or expected value. If your measurements are not very accurate or precise, then the uncertainty of your values will be very high. In more general terms, uncertainty can be thought of as a disclaimer for your measured values. For example, if someone asked you to provide the mileage on your car, you might say that it is 45,000 miles, plus or minus 500 miles. The plus or minus amount is the uncertainty in your value. That is, you are indicating that the actual mileage of your car might be as low as 44,500 miles or as high as 45,500 miles, or anywhere in between. All measurements contain some amount of uncertainty. In our example of measuring the length of the paper, we might say that the length of the paper is 11 in., plus or minus 0.2 in. The uncertainty in a measurement, A, is often denoted as δA ("delta A"), so the measurement result would be recorded as $A \pm \delta A$. In our paper example, the length of the paper could be expressed as $11 \text{ in.} \pm 0.2.$

Precision and accuracy is also is through mean and standard deviation. If your mean is close to the actual value, it is considered to be *accurate*, and if your standard deviation is small, it is considered to be *precise*. For example, let's say you were measuring the speed of a car driving by at 30 m/s. If the measured result is 60±0.1 m/s, the mean is far too high but your standard deviation is fairly low, so this result would be precise but not accurate. If the measured result is 29±40 m/s, the mean is fairly close to the actual speed, but the standard deviation is fairly high, so this result would be accurate, but not precise.

The factors contributing to uncertainty in a measurement include:

1. Limitations of the measuring device,
2. Irregularities in the object being measured,
3. Any other factors that affect the outcome (highly dependent on the situation).

In our example, such factors contributing to the uncertainty could be the following: the smallest division on the ruler is 0.1 in., the person using the ruler has bad eyesight, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be based on a careful consideration of all the factors that might contribute and their possible effects.

Making Connections: Real-World Connections – Fevers or Chills?

Uncertainty is a critical piece of information, both in physics and in many other real-world applications. Imagine you are caring for a sick child. You suspect the child has a fever, so you check his or her temperature with a thermometer. What if the uncertainty of the thermometer were 3.0°C? If the child's temperature reading was 37.0°C (which is normal body temperature), the "true" temperature could be anywhere from a hypothermic 34.0°C to a dangerously high 40.0°C. A thermometer with an uncertainty of $3.0 ^\circ\text{C}$ would be useless.

Check Your Understanding

A high school track coach has just purchased a new stopwatch. The stopwatch manual states that the stopwatch has an uncertainty of $\pm 0.05\text{ s}$. Runners on the track coach's team regularly clock 100-m sprints of 11.49 s to 15.01 s . At the school's last track meet, the first-place sprinter came in at 12.04 s and the second-place sprinter came in at 12.07 s . Will the coach's new stopwatch be helpful in timing the sprint team? Why or why not?

Solution
No, the uncertainty in the stopwatch is too great to effectively differentiate between the sprint times.

PhET Explorations: Estimation

Explore size estimation in one, two, and three dimensions! Multiple levels of difficulty allow for progressive skill improvement.

PhET Interactive Simulation

Figure 1.27 Estimation (http://legacy.cnx.org/content/m64071/1.4/estimation_en.jar)

Glossary

accuracy: the degree to which a measured value agrees with correct value for that measurement

classical physics: physics that was developed from the Renaissance to the end of the 19th century

conversion factor: a ratio expressing how many of one unit are equal to another unit

derived units: units that can be calculated using algebraic combinations of the fundamental units

English units: system of measurement used in the United States; includes units of measurement such as feet, gallons, and pounds

fundamental units: units that can only be expressed relative to the procedure used to measure them

kilogram: the SI unit for mass, abbreviated (kg)

law: a description, using concise language or a mathematical formula, a generalized pattern in nature that is supported by scientific evidence and repeated experiments

meter: the SI unit for length, abbreviated (m)

metric system: a system in which values can be calculated in factors of 10

model: representation of something that is often too difficult (or impossible) to display directly

modern physics: the study of relativity, quantum mechanics, or both

order of magnitude: refers to the size of a quantity as it relates to a power of 10

percent uncertainty: the ratio of the uncertainty of a measurement to the measured value, expressed as a percentage

physical quantity : a characteristic or property of an object that can be measured or calculated from other measurements

physics: the science concerned with describing the interactions of energy, matter, space, and time; it is especially interested in what fundamental mechanisms underlie every phenomenon

precision: the degree to which repeated measurements agree with each other

quantum mechanics: the study of objects smaller than can be seen with a microscope

relativity: the study of objects moving at speeds greater than about 1% of the speed of light, or of objects being affected by a strong gravitational field

scientific method: a method that typically begins with an observation and question that the scientist will research; next, the scientist typically performs some research about the topic and then devises a hypothesis; then, the scientist will test the hypothesis by performing an experiment; finally, the scientist analyzes the results of the experiment and draws a conclusion

second: the SI unit for time, abbreviated (s)

SI units : the international system of units that scientists in most countries have agreed to use; includes units such as meters, liters, and grams

theory: an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers

uncertainty: a quantitative measure of how much your measured values deviate from a standard or expected value

units : a standard used for expressing and comparing measurements

Section Summary

1.1 An Introduction to Physics

- Science seeks to discover and describe the underlying order and simplicity in nature.
- Physics is the most basic of the sciences, concerning itself with energy, matter, space and time, and their interactions.
- Scientific laws and theories express the general truths of nature and the body of knowledge they encompass. These laws of nature are rules that all natural processes appear to follow.

1.2 Physical Quantities and Units

- Physical quantities are a characteristic or property of an object that can be measured or calculated from other measurements.
- Units are standards for expressing and comparing the measurement of physical quantities. All units can be expressed as combinations of four fundamental units.
- The four fundamental units we will use in this text are the meter (for length), the kilogram (for mass), the second (for time), and the ampere (for electric current). These units are part of the metric system, which uses powers of 10 to relate quantities over the vast ranges encountered in nature.
- The four fundamental units are abbreviated as follows: meter, m; kilogram, kg; second, s; and ampere, A. The metric system also uses a standard set of prefixes to denote each order of magnitude greater than or lesser than the fundamental unit itself.
- Unit conversions involve changing a value expressed in one type of unit to another type of unit. This is done by using conversion factors, which are ratios relating equal quantities of different units.

1.3 Writing Numbers

- Use many enough digits to avoid unintended loss of information.
- Use few enough digits to be reasonably convenient.
- Avoid using significant figures

Conceptual Questions

1.1 An Introduction to Physics

1. Models are particularly useful in relativity and quantum mechanics, where conditions are outside those normally encountered by humans. What is a model?

2. How does a model differ from a theory?

3. If two different theories describe experimental observations equally well, can one be said to be more valid than the other (assuming both use accepted rules of logic)?

4. What determines the validity of a theory?

5. Certain criteria must be satisfied if a measurement or observation is to be believed. Will the criteria necessarily be as strict for an expected result as for an unexpected result?

6. Can the validity of a model be limited, or must it be universally valid? How does this compare to the required validity of a theory or a law?

7. Classical physics is a good approximation to modern physics under certain circumstances. What are they?

8. When is it *necessary* to use relativistic quantum mechanics?

9. Can classical physics be used to accurately describe a satellite moving at a speed of 7500 m/s? Explain why or why not.

1.2 Physical Quantities and Units

10. Identify some advantages of metric units.

1.5 Accuracy and Precision

11. What is the relationship between the accuracy and uncertainty of a measurement?

12. Prescriptions for vision correction are given in units called *diopters* (D). Determine the meaning of that unit. Obtain information (perhaps by calling an optometrist or performing an internet search) on the minimum uncertainty with which corrections in diopters are determined and the accuracy with which corrective lenses can be produced. Discuss the sources of uncertainties in both the prescription and accuracy in the manufacture of lenses.

Problems & Exercises

1.2 Physical Quantities and Units

1. The speed limit on some interstate highways is roughly 100 km/h. (a) What is this in meters per second? (b) How many miles per hour is this?

2. A car is traveling at a speed of 33 m/s. (a) What is its speed in kilometers per hour? (b) Is it exceeding the 90 km/h speed limit?

3. Show that $1.0 \text{ m/s} = 3.6 \text{ km/h}$. Hint: Show the explicit steps involved in converting $1.0 \text{ m/s} = 3.6 \text{ km/h}.$

4. American football is played on a 100-yd-long field, excluding the end zones. How long is the field in meters? (Assume that 1 meter equals 3.281 feet.)

5. Soccer fields vary in size. A large soccer field is 115 m long and 85 m wide. What are its dimensions in feet and inches? (Assume that 1 meter equals 3.281 feet.)

6. What is the height in meters of a person who is 6 ft 1.0 in. tall? (Assume that 1 meter equals 39.37 in.)

7. Mount Everest, at 29,028 feet, is the tallest mountain on the Earth. What is its height in kilometers? (Assume that 1 kilometer equals 3,281 feet.)

8. The speed of sound is measured to be 342 m/s on a certain day. What is this in km/h?

9. Tectonic plates are large segments of the Earth's crust that move slowly. Suppose that one such plate has an average speed of 4.0 cm/year. (a) What distance does it move in 1 s at this speed? (b) What is its speed in kilometers per million years?

10. (a) Refer to Table 1.3 to determine the average distance between the Earth and the Sun. Then calculate the average speed of the Earth in its orbit in kilometers per second. (b) What is this in meters per second?

1.5 Accuracy and Precision

Express your answers to problems in this section to the correct number of significant figures and proper units.

11. Suppose that your bathroom scale reads your mass as 65 kg with a 3% uncertainty. What is the uncertainty in your mass (in kilograms)?

12. A good-quality measuring tape can be off by 0.50 cm over a distance of 20 m. What is its percent uncertainty?

13. (a) A car speedometer has a 5.0% uncertainty. What is the range of possible speeds when it reads 90 km/h? (b) Convert this range to miles per hour. $(1 \text{ km} = 0.6214 \text{ mi})$

14. An infant's pulse rate is measured to be 130 ± 5 beats/min. What is the percent uncertainty in this measurement?

15. (a) Suppose that a person has an average heart rate of 72.0 beats/min. How many beats does he or she have in 2.0 y? (b) In 2.00 y? (c) In 2.000 y?

16. A can contains 375 mL of soda. How much is left after 308 mL is removed?

17. State how many significant figures are proper in the results of the following calculations: (a) $(106.7)(98.2) / (46.210)(1.01)$ (b) $(18.7)^2$ (c) $\left(1.60\times10^{-19}\right)(3712)$.

18. (a) How many significant figures are in the numbers 99 and 100? (b) If the uncertainty in each number is 1, what is the percent uncertainty in each? (c) Which is a more meaningful way to express the accuracy of these two numbers, significant figures or percent uncertainties?

19. (a) If your speedometer has an uncertainty of 2.0 km/h at a speed of 90 km/h, what is the percent uncertainty? (b) If it has the same percent uncertainty when it reads 60 km/h, what is the range of speeds you could be going?

20. (a) A person's blood pressure is measured to be $120 \pm 2 \text{ mm Hg}$. What is its percent uncertainty? (b) Assuming the same percent uncertainty, what is the uncertainty in a blood pressure measurement of 80 mm Hg?

21. A person measures his or her heart rate by counting the number of beats in 30 s. If 40 ± 1 beats are counted in $30.0 \pm 0.5 \text{ s}$, what is the heart rate and its uncertainty in beats per minute?

22. What is the area of a circle 3.102 cm in diameter?

23. If a marathon runner averages 9.5 mi/h, how long does it take him or her to run a 26.22-mi marathon?

24. A marathon runner completes a 42.188-km course in 2 h, 30 min, and 12 s. There is an uncertainty of 25 m in the distance traveled and an uncertainty of 1 s in the elapsed time. (a) Calculate the percent uncertainty in the distance. (b) Calculate the uncertainty in the elapsed time. (c) What is the average speed in meters per second? (d) What is the uncertainty in the average speed?

25. The sides of a small rectangular box are measured to be $1.80 \pm 0.01 \text{ cm}$, $2.05 \pm 0.02 \text{ cm, and } 3.1 \pm 0.1 \text{ cm}$ long. Calculate its volume and uncertainty in cubic centimeters.

26. When non-metric units were used in the United Kingdom, a unit of mass called the *pound-mass* (lbm) was employed, where $1 \text{ lbm} = 0.4539 \text{ kg}$. (a) If there is an uncertainty of 0.0001 kg in the pound-mass unit, what is its percent uncertainty? (b) Based on that percent uncertainty, what mass in pound-mass has an uncertainty of 1 kg when converted to kilograms?

27. The length and width of a rectangular room are measured to be $3.955 \pm 0.005 \text{ m}$ and $3.050 \pm 0.005 \text{ m}$. Calculate the area of the room and its uncertainty in square meters.

28. A car engine moves a piston with a circular cross section of $7.500 \pm 0.002 \text{ cm}$ diameter a distance of $3.250 \pm 0.001 \text{ cm}$ to compress the gas in the cylinder. (a) By what amount is the gas decreased in volume in cubic centimeters? (b) Find the uncertainty in this volume.

2 KINEMATICS

Figure 2.1 The motion of an American kestrel through the air can be described by the bird's displacement, speed, velocity, and acceleration. When it flies in a straight line without any change in direction, its motion is said to be one dimensional. (credit: Vince Maidens, Wikimedia Commons)

Chapter Outline

Introduction to One-Dimensional Kinematics

Objects are in motion everywhere we look. Everything from a tennis game to a space-probe flyby of the planet Neptune involves motion. When you are resting, your heart moves blood through your veins. And even in inanimate objects, there is continuous motion in the vibrations of atoms and molecules. Questions about motion are interesting in and of themselves: *How long will it take for a space probe to get to Mars? Where will a football land if it is thrown at a certain angle?* But an understanding of motion is also key to understanding other concepts in physics. An understanding of acceleration, for example, is crucial to the study of force.

Our formal study of physics begins with **kinematics** which is defined as the *study of motion without considering its causes*. The word "kinematics" comes from a Greek term meaning motion and is related to other English words such as "cinema" (movies) and “kinesiology” (the study of human motion). In one-dimensional kinematics and **Two-Dimensional Kinematics (https://legacy.cnx.org/content/m42126/latest/)** we will study only the *motion* of a football, for example, without worrying about what forces cause or change its motion. Such considerations come in other chapters. In this chapter, we examine the simplest type of motion—namely, motion along a straight line, or one-dimensional motion. In **Two-Dimensional Kinematics (https://legacy.cnx.org/content/m42126/latest/)** , we apply concepts developed here to study motion along curved paths (two- and three-dimensional motion); for example, that of a car rounding a curve.

2.1 Displacement

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- How position differs from displacement, and how distance differs from displacement
- When to use position and when to use displacement
- Identify that displacement has a direction

Figure 2.2 These cyclists in Vietnam can be described by their position relative to buildings and a canal. Their motion can be described by their change in position, or displacement, in the frame of reference. (credit: Suzan Black, Fotopedia)

UMASS AMHERST Instructor's Notes

While *position* and *displacement* might seem like they mean the same thing, there is a difference in what they mean that is important to understand. It's helpful to keep this in mind as you read about the two, and perhaps try to put what the difference is in your own words.

Position

In order to describe the motion of an object, you must first be able to describe its **position**—where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of an object as it relates to stationary objects in that reference frame. For example, a rocket launch would be described in terms of the position of the rocket with respect to the Earth as a whole, while a professor’s position could be described in terms of where she is in relation to the nearby white board. (See **Figure 2.3**.) In other cases, we use reference frames that are not stationary but are in motion relative to the Earth. To describe the position of a person in an airplane, for example, we use the airplane, not the Earth, as the reference frame. (See **Figure 2.4**.)

Displacement

If an object moves relative to a reference frame (for example, if a professor moves to the right relative to a white board or a passenger moves toward the rear of an airplane), then the object’s position changes. This change in position is known as **displacement**. The word “displacement” implies that an object has moved, or has been displaced.

Displacement

Displacement is the *change in position* of an object:

$$\Delta x = x_f - x_0, \tag{2.1}$$

where Δx is displacement, x_f is the final position, and x_0 is the initial position.

In this text the upper case Greek letter Δ (delta) always means "change in" whatever quantity follows it; thus, Δx means *change in position*. Always solve for displacement by subtracting initial position x_0 from final position x_f.

UMASS AMHERST Instructor's Notes

We will be using Δ quite a lot in this class. Whenever you see Δ in this class, it will always mean the change in whatever the Δ is in front of, or final minus initial.

Note that the SI unit for displacement is the meter (m) (see **Physical Quantities and Units (https://legacy.cnx.org/content/m42091/latest/)**), but sometimes kilometers, miles, feet, and other units of length are used. Keep in mind that when units other than the meter are used in a problem, you may need to convert them into meters to complete the calculation.

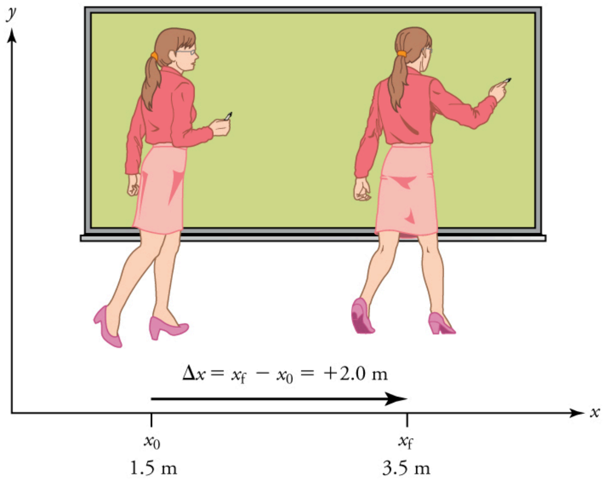


Figure 2.3 A professor paces left and right while lecturing. Her position relative to Earth is given by x. The $+2.0\text{ m}$ displacement of the professor relative to Earth is represented by an arrow pointing to the right.

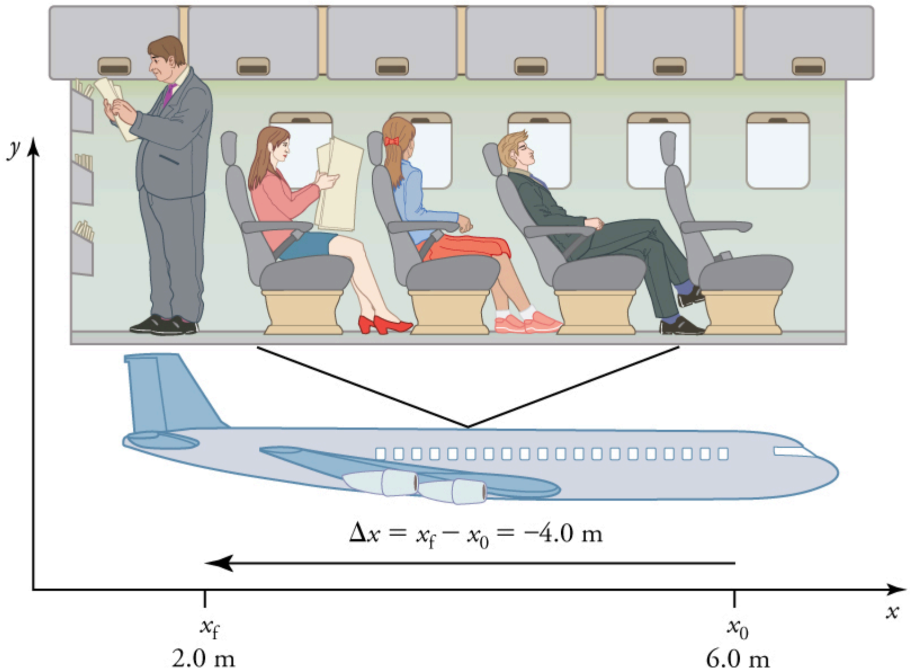


Figure 2.4 A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by x. The -4.0-m displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far) in **Figure 2.3**.

Note that displacement has a direction as well as a magnitude. The professor's displacement is 2.0 m to the right, and the airline passenger's displacement is 4.0 m toward the rear. In one-dimensional motion, direction can be specified with a plus or minus sign. When you begin a problem, you should select which direction is positive (usually that will be to the right or up, but you are free to select positive as being any direction). The professor's initial position is $x_0 = 1.5\text{ m}$ and her final position is $x_\text{f} = 3.5\text{ m}$. Thus her displacement is

$$\Delta x = x_\text{f} - x_0 = 3.5\text{ m} - 1.5\text{ m} = +2.0\text{ m}. \tag{2.2}$$

In this coordinate system, motion to the right is positive, whereas motion to the left is negative. Similarly, the airplane passenger's initial position is $x_0 = 6.0\text{ m}$ and his final position is $x_\text{f} = 2.0\text{ m}$, so his displacement is

$$\Delta x = x_\text{f} - x_0 = 2.0\text{ m} - 6.0\text{ m} = -4.0\text{ m}. \tag{2.3}$$

His displacement is negative because his motion is toward the rear of the plane, or in the negative x direction in our coordinate system.

Distance

Although displacement is described in terms of direction, distance is not. **Distance** is defined to be *the magnitude or size of displacement between two positions*. Note that the distance between two positions is not the same as the distance traveled between them. **Distance traveled** is *the total length of the path traveled between two positions*. Distance has no direction and, thus, no sign. For example, the distance the professor walks is 2.0 m. The distance the airplane passenger walks is 4.0 m.

UMASS AMHERST Instructor's Notes

Just like with position and displacement,there is a difference between the meaning of*distance* and *displacement*,even though they may seem similar, and it's important that you understand this difference. The following note discusses this in detail.

Misconception Alert: Distance Traveled vs. Magnitude of Displacement

It is important to note that the *distance traveled*, however, can be greater than the magnitude of the displacement (by magnitude, we mean just the size of the displacement without regard to its direction; that is, just a number with a unit). For example, the professor could pace back and forth many times, perhaps walking a distance of 150 m during a lecture, yet still end up only 2.0 m to the right of her starting point. In this case her displacement would be +2.0 m, the magnitude of her displacement would be 2.0 m, but the distance she traveled would be 150 m. In kinematics we nearly always deal with displacement and magnitude of displacement, and almost never with distance traveled. One way to think about this is to assume you marked the start of the motion and the end of the motion. The displacement is simply the difference in the position of the two marks and is independent of the path taken in traveling between the two marks. The distance traveled, however, is the total length of the path taken between the two marks.

Check Your Understanding

A cyclist rides 3 km west and then turns around and rides 2 km east. (a) What is her displacement? (b) What distance does she ride? (c) What is the magnitude of her displacement?

Solution

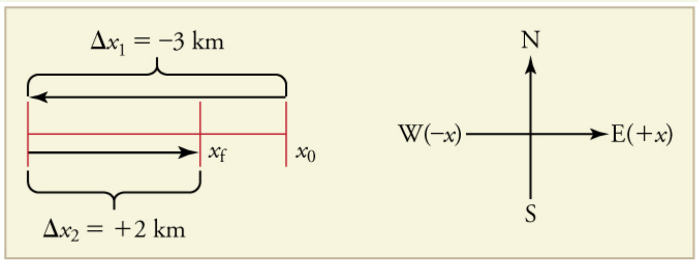


Figure 2.5

(a) The rider's displacement is $\Delta x = x_f - x_0 = -1\text{ km}$. (The displacement is negative because we take east to be positive and west to be negative.)

(b) The distance traveled is $3\text{ km} + 2\text{ km} = 5\text{ km}$.

(c) The magnitude of the displacement is 1 km.

2.2 Vectors, Scalars, and Coordinate Systems

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Definitions of vectors and scalars, and the difference between them

Other Things to Consider as You Read:

- In physics, often times you get to change the coordinate system to match the situation, such as deciding where the origin will be, so knowing how to manipulate the coordinate system is a useful tool. This idea also has a fundamental connection to physics that we'll explore later on.

Figure 2.6 The motion of this Eclipse Concept jet can be described in terms of the distance it has traveled (a scalar quantity) or its displacement in a specific direction (a vector quantity). In order to specify the direction of motion, its displacement must be described based on a coordinate system. In this case, it may be convenient to choose motion toward the left as positive motion (it is the forward direction for the plane), although in many cases, the x -coordinate runs from left to right, with motion to the right as positive and motion to the left as negative. (credit: Armchair Aviator, Flickr)

UMASS AMHERST Instructor's Notes

You may begin to notice that this chapter has a lot of basic definitions. It's important for you to understand these definitions in a fundamental way; they will play an important role in our discussion of physics. Also, a lot of these terms you may have already heard of, such as acceleration, but these words also have a more precise physics definition, so make sure that you understand these definitions in regards to physics.

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined only by magnitude. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A **vector** is any quantity with both *magnitude and direction*. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newtons straight down.

The direction of a vector in one-dimensional motion is given simply by a plus $(+)$ or minus $(-)$ sign. Vectors are represented graphically by arrows. An arrow used to represent a vector has a length proportional to the vector's magnitude (e.g., the larger the magnitude, the longer the length of the vector) and points in the same direction as the vector.

Some physical quantities, like distance, either have no direction or none is specified. A **scalar** is any quantity that has a magnitude, but no direction. For example, a 20°C temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars—quantities with no specified direction. Note, however, that a scalar can be negative, such as a -20°C temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows.

Coordinate Systems for One-Dimensional Motion

In order to describe the direction of a vector quantity, you must designate a coordinate system within the reference frame. For one-dimensional motion, this is a simple coordinate system consisting of a one-dimensional coordinate line. In general, when describing horizontal motion, motion to the right is usually considered positive, and motion to the left is considered negative. With vertical motion, motion up is usually positive and motion down is negative. In some cases, however, as with the jet in **Figure 2.6**, it can be more convenient to switch the positive and negative directions. For example, if you are analyzing the motion of falling objects, it can be useful to define downwards as the positive direction. If people in a race are running to the left, it is useful to define left as the positive direction. It does not matter as long as the system is clear and consistent. Once you assign a positive direction and start solving a problem, you cannot change it.

UMASS AMHERST Instructor's Notes

Keep in mind that last sentence; before you start working, you can change your coordinate system and what you define as positive and negative directions however you want, but once you start working with that system, you *cannot* change it.

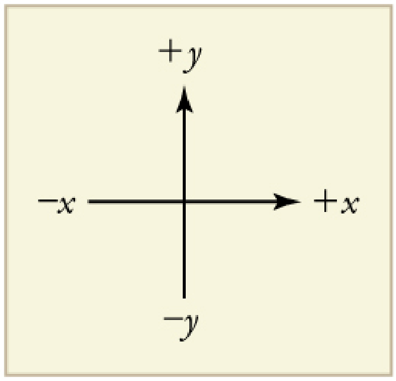


Figure 2.7 It is usually convenient to consider motion upward or to the right as positive $(+)$ and motion downward or to the left as negative $(-)$.

Check Your Understanding

A person's speed can stay the same as he or she rounds a corner and changes direction. Given this information, is speed a scalar or a vector quantity? Explain.

Solution
Speed is a scalar quantity. It does not change at all with direction changes; therefore, it has magnitude only. If it were a vector quantity, it would change as direction changes (even if its magnitude remained constant).

2.3 Time, Velocity, and Speed

UMASS AMHERST Instructor's Notes

Your Quiz would Cover
- Describing how velocity is different from speed
- Identifying that velocity has a direction
- Identifying that velocity is always parallel to the path
- Solving for position as a function of time, given velocity as a function of time. Eventually, we will be using this idea with simulations.

Other Things to Consider as You Read:
- As with the previous sections, there are a couple of definitions to pay attention to. There are some terms you may have heard before, but, again, it's the specific physics definition that's important to know.

Figure 2.8 The motion of these racing snails can be described by their speeds and their velocities. (credit: tobitasflickr, Flickr)

There is more to motion than distance and displacement. Questions such as, "How long does a foot race take?" and "What was the runner's speed?" cannot be answered without an understanding of other concepts. In this section we add definitions of time, velocity, and speed to expand our description of motion.

Time

As discussed in **Physical Quantities and Units (https://legacy.cnx.org/content/m42091/latest/)** , the most fundamental physical quantities are defined by how they are measured. This is the case with time. Every measurement of time involves measuring a change in some physical quantity. It may be a number on a digital clock, a heartbeat, or the position of the Sun in the sky. In physics, the definition of time is simple— **time** is *change*, or the interval over which change occurs. It is impossible to know that time has passed unless something changes.

The amount of time or change is calibrated by comparison with a standard. The SI unit for time is the second, abbreviated s. We might, for example, observe that a certain pendulum makes one full swing every 0.75 s. We could then use the pendulum to measure time by counting its swings or, of course, by connecting the pendulum to a clock mechanism that registers time on a dial. This allows us to not only measure the amount of time, but also to determine a sequence of events.

How does time relate to motion? We are usually interested in elapsed time for a particular motion, such as how long it takes an airplane passenger to get from his seat to the back of the plane. To find elapsed time, we note the time at the beginning and end of the motion and subtract the two. For example, a lecture may start at 11:00 A.M. and end at 11:50 A.M., so that the elapsed time would be 50 min. **Elapsed time** $\boldsymbol{\Delta t}$ is the difference between the ending time and beginning time,

$$\Delta t = t_{\mathrm{f}} - t_0, \tag{2.4}$$

where Δt is the change in time or elapsed time, t_{f} is the time at the end of the motion, and t_0 is the time at the beginning of the motion. (As usual, the delta symbol, Δ , means the change in the quantity that follows it.)

Life is simpler if the beginning time t_0 is taken to be zero, as when we use a stopwatch. If we were using a stopwatch, it would simply read zero at the start of the lecture and 50 min at the end. If $t_0 = 0$, then $\Delta t = t_{\mathrm{f}} \equiv t$.

In this text, for simplicity's sake,

- motion starts at time equal to zero $(t_0 = 0)$
- the symbol t is used for elapsed time unless otherwise specified $(\Delta t = t_{\mathrm{f}} \equiv t)$

UMASS AMHERST Instructor's Notes

We will also be using this simplification; t will always mean Δt , and initial time will always be zero, unless stated otherwise

Velocity

UMASS AMHERST Instructor's Notes

It's difficult to understand physics without understanding velocity, so try to internalize what velocity is, and think about what negative velocity means as well.

Your notion of velocity is probably the same as its scientific definition. You know that if you have a large displacement in a small amount of time you have a large velocity, and that velocity has units of distance divided by time, such as miles per hour or kilometers per hour.

Average Velocity

Average velocity is *displacement (change in position) divided by the time of travel*,

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_{\mathrm{f}} - x_0}{t_{\mathrm{f}} - t_0}, \tag{2.5}$$

where $\bar{v}$ is the *average* (indicated by the bar over the v) velocity, Δx is the change in position (or displacement), and x_{f} and x_0 are the final and beginning positions at times t_{f} and t_0 , respectively. If the starting time t_0 is taken to be zero, then the average velocity is simply

$$\bar{v} = \frac{\Delta x}{t}. \tag{2.6}$$

UMASS AMHERST Instructor's Notes

The equation above is a mathematical definition of velocity. If you think about physics as a series of different ideas, the equation is just another representation of the idea of velocity, along with the definition with words. It's important to know what velocity is in both words and the mathematics, as well as to connect those two definitions together.

Notice that this definition indicates that *velocity is a vector because displacement is a vector*. It has both magnitude and direction. The SI unit for velocity is meters per second or m/s, but many other units, such as km/h, mi/h (also written as mph), and cm/s, are in common use. Suppose, for example, an airplane passenger took 5 seconds to move −4 m (the negative sign indicates that displacement is toward the back of the plane). His average velocity would be

$$\bar{v} = \frac{\Delta x}{t} = \frac{-4\text{ m}}{5\text{ s}} = -0.8\text{ m/s}. \tag{2.7}$$

The minus sign indicates the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane passenger stops momentarily or backs up before he goes to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.

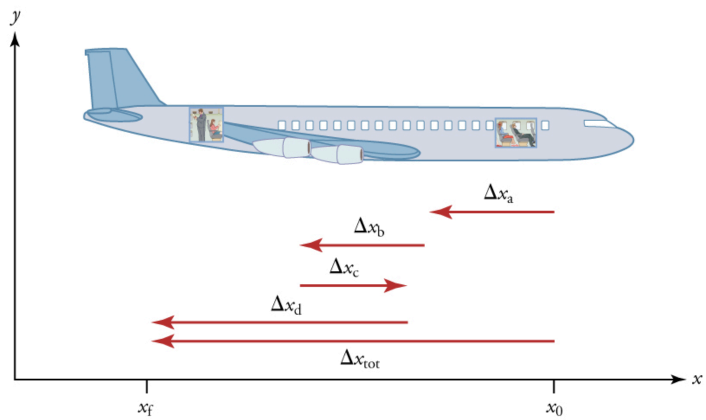


Figure 2.9 A more detailed record of an airplane passenger heading toward the back of the plane, showing smaller segments of his trip.

The smaller the time intervals considered in a motion, the more detailed the information. When we carry this process to its logical conclusion, we are left with an infinitesimally small interval. Over such an interval, the average velocity becomes the *instantaneous velocity* or the *velocity at a specific instant*. A car's speedometer, for example, shows the magnitude (but not the direction) of the instantaneous velocity of the car. (Police give tickets based on instantaneous velocity, but when calculating how long it will take to get from one place to another on a road trip, you need to use average velocity.) **Instantaneous velocity** v is the average velocity at a specific instant in time (or over an infinitesimally small time interval).

Mathematically, finding instantaneous velocity, v, at a precise instant t can involve taking a limit, a calculus operation beyond the scope of this text. However, under many circumstances, we can find precise values for instantaneous velocity without calculus.

Speed

UMASS AMHERST Instructor's Notes

As with position and displacement, speed and velocity are also two similar ideas but with distinct differences, so focus on what each means and how the differ in meaning.

In everyday language, most people use the terms "speed" and "velocity" interchangeably. In physics, however, they do not have the same meaning and they are distinct concepts. One major difference is that speed has no direction. Thus *speed is a scalar*. Just as we need to distinguish between instantaneous velocity and average velocity, we also need to distinguish between instantaneous speed and average speed.

Instantaneous speed is the magnitude of instantaneous velocity. For example, suppose the airplane passenger at one instant had an instantaneous velocity of −3.0 m/s (the minus meaning toward the rear of the plane). At that same time his instantaneous speed was 3.0 m/s. Or suppose that at one time during a shopping trip your instantaneous velocity is 40 km/h due north. Your instantaneous speed at that instant would be 40 km/h—the same magnitude but without a direction. Average speed, however, is very different from average velocity. **Average speed** is the distance traveled divided by elapsed time.

We have noted that distance traveled can be greater than displacement. So average speed can be greater than average velocity, which is displacement divided by time. For example, if you drive to a store and return home in half an hour, and your car's odometer shows the total distance traveled was 6 km, then your average speed was 12 km/h. Your average velocity, however, was zero, because your displacement for the round trip is zero. (Displacement is change in position and, thus, is zero for a round trip.) Thus average speed is *not* simply the magnitude of average velocity.

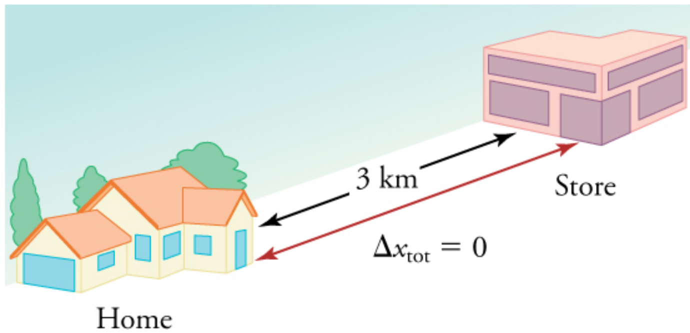


Figure 2.10 During a 30-minute round trip to the store, the total distance traveled is 6 km. The average speed is 12 km/h. The displacement for the round trip is zero, since there was no net change in position. Thus the average velocity is zero.

Another way of visualizing the motion of an object is to use a graph. A plot of position or of velocity as a function of time can be very useful. For example, for this trip to the store, the position, velocity, and speed-vs.-time graphs are displayed in **Figure 2.11**. (Note that these graphs depict a very simplified **model** of the trip. We are assuming that speed is constant during the trip, which is unrealistic given that we'll probably stop at the store. But for simplicity's sake, we will model it with no stops or changes in speed. We are also assuming that the route between the store and the house is a perfectly straight line.)

UMASS AMHERST Instructor's Notes

The idea of modelling a situation with a more simplified version is particularly useful. For example, if you want to model the velocity of a person walking forward at a speed, you could treat their motion as a single point moving forward in a straight line. Realistically, the person could be walking in a line that's slightly off being straight, or they could be walking at slightly varying speeds, and different parts of their body could be shifting around, like their arms could be swinging, but often times you don't really have to care. When you do care, you can factor in these facts later, but often times the model is close enough to reality, and is usually only off by a few percent. This idea of modeling will come up later in this course as well, especially with forces.

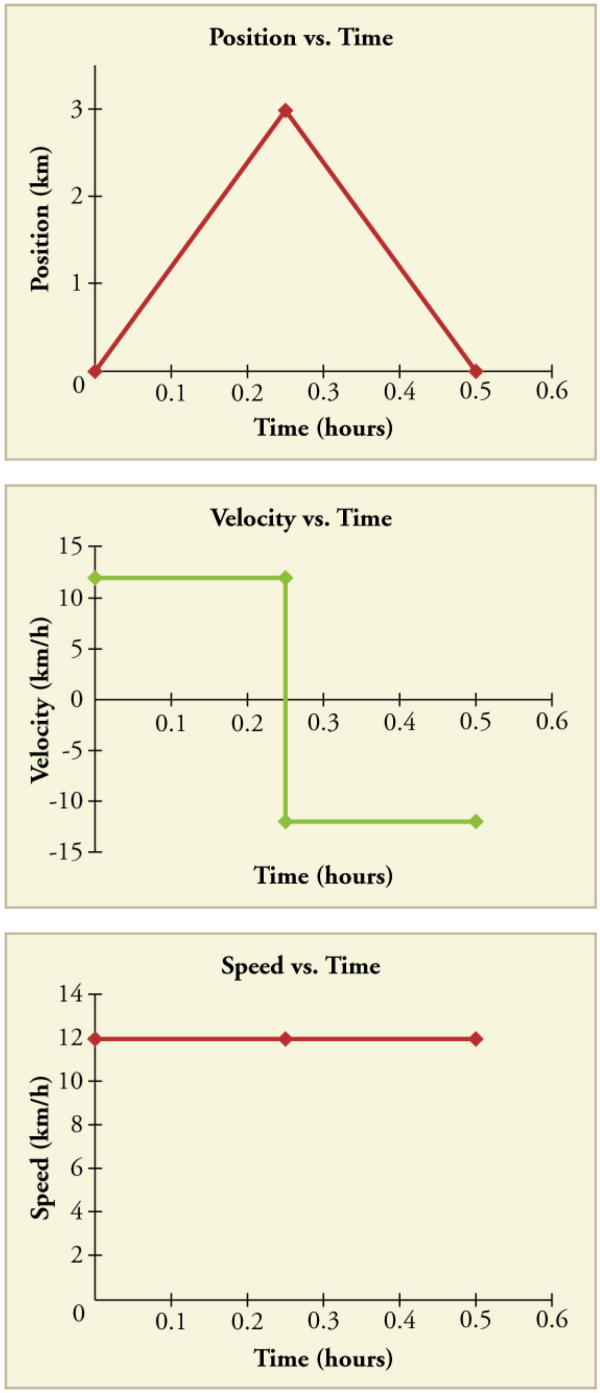


Figure 2.11 Position vs. time, velocity vs. time, and speed vs. time on a trip. Note that the velocity for the return trip is negative.

UMASS AMHERST **Instructor's Notes**

Just like how there is a mathematical representation of velocity and a representation of velocity in words, the above is a graphical represntation of velocity. This will apply to other concepts that we will discuss in class, and being able to connect all of these representations together to the idea will help your understanding.

Making Connections: Take-Home Investigation—Getting a Sense of Speed

If you have spent much time driving, you probably have a good sense of speeds between about 10 and 70 miles per hour. But what are these in meters per second? What do we mean when we say that something is moving at 10 m/s? To get a better sense of what these values really mean, do some observations and calculations on your own:

- calculate typical car speeds in meters per second
- estimate jogging and walking speed by timing yourself; convert the measurements into both m/s and mi/h
- determine the speed of an ant, snail, or falling leaf

Instructor's Notes

I would highly suggest trying out these 'Making Connections' sections; seeing physics in the real world will not only help you better understand the physics in this class, but will make you understanding and appreciation of nature much richer.

Check Your Understanding

A commuter train travels from Baltimore to Washington, DC, and back in 1 hour and 45 minutes. The distance between the two stations is approximately 40 miles. What is (a) the average velocity of the train, and (b) the average speed of the train in m/s?

Solution

(a) The average velocity of the train is zero because $x_\text{f} = x_0$; the train ends up at the same place it starts.

(b) The average speed of the train is calculated below. Note that the train travels 40 miles one way and 40 miles back, for a total distance of 80 miles.

$$\frac{\text{distance}}{\text{time}} = \frac{80\ \text{miles}}{105\ \text{minutes}} \tag{2.8}$$

$$\frac{80\ \text{miles}}{105\ \text{minutes}} \times \frac{5280\ \text{feet}}{1\ \text{mile}} \times \frac{1\ \text{meter}}{3.28\ \text{feet}} \times \frac{1\ \text{minute}}{60\ \text{seconds}} = 20\ \text{m/s} \tag{2.9}$$

2.4 Acceleration

Instructor's Notes

Your Quiz would Cover

- Identifying that acceleration has a direction
- Using the relationship between average velocity and position to solve problems about the motion of objects.
- Describing how velocity will change given acceleration.

Other Things to Consider as You Read:

- Acceleration is a more abstract concept to wrap your head around than velocity and position, and it might take some time and practice to understand.

Figure 2.12 A plane decelerates, or slows down, as it comes in for landing in St. Maarten. Its acceleration is opposite in direction to its velocity. (credit: Steve Conry, Flickr)

In everyday conversation, to accelerate means to speed up. The accelerator in a car can in fact cause it to speed up. The greater the **acceleration**, the greater the change in velocity over a given time. The formal definition of acceleration is consistent with these notions, but more inclusive.

Average Acceleration

Average Acceleration is *the rate at which velocity changes*,

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_{\mathrm{f}} - v_0}{t_{\mathrm{f}} - t_0}, \tag{2.10}$$

where $\bar{a}$ is average acceleration, v is velocity, and t is time. (The bar over the a means *average* acceleration.)

Because acceleration is velocity in m/s divided by time in s, the SI units for acceleration are $\mathrm{m/s^2}$, meters per second squared or meters per second per second, which literally means by how many meters per second the velocity changes every second.

Recall that velocity is a vector—it has both magnitude and direction. This means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in *direction*. For example, if a car turns a corner at constant speed, it is accelerating because its direction is changing. The quicker you turn, the greater the acceleration. So there is an acceleration when velocity changes either in magnitude (an increase or decrease in speed) or in direction, or both.

UMASS AMHERST Instructor's Notes

Like with velocity, try to internalize what average acceleration means, its mathematical representation above, and try to connect these two definitons.

Acceleration as a Vector

Acceleration is a vector in the same direction as the *change* in velocity, Δv. Since velocity is a vector, it can change either in magnitude or in direction. Acceleration is therefore a change in either speed or direction, or both.

Keep in mind that although acceleration is in the direction of the *change* in velocity, it is not always in the direction of *motion*. When an object slows down, its acceleration is opposite to the direction of its motion. This is known as **deceleration**.

Figure 2.13 A subway train in Sao Paulo, Brazil, decelerates as it comes into a station. It is accelerating in a direction opposite to its direction of motion. (credit: Yusuke Kawasaki, Flickr)

Misconception Alert: Deceleration vs. Negative Acceleration

Deceleration always refers to acceleration in the direction opposite to the direction of the velocity. Deceleration always reduces speed. Negative acceleration, however, is acceleration *in the negative direction in the chosen coordinate system*. Negative acceleration may or may not be deceleration, and deceleration may or may not be considered negative acceleration. For example, consider **Figure 2.14**.

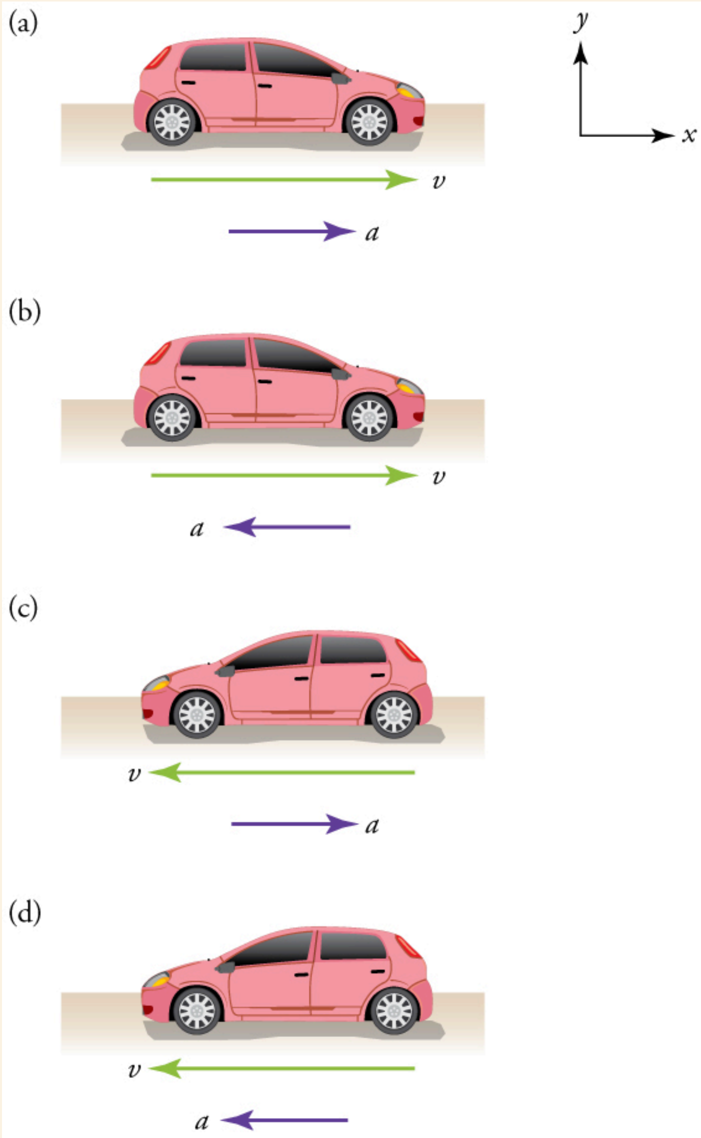


Figure 2.14 (a) This car is speeding up as it moves toward the right. It therefore has positive acceleration in our coordinate system. (b) This car is slowing down as it moves toward the right. Therefore, it has negative acceleration in our coordinate system, because its acceleration is toward the left. The car is also decelerating: the direction of its acceleration is opposite to its direction of motion. (c) This car is moving toward the left, but slowing down over time. Therefore, its acceleration is positive in our coordinate system because it is toward the right. However, the car is decelerating because its acceleration is opposite to its motion. (d) This car is speeding up as it moves toward the left. It has negative acceleration because it is accelerating toward the left. However, because its acceleration is in the same direction as its motion, it is speeding up (*not* decelerating).

UMASS AMHERST Instructor's Notes

This misconception is a pit many students fall into. Remember, there are precise definitions for terms such as acceleration that we want you to know, and knowing these will help you avoid these misunderstandings.

Example 2.1 Calculating Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?

Figure 2.15 (credit: Jon Sullivan, PD Photo.org)

Strategy

First we draw a sketch and assign a coordinate system to the problem. This is a simple problem, but it always helps to visualize it. Notice that we assign east as positive and west as negative. Thus, in this case, we have negative velocity.

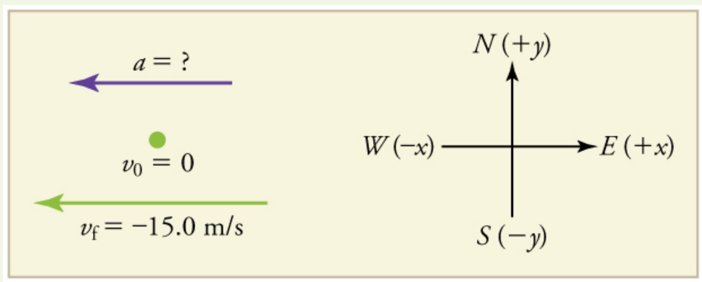


Figure 2.16

We can solve this problem by identifying Δv and Δt from the given information and then calculating the average acceleration directly from the equation $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$.

Solution

1. Identify the knowns. $v_0 = 0$, $v_f = -15.0 \text{ m/s}$ (the negative sign indicates direction toward the west), $\Delta t = 1.80 \text{ s}$.

2. Find the change in velocity. Since the horse is going from zero to -15.0 m/s, its change in velocity equals its final velocity: $\Delta v = v_f = -15.0 \text{ m/s}$.

3. Plug in the known values (Δv and Δt) and solve for the unknown $\bar{a}$.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-15.0 \text{ m/s}}{1.80 \text{ s}} = -8.33 \text{ m/s}^2. \tag{2.11}$$

Discussion

The negative sign for acceleration indicates that acceleration is toward the west. An acceleration of 8.33 m/s^2 due west means that the horse increases its velocity by 8.33 m/s due west each second, that is, 8.33 meters per second per second, which we write as 8.33 m/s^2. This is truly an average acceleration, because the ride is not smooth. We shall see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.

Instantaneous Acceleration

Instantaneous acceleration a, or the *acceleration at a specific instant in time*, is obtained by the same process as discussed for instantaneous velocity in **Time, Velocity, and Speed (https://legacy.cnx.org/content/m42096/latest/)** —that is, by considering an infinitesimally small interval of time. How do we find instantaneous acceleration using only algebra? The answer is that we choose an average acceleration that is representative of the motion. **Figure 2.17** shows graphs of instantaneous acceleration versus time for two very different motions. In **Figure 2.17**(a), the acceleration varies slightly and the average over the entire interval is nearly the same as the instantaneous acceleration at any time. In this case, we should treat this motion as if it had a constant acceleration equal to the average (in this case about 1.8 m/s^2). In **Figure 2.17**(b), the acceleration varies drastically over time. In such situations it is best to consider smaller time intervals and choose an average acceleration for each. For example, we could consider motion over the time intervals from 0 to 1.0 s and from 1.0 to 3.0 s as separate motions with accelerations of $+3.0 \text{ m/s}^2$ and -2.0 m/s^2, respectively.

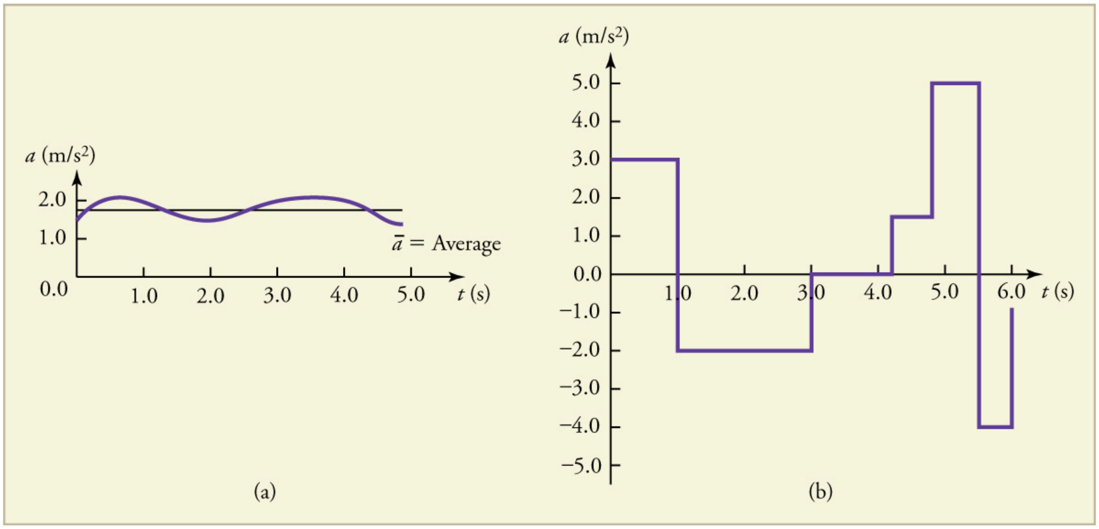


Figure 2.17 Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Here acceleration varies only slightly and is always in the same direction, since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Here the acceleration varies greatly, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0 to 1.0 s) with constant or nearly constant acceleration in such a situation.

The next several examples consider the motion of the subway train shown in **Figure 2.18**. In (a) the shuttle moves to the right, and in (b) it moves to the left. The examples are designed to further illustrate aspects of motion and to illustrate some of the reasoning that goes into solving problems.

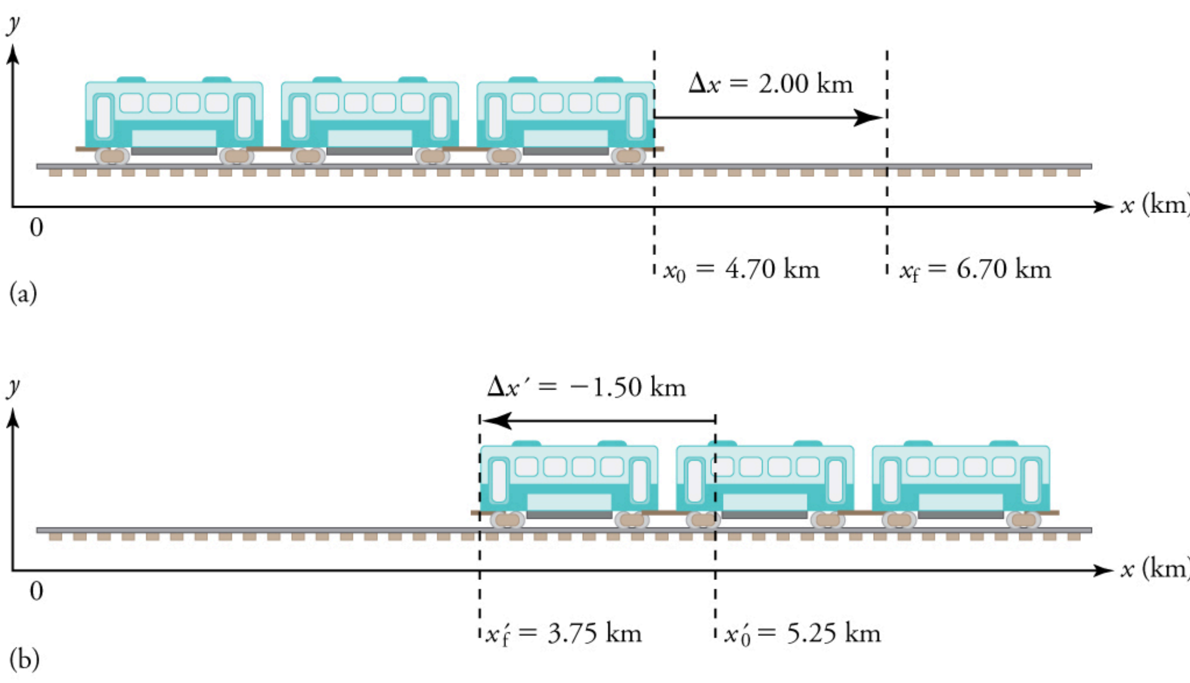


Figure 2.18 One-dimensional motion of a subway train considered in **Example 2.2**, **Example 2.3**, **Example 2.4**, **Example 2.5**, **Example 2.6**, and **Example 2.7**. Here we have chosen the x-axis so that + means to the right and $-$ means to the left for displacements, velocities, and accelerations. (a) The subway train moves to the right from x_0 to x_f. Its displacement Δx is +2.0 km. (b) The train moves to the left from x'_0 to x'_f. Its displacement $\Delta x'$ is -1.5 km. (Note that the prime symbol (′) is used simply to distinguish between displacement in the two different situations. The distances of travel and the size of the cars are on different scales to fit everything into the diagram.)

Example 2.2 Calculating Displacement: A Subway Train

What are the magnitude and sign of displacements for the motions of the subway train shown in parts (a) and (b) of **Figure 2.18**?

Strategy

A drawing with a coordinate system is already provided, so we don't need to make a sketch, but we should analyze it to make sure we understand what it is showing. Pay particular attention to the coordinate system. To find displacement, we use the equation $\Delta x = x_{\text{f}} - x_0$. This is straightforward since the initial and final positions are given.

Solution

1. Identify the knowns. In the figure we see that $x_{\text{f}} = 6.70\text{ km}$ and $x_0 = 4.70\text{ km}$ for part (a), and $x'_{\text{f}} = 3.75\text{ km}$ and $x'_0 = 5.25\text{ km}$ for part (b).

2. Solve for displacement in part (a).

$$\Delta x = x_{\text{f}} - x_0 = 6.70\text{ km} - 4.70\text{ km} = +2.00\text{ km} \tag{2.12}$$

3. Solve for displacement in part (b).

$$\Delta x' = x'_{\text{f}} - x'_0 = 3.75\text{ km} - 5.25\text{ km} = -1.50\text{ km} \tag{2.13}$$

Discussion

The direction of the motion in (a) is to the right and therefore its displacement has a positive sign, whereas motion in (b) is to the left and thus has a negative sign.

Example 2.3 Comparing Distance Traveled with Displacement: A Subway Train

What are the distances traveled for the motions shown in parts (a) and (b) of the subway train in **Figure 2.18**?

Strategy

To answer this question, think about the definitions of distance and distance traveled, and how they are related to displacement. Distance between two positions is defined to be the magnitude of displacement, which was found in **Example 2.2**. Distance traveled is the total length of the path traveled between the two positions. (See **Displacement (https://legacy.cnx.org/content/m42033/latest/)** .) In the case of the subway train shown in **Figure 2.18**, the distance traveled is the same as the distance between the initial and final positions of the train.

Solution

1. The displacement for part (a) was +2.00 km. Therefore, the distance between the initial and final positions was 2.00 km, and the distance traveled was 2.00 km.

2. The displacement for part (b) was $-1.5\text{ km}.$ Therefore, the distance between the initial and final positions was 1.50 km, and the distance traveled was 1.50 km.

Discussion

Distance is a scalar. It has magnitude but no sign to indicate direction.

Example 2.4 Calculating Acceleration: A Subway Train Speeding Up

Suppose the train in **Figure 2.18**(a) accelerates from rest to 30.0 km/h in the first 20.0 s of its motion. What is its average acceleration during that time interval?

Strategy

It is worth it at this point to make a simple sketch:

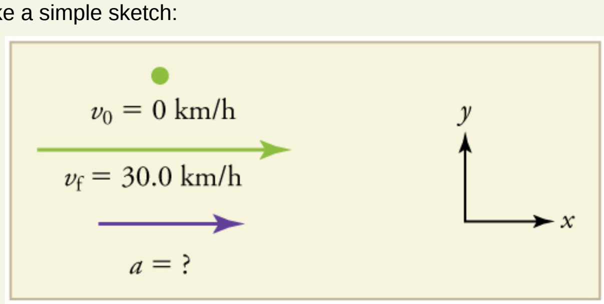


Figure 2.19

This problem involves three steps. First we must determine the change in velocity, then we must determine the change in time, and finally we use these values to calculate the acceleration.

Solution

1. Identify the knowns. $v_0 = 0$ (the trains starts at rest), $v_\text{f} = 30.0\text{ km/h}$, and $\Delta t = 20.0\text{ s}$.

2. Calculate Δv . Since the train starts from rest, its change in velocity is $\Delta v = +30.0\text{ km/h}$, where the plus sign means velocity to the right.

3. Plug in known values and solve for the unknown, $\bar{a}$.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{+30.0\text{ km/h}}{20.0\text{ s}} \quad (2.14)$$

4. Since the units are mixed (we have both hours and seconds for time), we need to convert everything into SI units of meters and seconds. (See **Physical Quantities and Units (https://legacy.cnx.org/content/m42091/latest/)** for more guidance.)

$$\bar{a} = \left(\frac{+30\text{ km/h}}{20.0\text{ s}}\right)\left(\frac{10^3\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = 0.417\text{ m/s}^2 \quad (2.15)$$

Discussion

The plus sign means that acceleration is to the right. This is reasonable because the train starts from rest and ends up with a velocity to the right (also positive). So acceleration is in the same direction as the *change* in velocity, as is always the case.

Example 2.5 Calculate Acceleration: A Subway Train Slowing Down

Now suppose that at the end of its trip, the train in **Figure 2.18**(a) slows to a stop from a speed of 30.0 km/h in 8.00 s. What is its average acceleration while stopping?

Strategy

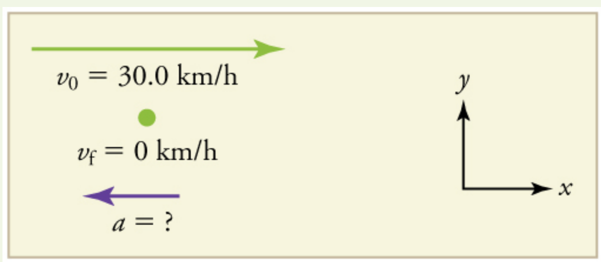


Figure 2.20

In this case, the train is decelerating and its acceleration is negative because it is toward the left. As in the previous example, we must find the change in velocity and the change in time and then solve for acceleration.

Solution

1. Identify the knowns. $v_0 = 30.0\text{ km/h}$, $v_\text{f} = 0\text{ km/h}$ (the train is stopped, so its velocity is 0), and $\Delta t = 8.00\text{ s}$.

2. Solve for the change in velocity, Δv .

$$\Delta v = v_\text{f} - v_0 = 0 - 30.0\text{ km/h} = -30.0\text{ km/h} \quad (2.16)$$

3. Plug in the knowns, Δv and Δt , and solve for $\bar{a}$.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-30.0\text{ km/h}}{8.00\text{ s}} \quad (2.17)$$

4. Convert the units to meters and seconds.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \left(\frac{-30.0\text{ km/h}}{8.00\text{ s}}\right)\left(\frac{10^3\text{ m}}{1\text{ km}}\right)\left(\frac{1\text{ h}}{3600\text{ s}}\right) = -1.04\text{ m/s}^2. \quad (2.18)$$

Discussion

The minus sign indicates that acceleration is to the left. This sign is reasonable because the train initially has a positive velocity in this problem, and a negative acceleration would oppose the motion. Again, acceleration is in the same direction as the *change* in velocity, which is negative here. This acceleration can be called a deceleration because it has a direction opposite to the velocity.

The graphs of position, velocity, and acceleration vs. time for the trains in **Example 2.4** and **Example 2.5** are displayed in **Figure 2.21**. (We have taken the velocity to remain constant from 20 to 40 s, after which the train decelerates.)

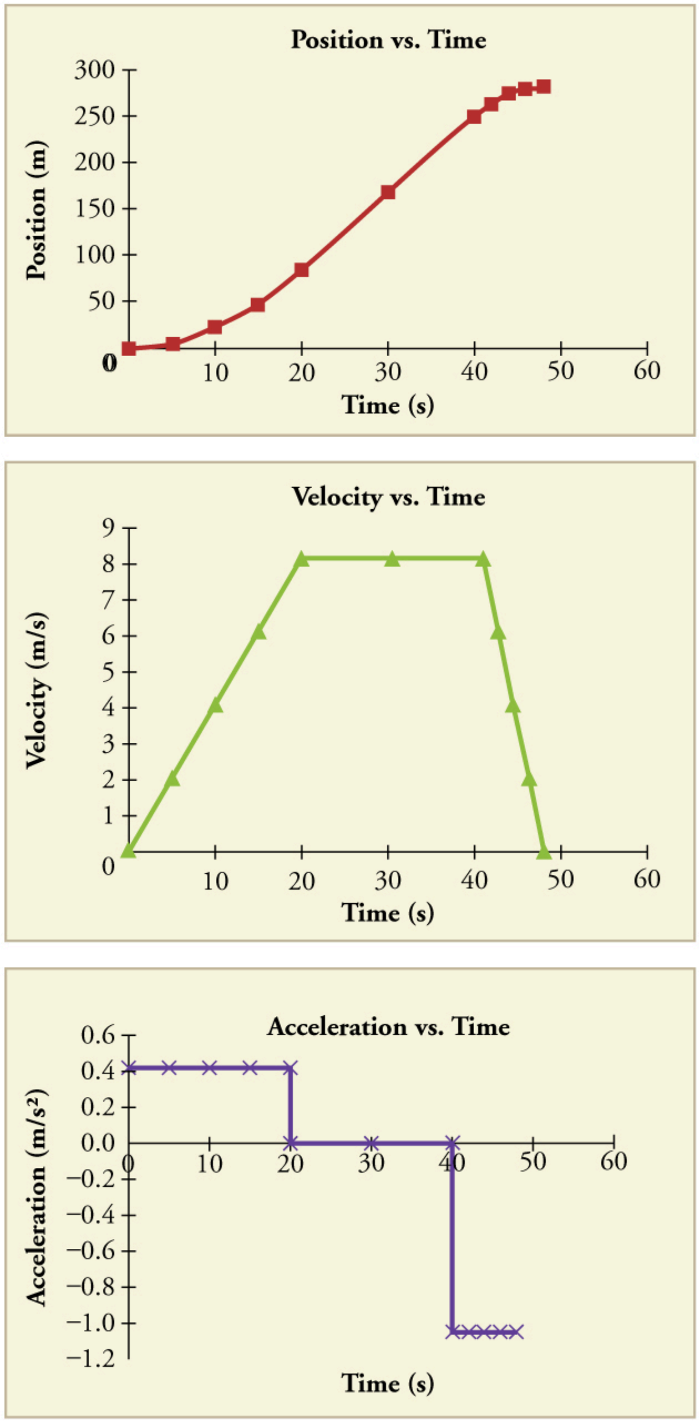


Figure 2.21 (a) Position of the train over time. Notice that the train's position changes slowly at the beginning of the journey, then more and more quickly as it picks up speed. Its position then changes more slowly as it slows down at the end of the journey. In the middle of the journey, while the velocity remains constant, the position changes at a constant rate. (b) Velocity of the train over time. The train's velocity increases as it accelerates at the beginning of the journey. It remains the same in the middle of the journey (where there is no acceleration). It decreases as the train decelerates at the end of the journey. (c) The acceleration of the train over time. The train has positive acceleration as it speeds up at the beginning of the journey. It has no acceleration as it travels at constant velocity in the middle of the journey. Its acceleration is negative as it slows down at the end of the journey.

Example 2.6 Calculating Average Velocity: The Subway Train

What is the average velocity of the train in part b of **Example 2.2**, and shown again below, if it takes 5.00 min to make its trip?

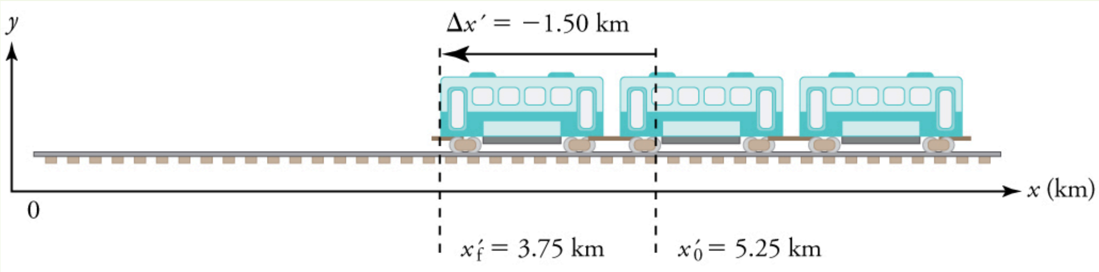


Figure 2.22

Strategy

Average velocity is displacement divided by time. It will be negative here, since the train moves to the left and has a negative displacement.

Solution

1. Identify the knowns. $x'_f = 3.75\text{ km}$, $x'_0 = 5.25\text{ km}$, $\Delta t = 5.00\text{ min}$.

2. Determine displacement, $\Delta x'$. We found $\Delta x'$ to be -1.5 km in Example 2.2.

3. Solve for average velocity.

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \frac{-1.50\text{ km}}{5.00\text{ min}} \tag{2.19}$$

4. Convert units.

$$\bar{v} = \frac{\Delta x'}{\Delta t} = \left(\frac{-1.50\text{ km}}{5.00\text{ min}}\right)\left(\frac{60\text{ min}}{1\text{ h}}\right) = -18.0\text{ km/h} \tag{2.20}$$

Discussion

The negative velocity indicates motion to the left.

Example 2.7 Calculating Deceleration: The Subway Train

Finally, suppose the train in Figure 2.22 slows to a stop from a velocity of 20.0 km/h in 10.0 s. What is its average acceleration?

Strategy

Once again, let's draw a sketch:

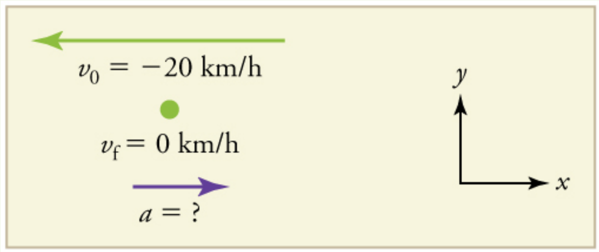


Figure 2.23

As before, we must find the change in velocity and the change in time to calculate average acceleration.

Solution

1. Identify the knowns. $v_0 = -20\text{ km/h}$, $v_f = 0\text{ km/h}$, $\Delta t = 10.0\text{ s}$.

2. Calculate Δv. The change in velocity here is actually positive, since

$$\Delta v = v_f - v_0 = 0 - (-20\text{ km/h}) = +20\text{ km/h}. \tag{2.21}$$

3. Solve for $\bar{a}$.

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{+20.0\text{ km/h}}{10.0\text{ s}} \tag{2.22}$$

4. Convert units.

$$\bar{a} = \left(\frac{+20.0\ \text{km/h}}{10.0\ \text{s}}\right)\left(\frac{10^3\ \text{m}}{1\ \text{km}}\right)\left(\frac{1\ \text{h}}{3600\ \text{s}}\right) = +0.556\ \text{m/s}^2 \tag{2.23}$$

Discussion

The plus sign means that acceleration is to the right. This is reasonable because the train initially has a negative velocity (to the left) in this problem and a positive acceleration opposes the motion (and so it is to the right). Again, acceleration is in the same direction as the *change* in velocity, which is positive here. As in **Example 2.5**, this acceleration can be called a deceleration since it is in the direction opposite to the velocity.

Sign and Direction

Perhaps the most important thing to note about these examples is the signs of the answers. In our chosen coordinate system, plus means the quantity is to the right and minus means it is to the left. This is easy to imagine for displacement and velocity. But it is a little less obvious for acceleration. Most people interpret negative acceleration as the slowing of an object. This was not the case in **Example 2.7**, where a positive acceleration slowed a negative velocity. The crucial distinction was that the acceleration was in the opposite direction from the velocity. In fact, a negative acceleration will *increase* a negative velocity. For example, the train moving to the left in **Figure 2.22** is sped up by an acceleration to the left. In that case, both v and a are negative. The plus and minus signs give the directions of the accelerations. If acceleration has the same sign as the velocity, the object is speeding up. If acceleration has the opposite sign as the velocity, the object is slowing down.

Check Your Understanding

An airplane lands on a runway traveling east. Describe its acceleration.

Solution

If we take east to be positive, then the airplane has negative acceleration, as it is accelerating toward the west. It is also decelerating: its acceleration is opposite in direction to its velocity.

PhET Explorations: Moving Man Simulation

Learn about position, velocity, and acceleration graphs. Move the little man back and forth with the mouse and plot his motion. Set the position, velocity, or acceleration and let the simulation move the man for you.

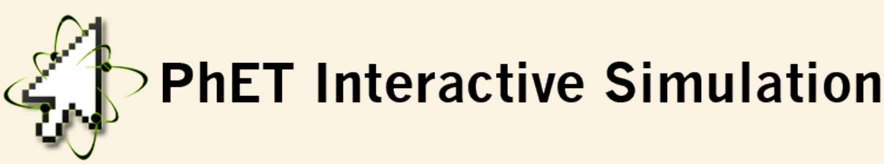


Figure 2.24 **Moving Man (http://legacy.cnx.org/content/m64042/1.5/moving-man_en.jar)**

2.5 Graphical Analysis of One-Dimensional Motion

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Describing the units of the value, slope, and the area under any graph
- Using slope of a position vs. time graph to sketch a velocity vs. time graph
- Using slope of a velocity vs. time graph to sketch an acceleration vs. time graph
- Using area under an acceleration vs. time graph to sketch a velocity vs. time graph
- Using the area under a velocity vs. time graph to sketch a position vs. time graph

Other Things to Consider as You Read:

- As with the other sections, there are definitions of basic concepts that we will be using throughout the course. We will also be working with graphs a lot as well.

A graph, like a picture, is worth a thousand words. Graphs not only contain numerical information; they also reveal relationships between physical quantities. This section uses graphs of displacement, velocity, and acceleration versus time to illustrate one-dimensional kinematics.

Slopes and General Relationships

First note that graphs in this text have perpendicular axes, one horizontal and the other vertical. When two physical quantities are plotted against one another in such a graph, the horizontal axis is usually considered to be an **independent variable** and the vertical axis a **dependent variable**. If we call the horizontal axis the x-axis and the vertical axis the y-axis, as in **Figure 2.25**, a straight-line graph has the general form

$$y = mx + b. \tag{2.24}$$

Here m is the **slope**, defined to be the rise divided by the run (as seen in the figure) of the straight line. The letter b is used for the ***y*-intercept**, which is the point at which the line crosses the vertical axis.

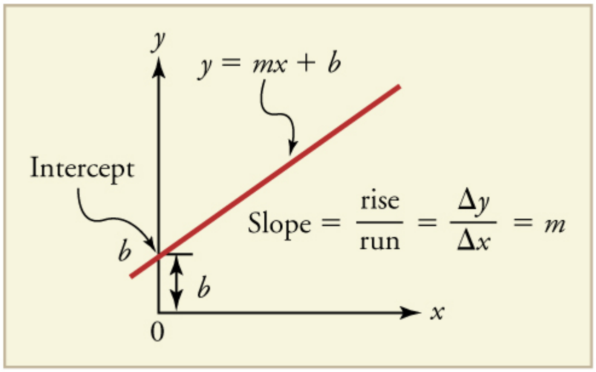


Figure 2.25 A straight-line graph. The equation for a straight line is $y = mx + b$.

Graph of Displacement vs. Time (*a* = 0, so *v* is constant)

Time is usually an independent variable that other quantities, such as displacement, depend upon. A graph of displacement versus time would, thus, have x on the vertical axis and t on the horizontal axis. **Figure 2.26** is just such a straight-line graph. It shows a graph of displacement versus time for a jet-powered car on a very flat dry lake bed in Nevada.

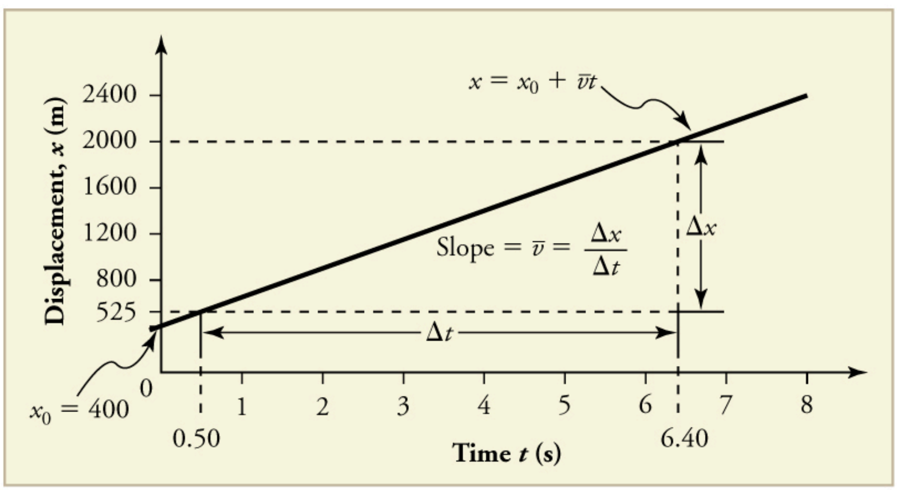


Figure 2.26 Graph of displacement versus time for a jet-powered car on the Bonneville Salt Flats.

Using the relationship between dependent and independent variables, we see that the slope in the graph above is average velocity $\bar{v}$ and the intercept is displacement at time zero—that is, x_0. Substituting these symbols into $y = mx + b$ gives

$$x = \bar{v}t + x_0 \tag{2.25}$$

or

$$x = x_0 + \bar{v}t. \tag{2.26}$$

Thus a graph of displacement versus time gives a general relationship among displacement, velocity, and time, as well as giving detailed numerical information about a specific situation.

The Slope of *x* vs. *t*

The slope of the graph of displacement x vs. time t is velocity v.

$$\text{slope} = \frac{\Delta x}{\Delta t} = v \tag{2.27}$$

Notice that this equation is the same as that derived algebraically from other motion equations in **Motion Equations for Constant Acceleration in One Dimension (https://legacy.cnx.org/content/m42099/latest/)** .

From the figure we can see that the car has a displacement of 25 m at 0.50 s and 2000 m at 6.40 s. Its displacement at other times can be read from the graph; furthermore, information about its velocity and acceleration can also be obtained from the graph.

Example 2.8 Determining Average Velocity from a Graph of Displacement versus Time: Jet Car

Find the average velocity of the car whose position is graphed in **Figure 2.26**.

Strategy

The slope of a graph of x vs. t is average velocity, since slope equals rise over run. In this case, rise = change in displacement and run = change in time, so that

$$\text{slope} = \frac{\Delta x}{\Delta t} = \bar{v}\,. \tag{2.28}$$

Since the slope is constant here, any two points on the graph can be used to find the slope. (Generally speaking, it is most accurate to use two widely separated points on the straight line. This is because any error in reading data from the graph is proportionally smaller if the interval is larger.)

Solution

1. Choose two points on the line. In this case, we choose the points labeled on the graph: (6.4 s, 2000 m) and (0.50 s, 525 m). (Note, however, that you could choose any two points.)

2. Substitute the x and t values of the chosen points into the equation. Remember in calculating change (Δ) we always use final value minus initial value.

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{2000\text{ m} - 525\text{ m}}{6.4\text{ s} - 0.50\text{ s}}, \tag{2.29}$$

yielding

$$\bar{v} = 250\text{ m/s}. \tag{2.30}$$

Discussion

This is an impressively large land speed (900 km/h, or about 560 mi/h): much greater than the typical highway speed limit of 60 mi/h (27 m/s or 96 km/h), but considerably shy of the record of 343 m/s (1234 km/h or 766 mi/h) set in 1997.

Graphs of Motion when a is constant but $a \neq 0$

The graphs in **Figure 2.27** below represent the motion of the jet-powered car as it accelerates toward its top speed, but only during the time when its acceleration is constant. Time starts at zero for this motion (as if measured with a stopwatch), and the displacement and velocity are initially 200 m and 15 m/s, respectively.

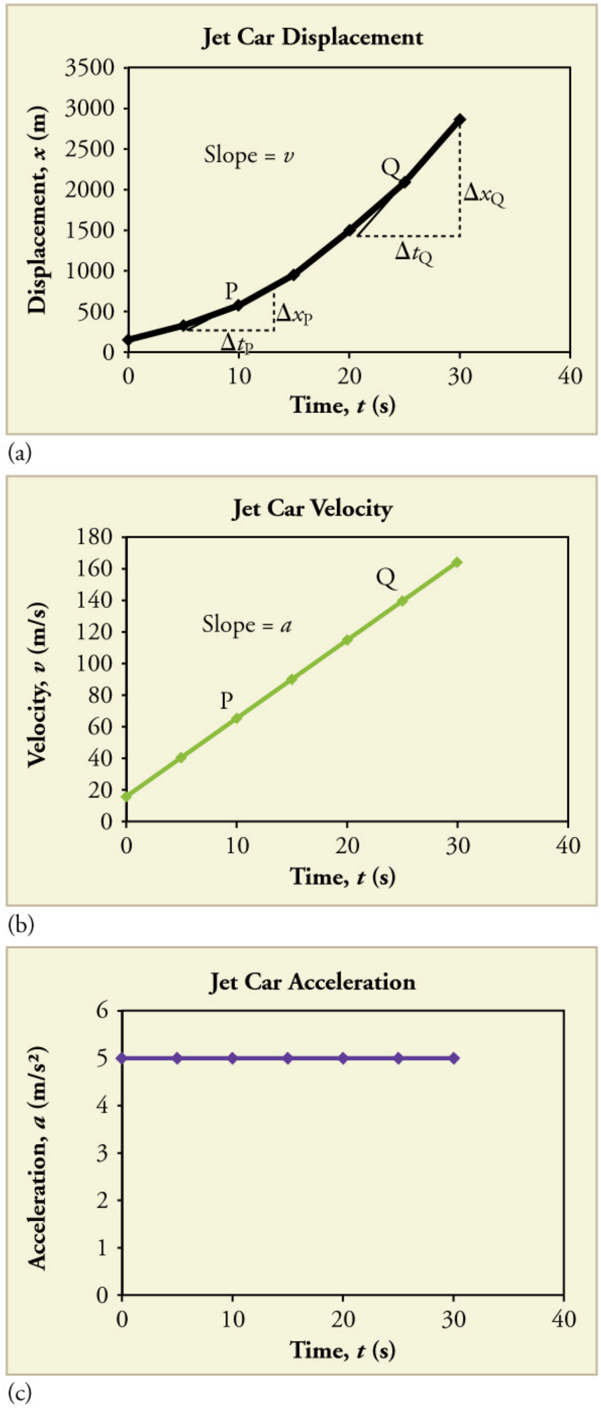


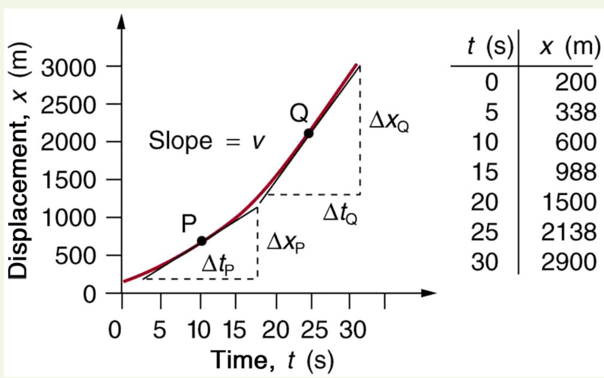
Figure 2.27 Graphs of motion of a jet-powered car during the time span when its acceleration is constant. (a) The slope of an x vs. t graph is velocity. This is shown at two points, and the instantaneous velocities obtained are plotted in the next graph. Instantaneous velocity at any point is the slope of the tangent at that point. (b) The slope of the v vs. t graph is constant for this part of the motion, indicating constant acceleration. (c) Acceleration has the constant value of 5.0 m/s^2 over the time interval plotted.

Figure 2.28 A U.S. Air Force jet car speeds down a track. (credit: Matt Trostle, Flickr)

The graph of displacement versus time in Figure 2.27(a) is a curve rather than a straight line. The slope of the curve becomes steeper as time progresses, showing that the velocity is increasing over time. The slope at any point on a displacement-versus-time graph is the instantaneous velocity at that point. It is found by drawing a straight line tangent to the curve at the point of interest and taking the slope of this straight line. Tangent lines are shown for two points in Figure 2.27(a). If this is done at every point on the curve and the values are plotted against time, then the graph of velocity versus time shown in Figure 2.27(b) is obtained. Furthermore, the slope of the graph of velocity versus time is acceleration, which is shown in Figure 2.27(c).

Example 2.9 Determining Instantaneous Velocity from the Slope at a Point: Jet Car

Calculate the velocity of the jet car at a time of 25 s by finding the slope of the x vs. t graph in the graph below.



t (s)	x (m)
0	200
5	338
10	600
15	988
20	1500
25	2138
30	2900

Figure 2.29 The slope of an x vs. t graph is velocity. This is shown at two points. Instantaneous velocity at any point is the slope of the tangent at that point.

Strategy

The slope of a curve at a point is equal to the slope of a straight line tangent to the curve at that point. This principle is illustrated in Figure 2.29, where Q is the point at $t = 25\text{ s}$.

Solution

1. Find the tangent line to the curve at $t = 25\text{ s}$.

2. Determine the endpoints of the tangent. These correspond to a position of 1300 m at time 19 s and a position of 3120 m at time 32 s.

3. Plug these endpoints into the equation to solve for the slope, v .

$$\text{slope} = v_\text{Q} = \frac{\Delta x_\text{Q}}{\Delta t_\text{Q}} = \frac{(3120\text{ m} - 1300\text{ m})}{(32\text{ s} - 19\text{ s})} \tag{2.31}$$

Thus,

$$v_\text{Q} = \frac{1820\text{ m}}{13\text{ s}} = 140\text{ m/s}. \tag{2.32}$$

Discussion

This is the value given in this figure's table for v at $t = 25\text{ s}$. The value of 140 m/s for v_Q is plotted in **Figure 2.29**. The entire graph of v vs. t can be obtained in this fashion.

Carrying this one step further, we note that the slope of a velocity versus time graph is acceleration. Slope is rise divided by run; on a v vs. t graph, rise = change in velocity Δv and run = change in time Δt.

The Slope of *v* vs. *t*

The slope of a graph of velocity v vs. time t is acceleration a.

$$\text{slope} = \frac{\Delta v}{\Delta t} = a \tag{2.33}$$

Since the velocity versus time graph in **Figure 2.27**(b) is a straight line, its slope is the same everywhere, implying that acceleration is constant. Acceleration versus time is graphed in **Figure 2.27**(c).

Additional general information can be obtained from **Figure 2.29** and the expression for a straight line, $y = mx + b$.

In this case, the vertical axis y is V, the intercept b is v_0, the slope m is a, and the horizontal axis x is t. Substituting these symbols yields

$$v = v_0 + at. \tag{2.34}$$

A general relationship for velocity, acceleration, and time has again been obtained from a graph. Notice that this equation was also derived algebraically from other motion equations in **Motion Equations for Constant Acceleration in One Dimension (https://legacy.cnx.org/content/m42099/latest/)**.

It is not accidental that the same equations are obtained by graphical analysis as by algebraic techniques. In fact, an important way to *discover* physical relationships is to measure various physical quantities and then make graphs of one quantity against another to see if they are correlated in any way. Correlations imply physical relationships and might be shown by smooth graphs such as those above. From such graphs, mathematical relationships can sometimes be postulated. Further experiments are then performed to determine the validity of the hypothesized relationships.

Graphs of Motion Where Acceleration is Not Constant

Now consider the motion of the jet car as it goes from 165 m/s to its top velocity of 250 m/s, graphed in **Figure 2.30**. Time again starts at zero, and the initial displacement and velocity are 2900 m and 165 m/s, respectively. (These were the final displacement and velocity of the car in the motion graphed in **Figure 2.27**.) Acceleration gradually decreases from 5.0 m/s^2 to zero when the car hits 250 m/s. The slope of the x vs. t graph increases until $t = 55\text{ s}$, after which time the slope is constant. Similarly, velocity increases until 55 s and then becomes constant, since acceleration decreases to zero at 55 s and remains zero afterward.

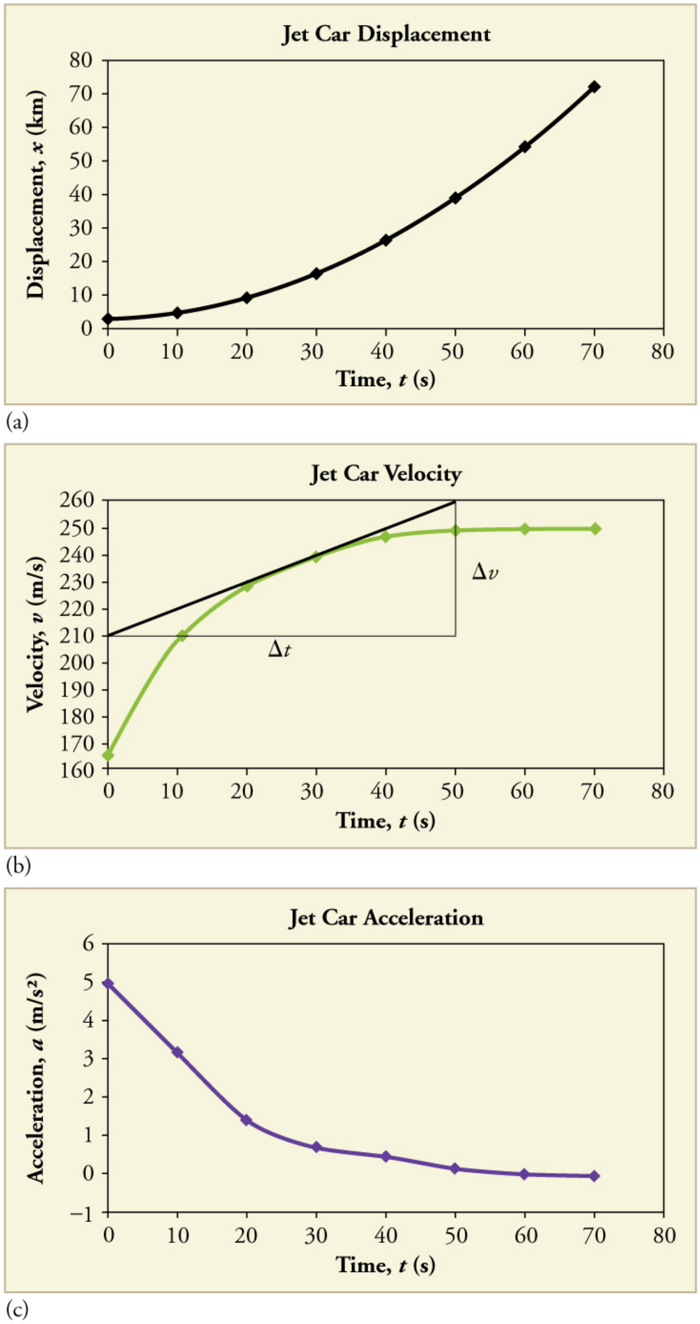


Figure 2.30 Graphs of motion of a jet-powered car as it reaches its top velocity. This motion begins where the motion in **Figure 2.27** ends. (a) The slope of this graph is velocity; it is plotted in the next graph. (b) The velocity gradually approaches its top value. The slope of this graph is acceleration; it is plotted in the final graph. (c) Acceleration gradually declines to zero when velocity becomes constant.

Example 2.10 Calculating Acceleration from a Graph of Velocity versus Time

Calculate the acceleration of the jet car at a time of 25 s by finding the slope of the v vs. t graph in **Figure 2.30**(b).

Strategy

The slope of the curve at $t = 25\text{ s}$ is equal to the slope of the line tangent at that point, as illustrated in **Figure 2.30**(b).

Solution

Determine endpoints of the tangent line from the figure, and then plug them into the equation to solve for slope, a.

$$\text{slope} = \frac{\Delta v}{\Delta t} = \frac{(260\text{ m/s} - 210\text{ m/s})}{(51\text{ s} - 1.0\text{ s})} \tag{2.35}$$

$$a = \frac{50\text{ m/s}}{50\text{ s}} = 1.0\text{ m/s}^2. \tag{2.36}$$

Discussion

Note that this value for a is consistent with the value plotted in **Figure 2.30**(c) at $t = 25\text{ s}$.

A graph of displacement versus time can be used to generate a graph of velocity versus time, and a graph of velocity versus time can be used to generate a graph of acceleration versus time. We do this by finding the slope of the graphs at every point. If the graph is linear (i.e., a line with a constant slope), it is easy to find the slope at any point and you have the slope for every point. Graphical analysis of motion can be used to describe both specific and general characteristics of kinematics. Graphs can also be used for other topics in physics. An important aspect of exploring physical relationships is to graph them and look for underlying relationships.

Check Your Understanding

A graph of velocity vs. time of a ship coming into a harbor is shown below. (a) Describe the motion of the ship based on the graph. (b)What would a graph of the ship's acceleration look like?

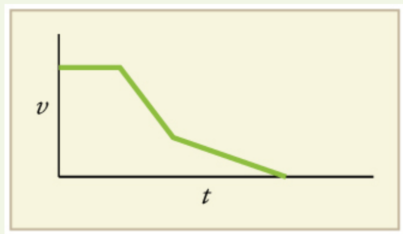


Figure 2.31

Solution

(a) The ship moves at constant velocity and then begins to decelerate at a constant rate. At some point, its deceleration rate decreases. It maintains this lower deceleration rate until it stops moving.

(b) A graph of acceleration vs. time would show zero acceleration in the first leg, large and constant negative acceleration in the second leg, and constant negative acceleration.

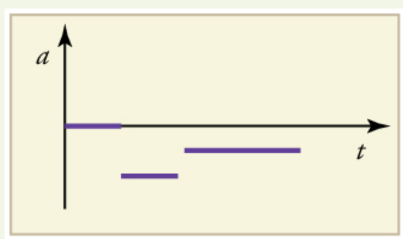


Figure 2.32

2.6 Simulations

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Given a velocity as a function of time, be able to solve for the position as a function of time iteratively.
- Use iterative methods to solve for the motion of an object given an arbitrary (non-constant) acceleration.

This section is based on videos from the UMass Physics 131 YouTube page. The video really adds something beyond this text though, and so for this section, it is **highly recommended** that you watch the videos instead. Here are the links to the videos:

- Theory on Solving Problems with Simulation: **https://www.youtube.com/watch?v=i1elRzD7HYQ (https://www.youtube.com/watch?v=i1elRzD7HYQ)**
- First Example of Solving a Problem with Simulation: **https://www.youtube.com/watch?v=j_HBtITs948 (https://www.youtube.com/watch?v=j_HBtITs948)**

- A More Complex Example of Solving a Problem with Simulation**https://www.youtube.com/watch?v=6FoKBtxuEhg (https://www.youtube.com/watch?v=6FoKBtxuEhg)**

Theory on Solving Problems Using Simulation

Simulation is going to be a tool that we use frequently in this class to solve problems. Why? Well, many physics problems are not solvable with just algebra. Now, you might have suspected this, and figured that, okay, maybe not with algebra, but maybe if I invoke some higher math such as calculus, I can start to do real physics problems. Well, that helps, but even with calculus and some of the most sophisticated math out there, you still can't solve most of the interesting real-world problems that we have. You can solve models, you can solve simplifications, but to try and get everything, the problems are actually undoable.

For example, just to make an extreme example, think about the Earth, the Sun, and the moon. The force on these three objects can be described by a force law from the 17th century, $F = \frac{GmM}{r^2}$, that says the force of gravity between two masses, say the earth and the moon, is their masses and the distance between them squared. That's it. For two objects, say, just the Earth and the moon, you can write an equation that describes their motion using this force law. However, you *cannot* write down an equation that describes the motion of the Sun, the Earth, and the moon; you need go to simulation. Moreover, the ideas of simulation that we're going to be discussing here and throughout this class are being used more and more in essentially all fields of science to solve complex problems. In the Fall 2015 semester, one of the SIs for Physics 131 was using these same ideas to solve problems in his life science based senior honors thesis as an undergraduate. Hopefully, this impresses upon you the relevance of this technique to all fields of science and medicine in the modern day.

What's the philosophy, the premise, behind the idea of simulation? Well, this is perhaps best done in the context of an example. Let's say we have a runner, and we're looking at the runner at some instant, and then some small time later, say, 0.000001 seconds later. Well, if we want to model the motion of the runner, we would think about the average velocity and how that's related to their position and the change in time. If we imagine a really small amount of time, then this runner's velocity from, say the first instant to the next instant, does not change very much. We're going to say that these two are super close together. If we make our time interval small enough, then the velocity essentially between both instants; it's essentially constant. The runner could be running at, say, 5 m/s, at the first instant, and 5.0001 at the next instant, but the change is small enough where we could just say that the runner is running at the average between those two velocities, which is essentially 5 m/s anyways.

Now we're going to do a little bit of algebra. Using the definition of velocity $\langle v \rangle = \frac{\Delta x}{\Delta t}$ and multiplying Δt to the other side gives us

$$\langle v \rangle \Delta t = \Delta x$$

or

$$\langle v \rangle \Delta t = x_f - x_i$$

Bringing over the initial position of the runner gives us the final position

$$\langle v \rangle \Delta t + x_i = x_f$$

We're assuming that the speed is essentially constant, so we can use the runners initial speed for average speed, and so we can solve for x_f , where they are at the end of this time interval. In other words, if we know where I am now, and my speed now, and I'm free to assume that my speed won't change because I'm considering just a tiny time interval, then I can use that information to predict where I'm going to be in the future. This is the idea of simulation. I use what I know about the system at any given instant to predict how things are going to be in the next small time, little bit of time later.

The same philosophy holds true for acceleration. I could have repeated the entire series of steps with acceleration. Let's start with the definition of acceleration, $\langle a \rangle = \frac{\Delta v}{\Delta t}$. If I'm thinking about a very small time interval then the average acceleration is going to be the acceleration; the acceleration is not going to change very much as long as this Δt is really, really small. Doing a little bit of algebra,

$$\langle a \rangle \Delta t = \Delta v$$

$$\langle a \rangle \Delta t = v_f - v_i$$

$$\langle a \rangle \Delta t + v_i = v_f$$

Again, if I know my speed at some instant, my acceleration at some instant, and my Δt is small enough, then I can use that information to predict what my Δv is going to be some small time later. We'll be doing this throughout the course with a wide variety of concepts, from forces to temperature to entropy. In all cases, you're using what you know now to predict what will happen a small time later.

Example of Solving a Problem Using Simulation

Let's take a very simple situation and work our way up.

A ball is dropped from 10 meters above the ground, and as we'll see later in the course, a falling object undergoes a constant acceleration of 9.81 m/s^2. We'll discuss this in more detail later, and we'll get into the physics of freefall at length, but for now, we'll just show you how to run the mechanics of the simulation. How far above the ground is the ball 0.02 seconds later?

We're dealing with very small amounts of time, so our assumption that things are constant will be true. To get started, we'll set up a table:

Table 2.1

Time (t)	Position (x)	Velocity (v)	Acceleration (a)
0	10	0	-9.81

Our table has position, velocity, acceleration, and time, as these will all play into the motion of the ball. We'll define the beginning of the drop at t=0, with an initial position of 10 meters, and the initial velocity of 0 m/s, as the ball isn't moving at the very beginning of the drop. For the acceleration, as stated earlier, it's -9.81 m/s^2. Again, don't get too engrossed in the physics right now, we'll talk about why it's a negative and a positive much later. For now, I just want to go through the mechanics.

Now we'll move on to some time later. We're going to go some small amount of time later, so let's pick our change in time to be 0.01, since it's small and fits into 0.02 seconds nicely. Remember, this whole idea is predicated on the assumption that velocity and acceleration aren't changing very much over the time, and the only way that can be true is if the time is small, so we need to take small time steps. For the next time step, the time is the previous time plus the change in time, or $t+\Delta t$, which comes out to be 0.01. Acceleration won't change, so we can leave that as is.

Now what about the velocity and the position? From above, we solved for final position and final velocity as

$$\langle v\rangle\Delta t + x_i = x_f$$

$$\langle a\rangle\Delta t + v_i = v_f$$

We can use the initial position, initial velocity, and acceleration to solve for the final velocity and final position. Remember, since we're taking a small time step, the velocity won't change much, so we can replace the average velocity with just the initial velocity. Acceleration doesn't change, so the average acceleration is just what it started as in the beginning. Plugging in the numbers gives us: $\left(0\frac{m}{s}\right)(0.01s) + (10m) = 10m$

$$\left(-9.81\frac{m}{s^2}\right)(0.01s) + 0\frac{m}{s} = -0.0981\frac{m}{s}$$

We can use these numbers to continue our table:

Table 2.2

Time (t)	Position (x)	Velocity (v)	Acceleration (a)
0	10	0	-9.8
0.01	10	-0.0981	-9.8

We can repeat this process for the next step in time, using the values we found for the next time step as our new initial conditions. So, plugging in the numbers into the equations:

$$\left(-0.0981\frac{m}{s}\right)(0.01s) + (10m) = 9.99902m$$

$$\left(-9.81\frac{m}{s^2}\right)(0.01s) + -0.0981\frac{m}{s} = -0.1962\frac{m}{s}$$

And putting out new values into the table:

Table 2.3

Time (t)	Position (x)	Velocity (v)	Acceleration (a)
0	10	0	-9.81
0.01	10	-0.0981	-9.81
0.02	9.99902	9.99902	-9.81

So the answer to our initial problem is that the ball is 9.99902 meters off the ground 0.02 seconds later.

The ball doesn't move very far in the short amount of time; we could have figured that out probably by intuition. However, using simulations can better your understanding of concepts such as acceleration and velocity. For example, notice that the ball's position doesn't change in the first time step. Through this simulation, you can see that the acceleration does not result in a change in position immediately, but rather a change in velocity, and it's this velocity that causes the change in position.

More Complex Example

Let's say you have a car, starting at rest, with an acceleration defined to be: $a(t) = (5\frac{m}{s^2})t^2$

A. How fast is it moving after 0.02s?
B. Where is it after 5s?
C. What if it changed to $a(t) = (2\frac{m}{s^2})t^2$, where would it be after 10s?

This example has a few different parts, where our acceleration is not the nice simple constant. In the previous example, the acceleration was constant, so you could solve it without using simulation. However, with an acceleration that's not constant, the only way you can solve it is through simulation.

Let's get started with part a). First, we make our table and put in our initial conditions. Again, we use time, position, velocity, and acceleration for our table. We can set time to start at 0 and position to start at 0, since the problem doesn't specify a starting position. Initial velocity is 0, since the car starts at rest. To find the initial acceleration, we plug in our time into the acceleration given by the problem, so:

$$a(0) = \left(5\frac{m}{s^2}\right)0^2 = 0\frac{m}{s^2}$$

So the initial acceleration is 0 as well.

Table 2.4

Time (t)	Position (x)	Velocity (v)	Acceleration (a)
0	0	0	0

Now let's take our first step and go to 0.01 seconds, where we're still going to be using these two expressions.

$$\langle v\rangle\Delta t + x_i = x_f$$

$$\langle a\rangle\Delta t + v_i = v_f$$

I don't want you to memorize these now, remember, these just come from the definitions of velocity and acceleration. These just the definitions of these quantities rewritten, so it's not a new equation, it's the same definitions we've been exploring in this entire preparation. Again, we plug in the initial values into the two expressions, so: $(0\frac{m}{s})(0.01s) + (0m) = 0m$

$$\left(0\frac{m}{s^2}\right)(0.01s) + 0\frac{m}{s} = 0\frac{m}{s}$$

We also have to solve for our new acceleration using the formula given in the problem. Before we move on, notice that there's a Δt in our expressions for final position and velocity, but there is a t in the formula for acceleration. Normally, these are interchangeable, but in this problem, they mean two separate things. The change in time Δt , so the time steps, while t is the total time up to that step, so it's important to keep in mind this distinction. Solving for the new acceleration gives us:

$$a(0.01) = \left(5\frac{m}{s^2}\right)0.01^2 = 0.0005\frac{m}{s^2}$$

And our new table looks like:

Table 2.5

Time (t)	Position (x)	Velocity (v)	Acceleration (a)
0	0	0	0
0.01	0	0	0.0005

Repeating the process for 0.02 gives us:

$$\left(0\frac{m}{s}\right)(0.01s) + (0m) = 0m$$

$$\left(0.0005\frac{m}{s^2}\right)(0.01s) + 0\frac{m}{s} = 0.000005\frac{m}{s}$$

$$a(0.02) = \left(5\frac{m}{s^2}\right)0.02^2 = 0.002\frac{m}{s^2}$$

Notice that 0.02 is used for the time in the acceleration. Again, keep in mind the distinction between Δt and t in the problem. Filling out the table gives us:

Table 2.6

Time (t)	Position (x)	Velocity (v)	Acceleration (a)
0	0	0	0
0.01	0	0	0.0005
0.02	0	0.000005	0.002

So the answer to the question is that the car is moving 0.000005 m/s after 0.02 seconds.

Alright, so now we've solved this problem by hand for up to .02 seconds, great. The next question is, where is it after five seconds? Well, doing this by hand in one one-hundredth of a second increments for five seconds is going to take us a really long time. You could do it, but you'd be at it for quite a while. This is where the benefit of using a computer to solve the problem will come into play. Since simulation is mostly just a process, you can have a computer program, like Excel or Google Spreadsheets, run through the process for you. We'll be going over how to do this in class.

Glossary

acceleration: the rate of change in velocity; the change in velocity over time

average acceleration: the change in velocity divided by the time over which it changes

average speed: distance traveled divided by time during which motion occurs

average velocity: displacement divided by time over which displacement occurs

deceleration: acceleration in the direction opposite to velocity; acceleration that results in a decrease in velocity

dependent variable: the variable that is being measured; usually plotted along the y -axis

displacement: the change in position of an object

distance: the magnitude of displacement between two positions

distance traveled: the total length of the path traveled between two positions

elapsed time: the difference between the ending time and beginning time

independent variable: the variable that the dependent variable is measured with respect to; usually plotted along the x -axis

instantaneous acceleration: acceleration at a specific point in time

instantaneous speed: magnitude of the instantaneous velocity

instantaneous velocity: velocity at a specific instant, or the average velocity over an infinitesimal time interval

kinematics: the study of motion without considering its causes

model: simplified description that contains only those elements necessary to describe the physics of a physical situation

position: the location of an object at a particular time

scalar: a quantity that is described by magnitude, but not direction

slope: the difference in y -value (the rise) divided by the difference in x -value (the run) of two points on a straight line

time: change, or the interval over which change occurs

vector: a quantity that is described by both magnitude and direction

y-intercept: the y- value when x = 0, or when the graph crosses the y -axis

Section Summary

2.1 Displacement

- Kinematics is the study of motion without considering its causes. In this chapter, it is limited to motion along a straight line, called one-dimensional motion.
- Displacement is the change in position of an object.
- In symbols, displacement Δx is defined to be

$$\Delta x = x_{\text{f}} - x_0,$$

where x_0 is the initial position and x_{f} is the final position. In this text, the Greek letter Δ (delta) always means "change in" whatever quantity follows it. The SI unit for displacement is the meter (m). Displacement has a direction as well as a magnitude.
- When you start a problem, assign which direction will be positive.
- Distance is the magnitude of displacement between two positions.
- Distance traveled is the total length of the path traveled between two positions.

2.2 Vectors, Scalars, and Coordinate Systems

- A vector is any quantity that has magnitude and direction.
- A scalar is any quantity that has magnitude but no direction.
- Displacement and velocity are vectors, whereas distance and speed are scalars.
- In one-dimensional motion, direction is specified by a plus or minus sign to signify left or right, up or down, and the like.

2.3 Time, Velocity, and Speed

- Time is measured in terms of change, and its SI unit is the second (s). Elapsed time for an event is

$$\Delta t = t_{\text{f}} - t_0,$$

where t_{f} is the final time and t_0 is the initial time. The initial time is often taken to be zero, as if measured with a stopwatch; the elapsed time is then just t .
- Average velocity $\bar{v}$ is defined as displacement divided by the travel time. In symbols, average velocity is

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_{\text{f}} - x_0}{t_{\text{f}} - t_0}.$$

- The SI unit for velocity is m/s.
- Velocity is a vector and thus has a direction.
- Instantaneous velocity v is the velocity at a specific instant or the average velocity for an infinitesimal interval.
- Instantaneous speed is the magnitude of the instantaneous velocity.
- Instantaneous speed is a scalar quantity, as it has no direction specified.
- Average speed is the total distance traveled divided by the elapsed time. (Average speed is *not* the magnitude of the average velocity.) Speed is a scalar quantity; it has no direction associated with it.

2.4 Acceleration

- Acceleration is the rate at which velocity changes. In symbols, **average acceleration** $\bar{a}$ is

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_{\text{f}} - v_0}{t_{\text{f}} - t_0}.$$

- The SI unit for acceleration is m/s^2 .
- Acceleration is a vector, and thus has a both a magnitude and direction.

- Acceleration can be caused by either a change in the magnitude or the direction of the velocity.
- Instantaneous acceleration a is the acceleration at a specific instant in time.
- Deceleration is an acceleration with a direction opposite to that of the velocity.

2.5 Graphical Analysis of One-Dimensional Motion

- Graphs of motion can be used to analyze motion.
- Graphical solutions yield identical solutions to mathematical methods for deriving motion equations.
- The slope of a graph of displacement x vs. time t is velocity v.
- The slope of a graph of velocity v vs. time t graph is acceleration a.
- Average velocity, instantaneous velocity, and acceleration can all be obtained by analyzing graphs.

2.6 Simulations

- Simulation is an iterative process that can be used to solve problems, especially in the real world, that can't be solved purely mathematically.

Conceptual Questions

2.1 Displacement

1. Give an example in which there are clear distinctions among distance traveled, displacement, and magnitude of displacement. Specifically identify each quantity in your example.

2. Under what circumstances does distance traveled equal magnitude of displacement? What is the only case in which magnitude of displacement and displacement are exactly the same?

3. Bacteria move back and forth by using their flagella (structures that look like little tails). Speeds of up to $50\ \mu\text{m/s}\ \left(50\times10^{-6}\ \text{m/s}\right)$ have been observed. The total distance traveled by a bacterium is large for its size, while its displacement is small. Why is this?

2.2 Vectors, Scalars, and Coordinate Systems

4. A student writes, "*A bird that is diving for prey has a speed of* $-10\ m/s$." What is wrong with the student's statement? What has the student actually described? Explain.

5. What is the speed of the bird in **Exercise 2.4**?

6. Acceleration is the change in velocity over time. Given this information, is acceleration a vector or a scalar quantity? Explain.

7. A weather forecast states that the temperature is predicted to be -5°C the following day. Is this temperature a vector or a scalar quantity? Explain.

2.3 Time, Velocity, and Speed

8. Give an example (but not one from the text) of a device used to measure time and identify what change in that device indicates a change in time.

9. There is a distinction between average speed and the magnitude of average velocity. Give an example that illustrates the difference between these two quantities.

10. Does a car's odometer measure position or displacement? Does its speedometer measure speed or velocity?

11. If you divide the total distance traveled on a car trip (as determined by the odometer) by the time for the trip, are you calculating the average speed or the magnitude of the average velocity? Under what circumstances are these two quantities the same?

12. How are instantaneous velocity and instantaneous speed related to one another? How do they differ?

2.4 Acceleration

13. Is it possible for speed to be constant while acceleration is not zero? Give an example of such a situation.

14. Is it possible for velocity to be constant while acceleration is not zero? Explain.

15. Give an example in which velocity is zero yet acceleration is not.

16. If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?

17. Plus and minus signs are used in one-dimensional motion to indicate direction. What is the sign of an acceleration that reduces the magnitude of a negative velocity? Of a positive velocity?

2.5 Graphical Analysis of One-Dimensional Motion

18. (a) Explain how you can use the graph of position versus time in **Figure 2.33** to describe the change in velocity over time. Identify (b) the time (t_a , t_b , t_c , t_d , or t_e) at which the instantaneous velocity is greatest, (c) the time at which it is zero, and (d) the time at which it is negative.

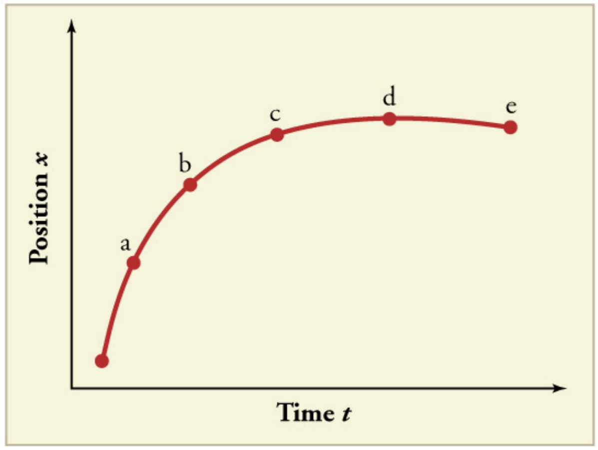


Figure 2.33

19. (a) Sketch a graph of velocity versus time corresponding to the graph of displacement versus time given in **Figure 2.34**. (b) Identify the time or times (t_a , t_b , t_c , etc.) at which the instantaneous velocity is greatest. (c) At which times is it zero? (d) At which times is it negative?

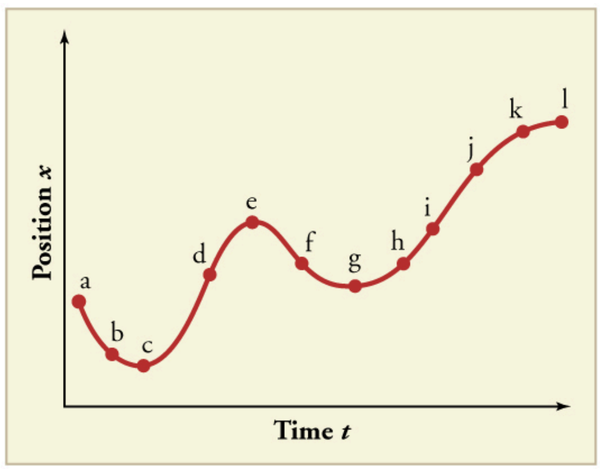


Figure 2.34

20. (a) Explain how you can determine the acceleration over time from a velocity versus time graph such as the one in **Figure 2.35**. (b) Based on the graph, how does acceleration change over time?

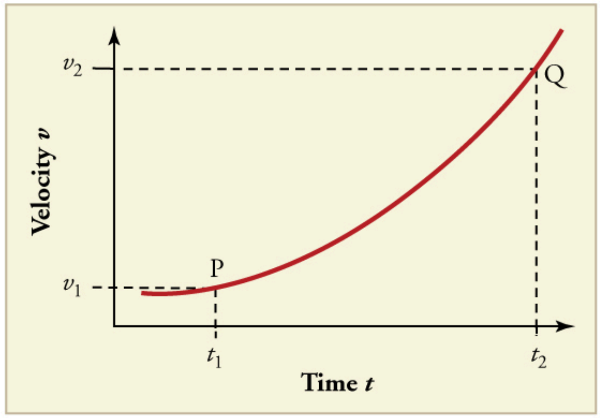


Figure 2.35

21. (a) Sketch a graph of acceleration versus time corresponding to the graph of velocity versus time given in **Figure 2.36**. (b) Identify the time or times (t_a , t_b , t_c , etc.) at which the acceleration is greatest. (c) At which times is it zero? (d) At which times is it negative?

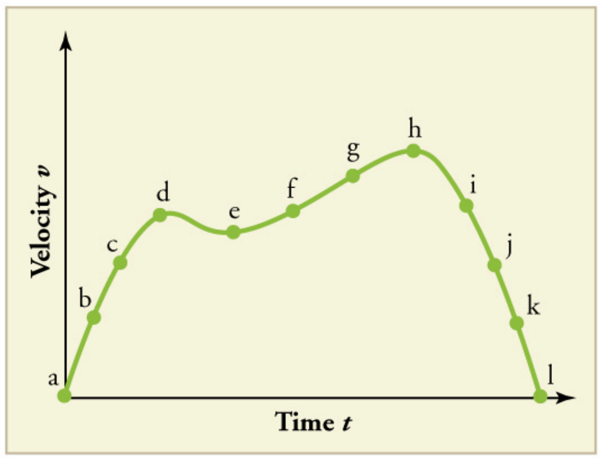


Figure 2.36

22. Consider the velocity vs. time graph of a person in an elevator shown in **Figure 2.37**. Suppose the elevator is initially at rest. It then accelerates for 3 seconds, maintains that velocity for 15 seconds, then decelerates for 5 seconds until it stops. The acceleration for the entire trip is not constant so we cannot use the equations of motion from **Motion Equations for Constant Acceleration in One Dimension (https://legacy.cnx.org/content/m42099/latest/)** for the complete trip. (We could, however, use them in the three individual sections where acceleration is a constant.) Sketch graphs of (a) position vs. time and (b) acceleration vs. time for this trip.

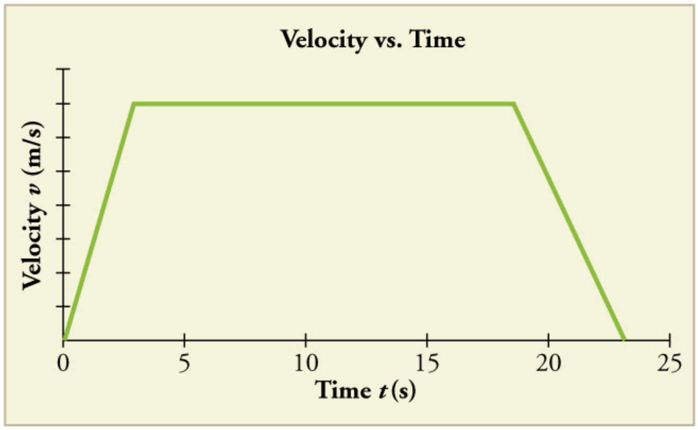


Figure 2.37

23. A cylinder is given a push and then rolls up an inclined plane. If the origin is the starting point, sketch the position, velocity, and acceleration of the cylinder vs. time as it goes up and then down the plane.

Problems & Exercises

2.1 Displacement

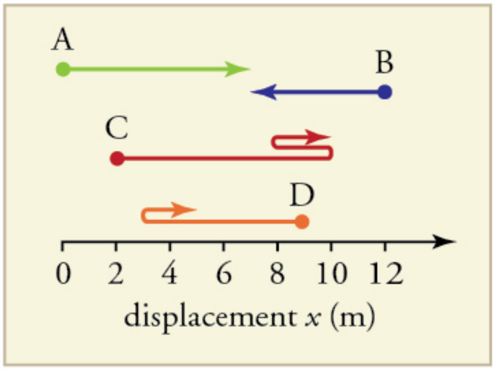


Figure 2.38

1. Find the following for path A in Figure 2.38: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

2. Find the following for path B in Figure 2.38: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

3. Find the following for path C in Figure 2.38: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

4. Find the following for path D in Figure 2.38: (a) The distance traveled. (b) The magnitude of the displacement from start to finish. (c) The displacement from start to finish.

2.3 Time, Velocity, and Speed

5. (a) Calculate Earth's average speed relative to the Sun. (b) What is its average velocity over a period of one year?

6. A helicopter blade spins at exactly 100 revolutions per minute. Its tip is 5.00 m from the center of rotation. (a) Calculate the average speed of the blade tip in the helicopter's frame of reference. (b) What is its average velocity over one revolution?

7. The North American and European continents are moving apart at a rate of about 3 cm/y. At this rate how long will it take them to drift 500 km farther apart than they are at present?

8. Land west of the San Andreas fault in southern California is moving at an average velocity of about 6 cm/y northwest relative to land east of the fault. Los Angeles is west of the fault and may thus someday be at the same latitude as San Francisco, which is east of the fault. How far in the future will this occur if the displacement to be made is 590 km northwest, assuming the motion remains constant?

9. On May 26, 1934, a streamlined, stainless steel diesel train called the Zephyr set the world's nonstop long-distance speed record for trains. Its run from Denver to Chicago took 13 hours, 4 minutes, 58 seconds, and was witnessed by more than a million people along the route. The total distance traveled was 1633.8 km. What was its average speed in km/h and m/s?

10. Tidal friction is slowing the rotation of the Earth. As a result, the orbit of the Moon is increasing in radius at a rate of approximately 4 cm/year. Assuming this to be a constant rate, how many years will pass before the radius of the Moon's orbit increases by $3.84\times10^6\ \text{m}$ (1%)?

11. A student drove to the university from her home and noted that the odometer reading of her car increased by 12.0 km. The trip took 18.0 min. (a) What was her average speed? (b) If the straight-line distance from her home to the university is 10.3 km in a direction 25.0° south of east, what was her average velocity? (c) If she returned home by the same path 7 h 30 min after she left, what were her average speed and velocity for the entire trip?

12. The speed of propagation of the action potential (an electrical signal) in a nerve cell depends (inversely) on the diameter of the axon (nerve fiber). If the nerve cell connecting the spinal cord to your feet is 1.1 m long, and the nerve impulse speed is 18 m/s, how long does it take for the nerve signal to travel this distance?

13. Conversations with astronauts on the lunar surface were characterized by a kind of echo in which the earthbound person's voice was so loud in the astronaut's space helmet that it was picked up by the astronaut's microphone and transmitted back to Earth. It is reasonable to assume that the echo time equals the time necessary for the radio wave to travel from the Earth to the Moon and back (that is, neglecting any time delays in the electronic equipment). Calculate the distance from Earth to the Moon given that the echo time was 2.56 s and that radio waves travel at the speed of light $(3.00\times10^8\ \text{m/s})$.

14. A football quarterback runs 15.0 m straight down the playing field in 2.50 s. He is then hit and pushed 3.00 m straight backward in 1.75 s. He breaks the tackle and runs straight forward another 21.0 m in 5.20 s. Calculate his average velocity (a) for each of the three intervals and (b) for the entire motion.

15. The planetary model of the atom pictures electrons orbiting the atomic nucleus much as planets orbit the Sun. In this model you can view hydrogen, the simplest atom, as having a single electron in a circular orbit $1.06\times10^{-10}\ \text{m}$ in diameter. (a) If the average speed of the electron in this orbit is known to be $2.20\times10^6\ \text{m/s}$, calculate the number of revolutions per second it makes about the nucleus. (b) What is the electron's average velocity?

2.4 Acceleration

16. A cheetah can accelerate from rest to a speed of 30.0 m/s in 7.00 s. What is its acceleration?

17. Professional Application

Dr. John Paul Stapp was U.S. Air Force officer who studied the effects of extreme deceleration on the human body. On December 10, 1954, Stapp rode a rocket sled, accelerating from rest to a top speed of 282 m/s (1015 km/h) in 5.00 s, and was brought jarringly back to rest in only 1.40 s! Calculate his (a) acceleration and (b) deceleration. Express each in multiples of g $(9.80\ \text{m/s}^2)$ by taking its ratio to the acceleration of gravity.

18. A commuter backs her car out of her garage with an acceleration of $1.40\ \text{m/s}^2$. (a) How long does it take her to reach a speed of 2.00 m/s? (b) If she then brakes to a stop in 0.800 s, what is her deceleration?

19. Assume that an intercontinental ballistic missile goes from rest to a suborbital speed of 6.50 km/s in 60.0 s (the actual speed and time are classified). What is its average acceleration in $\mathrm{m/s^2}$ and in multiples of g $(9.80\ \mathrm{m/s^2})$?

2.5 Graphical Analysis of One-Dimensional Motion

Note: There is always uncertainty in numbers taken from graphs. If your answers differ from expected values, examine them to see if they are within data extraction uncertainties estimated by you.

20. (a) By taking the slope of the curve in **Figure 2.39**, verify that the velocity of the jet car is 115 m/s at $t = 20\ \mathrm{s}$. (b) By taking the slope of the curve at any point in **Figure 2.40**, verify that the jet car's acceleration is $5.0\ \mathrm{m/s^2}$.

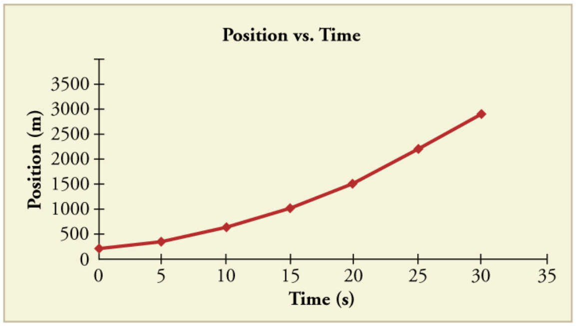


Figure 2.39

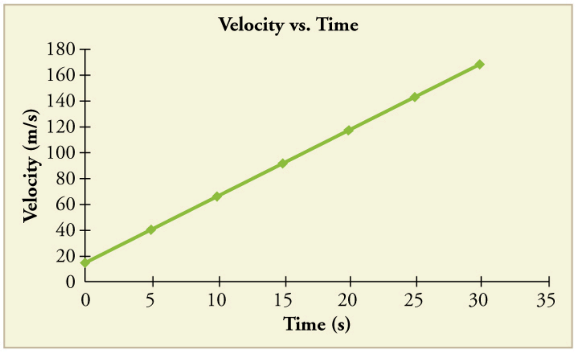


Figure 2.40

21. Using approximate values, calculate the slope of the curve in **Figure 2.41** to verify that the velocity at $t = 10.0\ \mathrm{s}$ is 0.208 m/s. Assume all values are known to 3 significant figures.

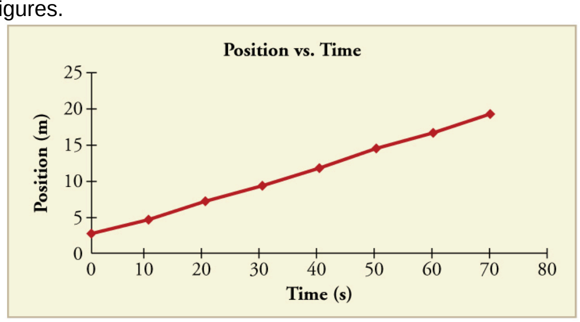


Figure 2.41

22. Using approximate values, calculate the slope of the curve in **Figure 2.41** to verify that the velocity at $t = 30.0\ \mathrm{s}$ is 0.238 m/s. Assume all values are known to 3 significant figures.

23. By taking the slope of the curve in **Figure 2.42**, verify that the acceleration is $3.2\ \mathrm{m/s^2}$ at $t = 10\ \mathrm{s}$.

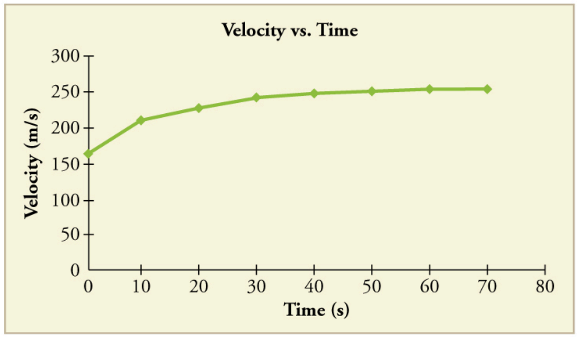


Figure 2.42

24. Construct the displacement graph for the subway shuttle train as shown in **m42100 (https://legacy.cnx.org/content/m42100/latest/#import-auto-id2590556)** (a). Your graph should show the position of the train, in kilometers, from t = 0 to 20 s. You will need to use the information on acceleration and velocity given in the examples for this figure.

25. (a) Take the slope of the curve in **Figure 2.43** to find the jogger's velocity at $t = 2.5\text{ s}$. (b) Repeat at 7.5 s. These values must be consistent with the graph in **Figure 2.44**.

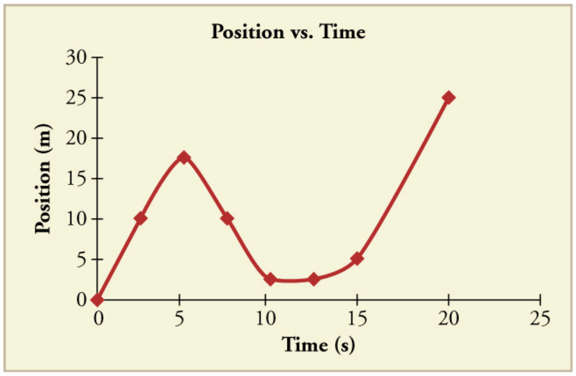


Figure 2.43

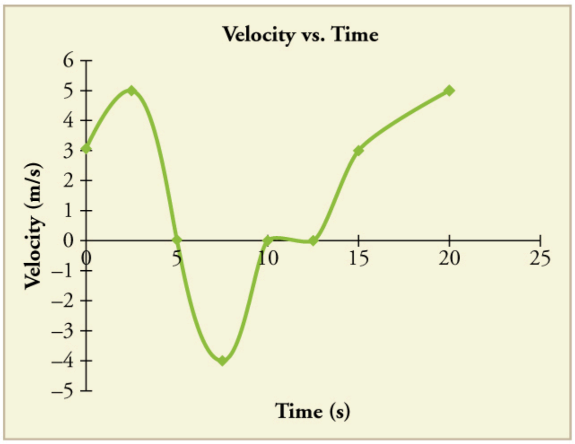


Figure 2.44

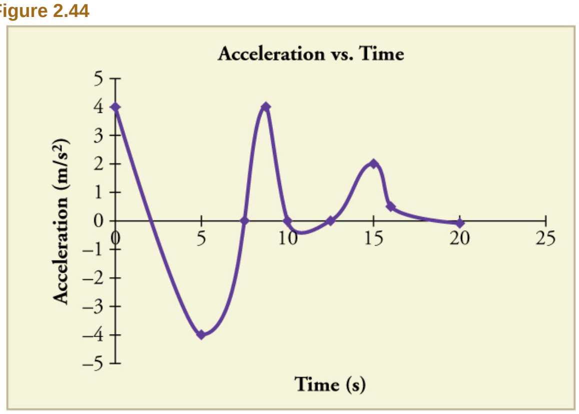


Figure 2.45

26. A graph of $v(t)$ is shown for a world-class track sprinter in a 100-m race. (See **Figure 2.46**). (a) What is his average velocity for the first 4 s? (b) What is his instantaneous velocity at $t = 5\text{ s}$? (c) What is his average acceleration between 0 and 4 s? (d) What is his time for the race?

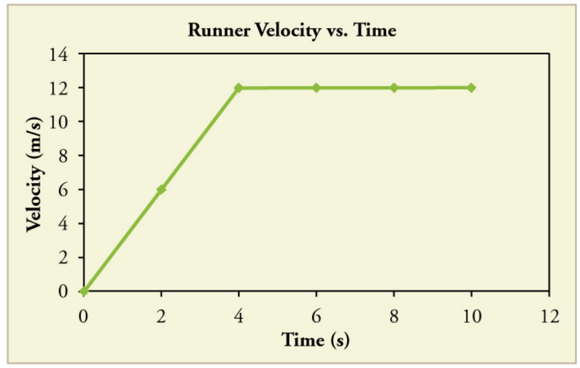


Figure 2.46

27. **Figure 2.47** shows the displacement graph for a particle for 5 s. Draw the corresponding velocity and acceleration graphs.

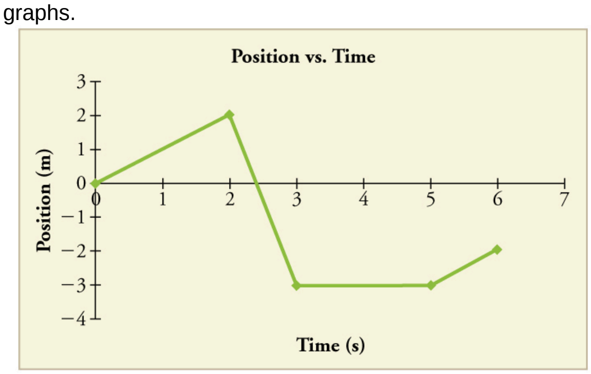


Figure 2.47

3 TWO-DIMENSIONAL KINEMATICS

Figure 3.1 Everyday motion that we experience is, thankfully, rarely as tortuous as a rollercoaster ride like this—the Dragon Khan in Spain's Universal Port Aventura Amusement Park. However, most motion is in curved, rather than straight-line, paths. Motion along a curved path is two- or three-dimensional motion, and can be described in a similar fashion to one-dimensional motion. (credit: Boris23/Wikimedia Commons)

Chapter Outline

3.1. Kinematics in Two Dimensions-An introduction
- A vector is a quantity with a magnitude and direction
- Converting between magnitude/direction and the component form for any vector. This ties into the Pythagorean Theorem

3.2. Vector Addition and Subtraction: Graphical Methods
- Given two graphical representations of vectors, be able to draw the sum or difference.
- Describe both visually and mathematically what happens when a scalar is multiplied by a vector.
- Convert between magnitude/direction and component form for any vector

3.3. Vector Addition and Subtraction-Analytical Methods
- Adding vectors by components

3.4. Addition of Velocities

Introduction to Two-Dimensional Kinematics

The arc of a basketball, the orbit of a satellite, a bicycle rounding a curve, a swimmer diving into a pool, blood gushing out of a wound, and a puppy chasing its tail are but a few examples of motions along curved paths. In fact, most motions in nature follow curved paths rather than straight lines. Motion along a curved path on a flat surface or a plane (such as that of a ball on a pool table or a skater on an ice rink) is two-dimensional, and thus described by two-dimensional kinematics. Motion not confined to a plane, such as a car following a winding mountain road, is described by three-dimensional kinematics. Both two- and three-dimensional kinematics are simple extensions of the one-dimensional kinematics developed for straight-line motion in the previous chapter. This simple extension will allow us to apply physics to many more situations, and it will also yield unexpected insights about nature.

3.1 Kinematics in Two Dimensions-An introduction

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- A vector is a quantity with a magnitude and direction
- Converting between magnitude/direction and the component form for any vector. This ties into the Pythagorean Theorem

Figure 3.2 Walkers and drivers in a city like New York are rarely able to travel in straight lines to reach their destinations. Instead, they must follow roads and sidewalks, making two-dimensional, zigzagged paths. (credit: Margaret W. Carruthers)

Two-Dimensional Motion: Walking in a City

Suppose you want to walk from one point to another in a city with uniform square blocks, as pictured in Figure 3.3.

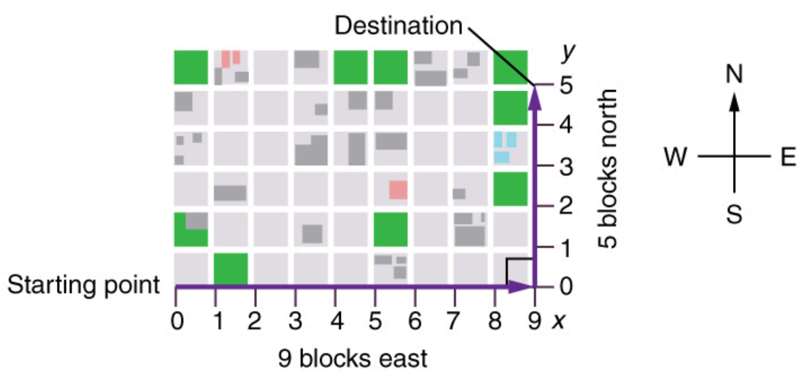


Figure 3.3 A pedestrian walks a two-dimensional path between two points in a city. In this scene, all blocks are square and are the same size.

The straight-line path that a helicopter might fly is blocked to you as a pedestrian, and so you are forced to take a two-dimensional path, such as the one shown. You walk 14 blocks in all, 9 east followed by 5 north. What is the straight-line distance?

An old adage states that the shortest distance between two points is a straight line. The two legs of the trip and the straight-line path form a right triangle, and so the Pythagorean theorem, $a^2 + b^2 = c^2$, can be used to find the straight-line distance.

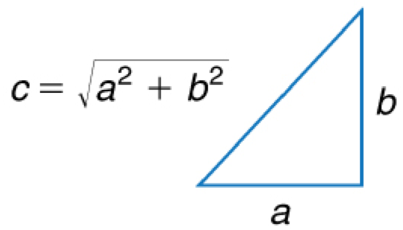


Figure 3.4 The Pythagorean theorem relates the length of the legs of a right triangle, labeled a and b, with the hypotenuse, labeled c. The relationship is given by: $a^2 + b^2 = c^2$. This can be rewritten, solving for c : $c = \sqrt{a^2 + b^2}$.

Instructor's Notes

We will be using the pythagorean theorem all throughout two-dimensional kinematics, as well as throughout this entire course. If you are uncomfortable or unfamiliar with the Pythagorean Theorem, or even if it's just been a long time since you've used it, please come see your instructor as soon as possible and they will get you up to speed.

The hypotenuse of the triangle is the straight-line path, and so in this case its length in units of city blocks is $\sqrt{(9\text{ blocks})^2 + (5\text{ blocks})^2} = 10.3\text{ blocks}$, considerably shorter than the 14 blocks you walked. (Note that we are using three significant figures in the answer. Although it appears that “9” and “5” have only one significant digit, they are discrete numbers. In this case “9 blocks” is the same as “9.0 or 9.00 blocks.” We have decided to use three significant figures in the answer in order to show the result more precisely.)

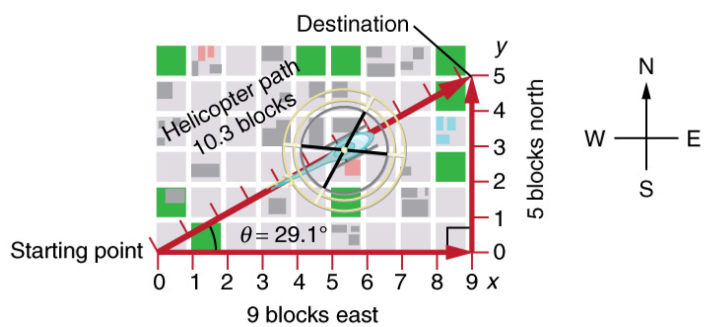


Figure 3.5 The straight-line path followed by a helicopter between the two points is shorter than the 14 blocks walked by the pedestrian. All blocks are square and the same size.

The fact that the straight-line distance (10.3 blocks) in **Figure 3.5** is less than the total distance walked (14 blocks) is one example of a general characteristic of vectors. (Recall that **vectors** are quantities that have both magnitude and direction.)

As for one-dimensional kinematics, we use arrows to represent vectors. The length of the arrow is proportional to the vector’s magnitude. The arrow’s length is indicated by hash marks in **Figure 3.3** and **Figure 3.5**. The arrow points in the same direction as the vector. For two-dimensional motion, the path of an object can be represented with three vectors: one vector shows the straight-line path between the initial and final points of the motion, one vector shows the horizontal component of the motion, and one vector shows the vertical component of the motion. The horizontal and vertical components of the motion add together to give the straight-line path. For example, observe the three vectors in **Figure 3.5**. The first represents a 9-block displacement east. The second represents a 5-block displacement north. These vectors are added to give the third vector, with a 10.3-block total displacement. The third vector is the straight-line path between the two points. Note that in this example, the vectors that we are adding are perpendicular to each other and thus form a right triangle. This means that we can use the Pythagorean theorem to calculate the magnitude of the total displacement. (Note that we cannot use the Pythagorean theorem to add vectors that are not perpendicular. We will develop techniques for adding vectors having any direction, not just those perpendicular to one another, in **Vector Addition and Subtraction: Graphical Methods (https://legacy.cnx.org/content/m42127/latest/)** and **Vector Addition and Subtraction: Analytical Methods (https://legacy.cnx.org/content/m42128/latest/)** .)

The Independence of Perpendicular Motions

UMASS AMHERST Instructor's Notes

The idea of the independence of perpendicular motion is a fundamental one that you should take some time to think about, and there are some questions about this on the homework.

The person taking the path shown in **Figure 3.5** walks east and then north (two perpendicular directions). How far he or she walks east is only affected by his or her motion eastward. Similarly, how far he or she walks north is only affected by his or her motion northward.

Independence of Motion

The horizontal and vertical components of two-dimensional motion are independent of each other. Any motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

This is true in a simple scenario like that of walking in one direction first, followed by another. It is also true of more complicated motion involving movement in two directions at once. For example, let’s compare the motions of two baseballs. One baseball is dropped from rest. At the same instant, another is thrown horizontally from the same height and follows a curved path. A stroboscope has captured the positions of the balls at fixed time intervals as they fall.

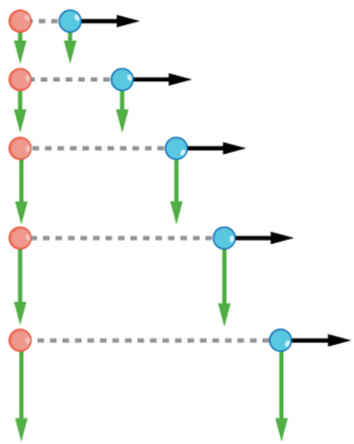


Figure 3.6 This shows the motions of two identical balls—one falls from rest, the other has an initial horizontal velocity. Each subsequent position is an equal time interval. Arrows represent horizontal and vertical velocities at each position. The ball on the right has an initial horizontal velocity, while the ball on the left has no horizontal velocity. Despite the difference in horizontal velocities, the vertical velocities and positions are identical for both balls. This shows that the vertical and horizontal motions are independent.

UMASS AMHERST Instructor's Notes

This graphic displays this concept quite nicely; notice how both balls fall downward at the same speed at each point, even though one of the balls has a horizontal velocity. Basically, the velocity of the ball in the x-direction has no effect on the velocity in the y-direction, and vice-versa. This will be an important idea, especially when working with vectors.

It is remarkable that for each flash of the strobe, the vertical positions of the two balls are the same. This similarity implies that the vertical motion is independent of whether or not the ball is moving horizontally. (Assuming no air resistance, the vertical motion of a falling object is influenced by gravity only, and not by any horizontal forces.) Careful examination of the ball thrown horizontally shows that it travels the same horizontal distance between flashes. This is due to the fact that there are no additional forces on the ball in the horizontal direction after it is thrown. This result means that the horizontal velocity is constant, and affected neither by vertical motion nor by gravity (which is vertical). Note that this case is true only for ideal conditions. In the real world, air resistance will affect the speed of the balls in both directions.

The two-dimensional curved path of the horizontally thrown ball is composed of two independent one-dimensional motions (horizontal and vertical). The key to analyzing such motion, called *projectile motion*, is to *resolve* (break) it into motions along perpendicular directions. Resolving two-dimensional motion into perpendicular components is possible because the components are independent. We shall see how to resolve vectors in **Vector Addition and Subtraction: Graphical Methods (https://legacy.cnx.org/content/m42127/latest/)** and **Vector Addition and Subtraction: Analytical Methods (https://legacy.cnx.org/content/m42128/latest/)** . We will find such techniques to be useful in many areas of physics.

PhET Explorations: Ladybug Motion 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

PhET Interactive Simulation

Figure 3.7 Ladybug Motion 2D (http://legacy.cnx.org/content/m64153/1.3/ladybug-motion-2d_en.jar)

3.2 Vector Addition and Subtraction: Graphical Methods

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Given two graphical representations of vectors, be able to draw the sum or difference. There are some simple procedures to follow. Solidify your understanding of these procedures and we can work on why this makes sense in class
- Describe both visually and mathematically what happens when a scalar is multiplied by a vector. If I give you a vector and a number, you should be able to turn the crank and multiply them mathematically. I am NOT expecting you to be able to do this graphically and will not ask you what it means. Just focus on the mechanics of how to do it.
- Convert between magnitude/direction and component form for any vector

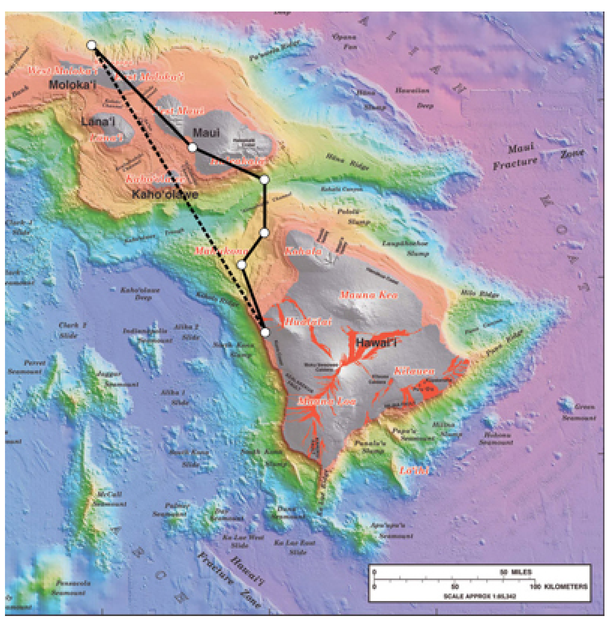


Figure 3.8 Displacement can be determined graphically using a scale map, such as this one of the Hawaiian Islands. A journey from Hawai'i to Moloka'i has a number of legs, or journey segments. These segments can be added graphically with a ruler to determine the total two-dimensional displacement of the journey. (credit: US Geological Survey)

Vectors in Two Dimensions

A **vector** is a quantity that has magnitude and direction. Displacement, velocity, acceleration, and force, for example, are all vectors. In one-dimensional, or straight-line, motion, the direction of a vector can be given simply by a plus or minus sign. In two dimensions (2-d), however, we specify the direction of a vector relative to some reference frame (i.e., coordinate system), using an arrow having length proportional to the vector's magnitude and pointing in the direction of the vector.

Figure 3.9 shows such a *graphical representation of a vector*, using as an example the total displacement for the person walking in a city considered in **Kinematics in Two Dimensions: An Introduction (https://legacy.cnx.org/content/m42104/latest/)** . We shall use the notation that a boldface symbol, such as $\mathbf{D}$, stands for a vector. Its magnitude is represented by the symbol in italics, D , and its direction by θ .

Instructor's Notes

There's some notation in the following note that would be useful to pay attention too.

Vectors in this Text

In this text, we will represent a vector with a boldface variable. For example, we will represent the quantity force with the vector $\mathbf{F}$, which has both magnitude and direction. The magnitude of the vector will be represented by a variable in italics, such as F , and the direction of the variable will be given by an angle θ .

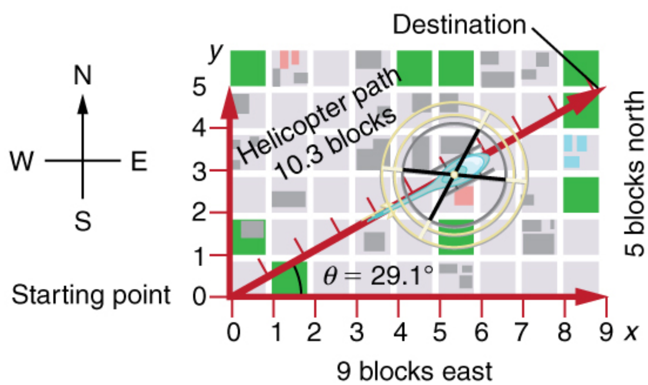


Figure 3.9 A person walks 9 blocks east and 5 blocks north. The displacement is 10.3 blocks at an angle 29.1° north of east.

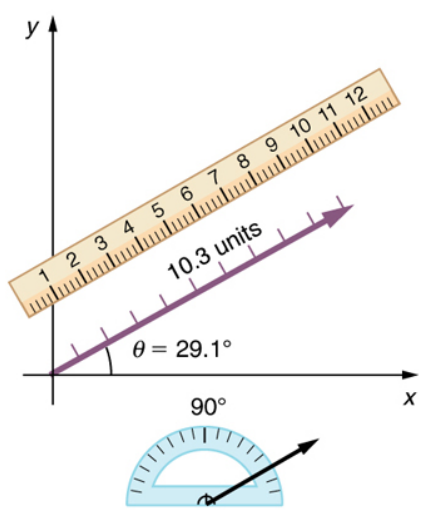


Figure 3.10 To describe the resultant vector for the person walking in a city considered in **Figure 3.9** graphically, draw an arrow to represent the total displacement vector $\mathbf{D}$. Using a protractor, draw a line at an angle θ relative to the east-west axis. The length D of the arrow is proportional to the vector's magnitude and is measured along the line with a ruler. In this example, the magnitude D of the vector is 10.3 units, and the direction θ is 29.1° north of east.

UMASS AMHERST Instructor's Notes

Taking some time to understand and practice the head-to-tail method is recommended, you'll notice that there's a series of algorithmic steps, so you just need to learn the process, and it will work for any two vectors.

Vector Addition: Head-to-Tail Method

The **head-to-tail method** is a graphical way to add vectors, described in Figure 3.11 below and in the steps following. The **tail** of the vector is the starting point of the vector, and the **head** (or tip) of a vector is the final, pointed end of the arrow.

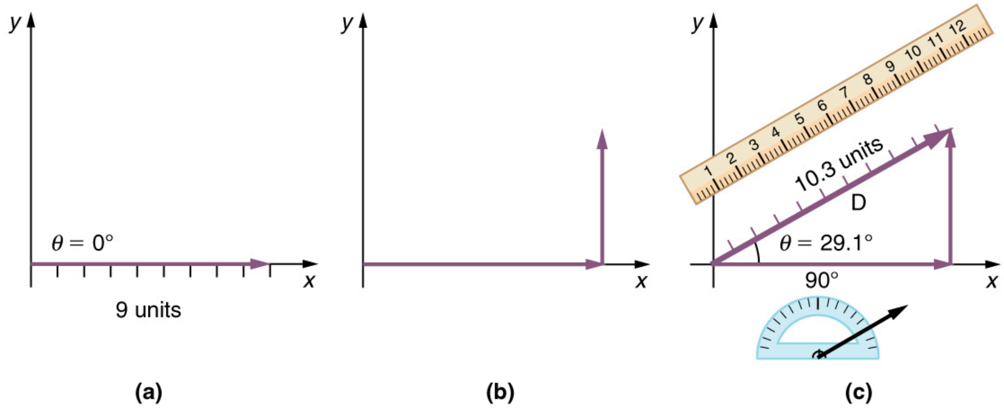


Figure 3.11 **Head-to-Tail Method:** The head-to-tail method of graphically adding vectors is illustrated for the two displacements of the person walking in a city considered in Figure 3.9. (a) Draw a vector representing the displacement to the east. (b) Draw a vector representing the displacement to the north. The tail of this vector should originate from the head of the first, east-pointing vector. (c) Draw a line from the tail of the east-pointing vector to the head of the north-pointing vector to form the sum or **resultant vector** $\mathbf{D}$. The length of the arrow $\mathbf{D}$ is proportional to the vector's magnitude and is measured to be 10.3 units . Its direction, described as the angle with respect to the east (or horizontal axis) θ is measured with a protractor to be 29.1°.

Step 1. *Draw an arrow to represent the first vector (9 blocks to the east) using a ruler and protractor*.

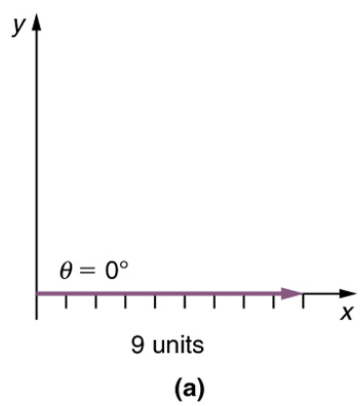
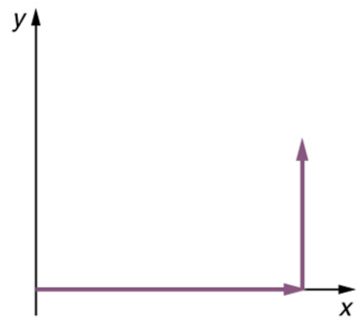


Figure 3.12

Step 2. Now draw an arrow to represent the second vector (5 blocks to the north). *Place the tail of the second vector at the head of the first vector*.

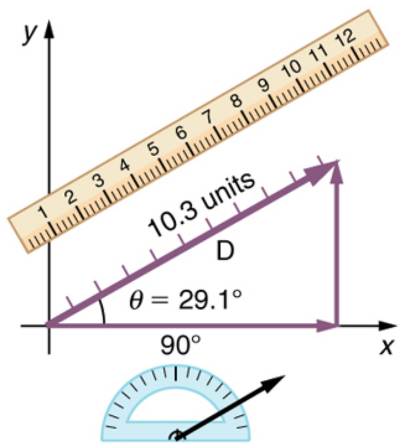


(b)

Figure 3.13

Step 3. *If there are more than two vectors, continue this process for each vector to be added. Note that in our example, we have only two vectors, so we have finished placing arrows tip to tail*.

Step 4. *Draw an arrow from the tail of the first vector to the head of the last vector*. This is the **resultant**, or the sum, of the other vectors.



(c)

Figure 3.14

Step 5. To get the **magnitude** of the resultant, *measure its length with a ruler. (Note that in most calculations, we will use the Pythagorean theorem to determine this length.)*

Step 6. To get the **direction** of the resultant, *measure the angle it makes with the reference frame using a protractor. (Note that in most calculations, we will use trigonometric relationships to determine this angle.)*

The graphical addition of vectors is limited in accuracy only by the precision with which the drawings can be made and the precision of the measuring tools. It is valid for any number of vectors.

Example 3.1 Adding Vectors Graphically Using the Head-to-Tail Method: A Woman Takes a Walk

Use the graphical technique for adding vectors to find the total displacement of a person who walks the following three paths (displacements) on a flat field. First, she walks 25.0 m in a direction 49.0° north of east. Then, she walks 23.0 m heading 15.0° north of east. Finally, she turns and walks 32.0 m in a direction 68.0° south of east.

Strategy

Represent each displacement vector graphically with an arrow, labeling the first $\mathbf{A}$, the second $\mathbf{B}$, and the third $\mathbf{C}$, making the lengths proportional to the distance and the directions as specified relative to an east-west line. The head-to-tail method outlined above will give a way to determine the magnitude and direction of the resultant displacement, denoted $\mathbf{R}$.

Solution

(1) Draw the three displacement vectors.

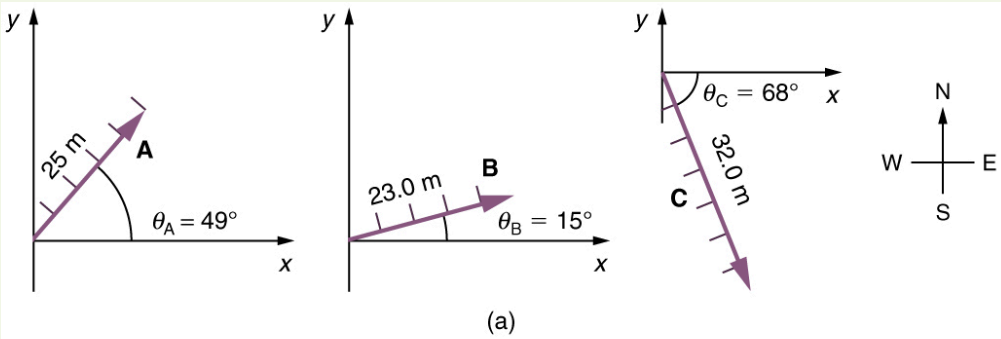


Figure 3.15

(2) Place the vectors head to tail retaining both their initial magnitude and direction.

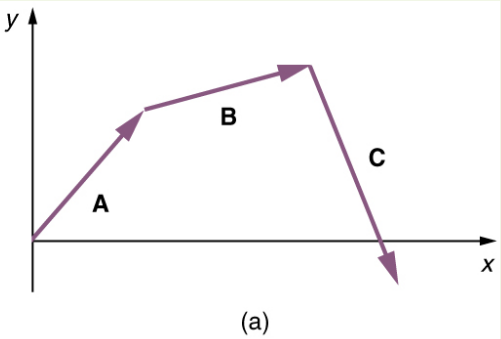


Figure 3.16

(3) Draw the resultant vector, $\mathbf{R}$.

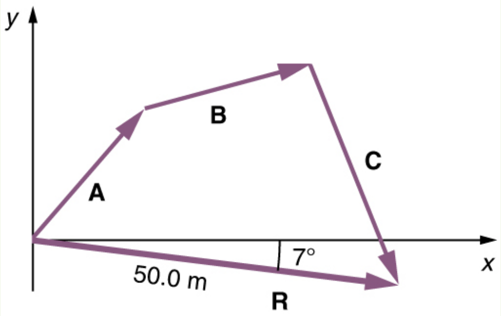


Figure 3.17

(4) Use a ruler to measure the magnitude of $\mathbf{R}$, and a protractor to measure the direction of $\mathbf{R}$. While the direction of the vector can be specified in many ways, the easiest way is to measure the angle between the vector and the nearest horizontal or vertical axis. Since the resultant vector is south of the eastward pointing axis, we flip the protractor upside down and measure the angle between the eastward axis and the vector.

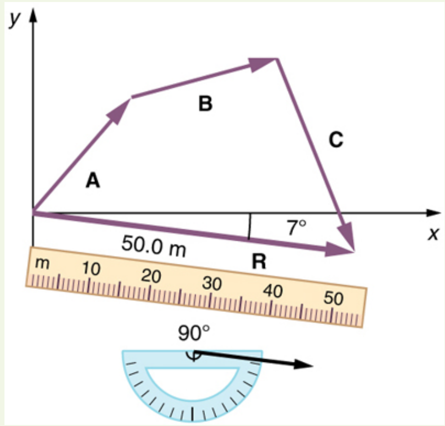


Figure 3.18

In this case, the total displacement $\mathbf{R}$ is seen to have a magnitude of 50.0 m and to lie in a direction 7.0^{o} south of east. By using its magnitude and direction, this vector can be expressed as $R = 50.0\text{ m}$ and $\theta = 7.0^{\text{o}}$ south of east.

Discussion

The head-to-tail graphical method of vector addition works for any number of vectors. It is also important to note that the resultant is independent of the order in which the vectors are added. Therefore, we could add the vectors in any order as illustrated in **Figure 3.19** and we will still get the same solution.

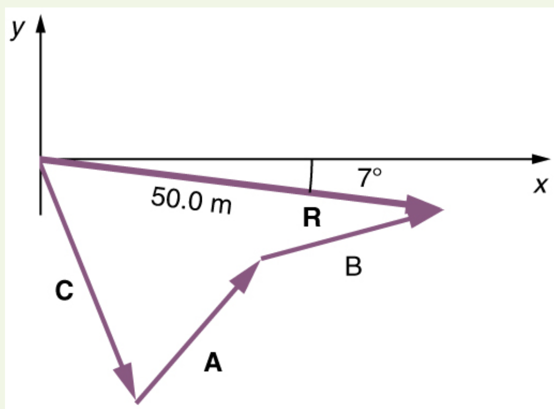


Figure 3.19

Here, we see that when the same vectors are added in a different order, the result is the same. This characteristic is true in every case and is an important characteristic of vectors. Vector addition is **commutative**. Vectors can be added in any order.

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}. \tag{3.1}$$

(This is true for the addition of ordinary numbers as well—you get the same result whether you add $\mathbf{2} + \mathbf{3}$ or $\mathbf{3} + \mathbf{2}$, for example).

UMASS AMHERST Instructor's Notes

Understanding vector subtraction is necessary to understand other physics ideas. For example, acceleration is Δv/Δt, and Δv is v_f - v_i. Velocity is a vector, so you're looking at a vector subtraction whenever you're working with acceleration.

Vector Subtraction

Vector subtraction is a straightforward extension of vector addition. To define subtraction (say we want to subtract $\mathbf{B}$ from $\mathbf{A}$, written $\mathbf{A} - \mathbf{B}$, we must first define what we mean by subtraction. The *negative* of a vector $\mathbf{B}$ is defined to be $-\mathbf{B}$; that is, graphically *the negative of any vector has the same magnitude but the opposite direction*, as shown in **Figure 3.20**. In other words, $\mathbf{B}$ has the same length as $-\mathbf{B}$, but points in the opposite direction. Essentially, we just flip the vector so it points in the opposite direction.

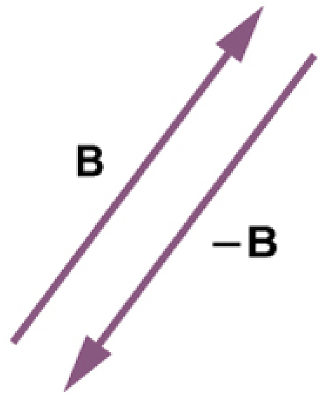


Figure 3.20 The negative of a vector is just another vector of the same magnitude but pointing in the opposite direction. So $\mathbf{B}$ is the negative of $-\mathbf{B}$; it has the same length but opposite direction.

The *subtraction* of vector $\mathbf{B}$ from vector $\mathbf{A}$ is then simply defined to be the addition of $-\mathbf{B}$ to $\mathbf{A}$. Note that vector subtraction is the addition of a negative vector. The order of subtraction does not affect the results.

$$\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}). \tag{3.2}$$

This is analogous to the subtraction of scalars (where, for example, $5 - 2 = 5 + (-2)$). Again, the result is independent of the order in which the subtraction is made. When vectors are subtracted graphically, the techniques outlined above are used, as the following example illustrates.

Example 3.2 Subtracting Vectors Graphically: A Woman Sailing a Boat

A woman sailing a boat at night is following directions to a dock. The instructions read to first sail 27.5 m in a direction 66.0° north of east from her current location, and then travel 30.0 m in a direction 112° north of east (or 22.0° west of north). If the woman makes a mistake and travels in the *opposite* direction for the second leg of the trip, where will she end up? Compare this location with the location of the dock.

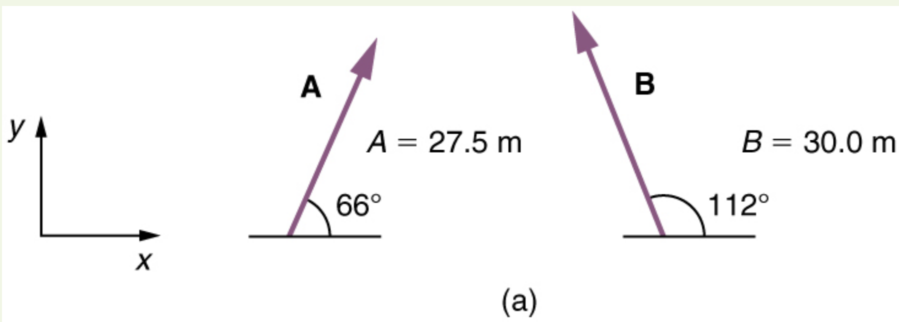


Figure 3.21

Strategy

We can represent the first leg of the trip with a vector $\mathbf{A}$, and the second leg of the trip with a vector $\mathbf{B}$. The dock is located at a location $\mathbf{A} + \mathbf{B}$. If the woman mistakenly travels in the *opposite* direction for the second leg of the journey, she will travel a distance B (30.0 m) in the direction $180^\circ - 112^\circ = 68^\circ$ south of east. We represent this as $-\mathbf{B}$, as shown below. The vector $-\mathbf{B}$ has the same magnitude as $\mathbf{B}$ but is in the opposite direction. Thus, she will end up at a location $\mathbf{A} + (-\mathbf{B})$, or $\mathbf{A} - \mathbf{B}$.

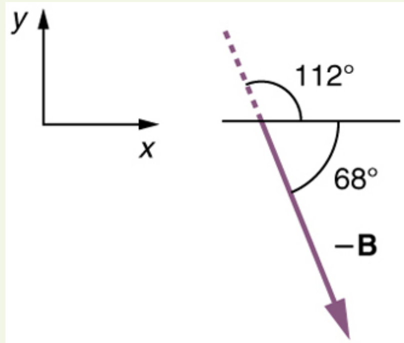


Figure 3.22

We will perform vector addition to compare the location of the dock, $\mathbf{A} + \mathbf{B}$, with the location at which the woman mistakenly arrives, $\mathbf{A} + (-\mathbf{B})$.

Solution

(1) To determine the location at which the woman arrives by accident, draw vectors $\mathbf{A}$ and $-\mathbf{B}$.

(2) Place the vectors head to tail.

(3) Draw the resultant vector $\mathbf{R}$.

(4) Use a ruler and protractor to measure the magnitude and direction of $\mathbf{R}$.

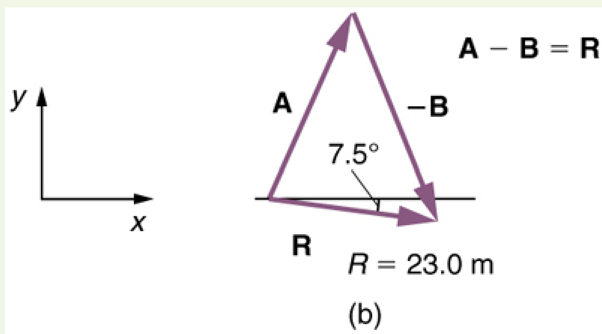


Figure 3.23

In this case, $R = 23.0 \text{ m}$ and $\theta = 7.5^\circ$ south of east.

(5) To determine the location of the dock, we repeat this method to add vectors $\mathbf{A}$ and $\mathbf{B}$. We obtain the resultant vector $\mathbf{R}'$:

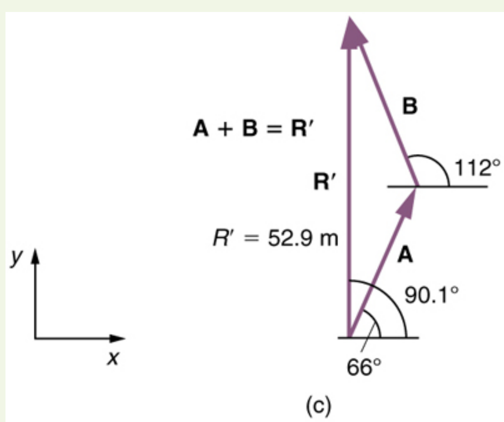


Figure 3.24

In this case $R = 52.9 \text{ m}$ and $\theta = 90.1^\circ$ north of east.

We can see that the woman will end up a significant distance from the dock if she travels in the opposite direction for the second leg of the trip.

Discussion

Because subtraction of a vector is the same as addition of a vector with the opposite direction, the graphical method of subtracting vectors works the same as for addition.

UMASS AMHERST Instructor's Notes

If you've taken another physics course, you've probably seen **F** = *m***a**. This equation will play a significant role in this class, and you'll notice that mass is a scalar, and acceleration is a vector, so understanding how scalars and vectors multiply will be important.

Multiplication of Vectors and Scalars

If we decided to walk three times as far on the first leg of the trip considered in the preceding example, then we would walk $3 \times 27.5 \text{ m}$, or 82.5 m, in a direction 66.0° north of east. This is an example of multiplying a vector by a positive **scalar**. Notice that the magnitude changes, but the direction stays the same.

If the scalar is negative, then multiplying a vector by it changes the vector's magnitude and gives the new vector the *opposite* direction. For example, if you multiply by –2, the magnitude doubles but the direction changes. We can summarize these rules in the following way: When vector $\mathbf{A}$ is multiplied by a scalar c,

- the magnitude of the vector becomes the absolute value of $c\ A$,
- if c is positive, the direction of the vector does not change,
- if c is negative, the direction is reversed.

In our case, $c = 3$ and $A = 27.5 \text{ m}$. Vectors are multiplied by scalars in many situations. Note that division is the inverse of multiplication. For example, dividing by 2 is the same as multiplying by the value (1/2). The rules for multiplication of vectors by scalars are the same for division; simply treat the divisor as a scalar between 0 and 1.

UMASS AMHERST Instructor's Notes

When dealing with vectors using analytic methods (which is covered in the next section), you need to break down vectors into essentially x-components and y-components. This next part covers this idea, so try to familiarize yourself with breaking down vectors as you read.

Resolving a Vector into Components

In the examples above, we have been adding vectors to determine the resultant vector. In many cases, however, we will need to do the opposite. We will need to take a single vector and find what other vectors added together produce it. In most cases, this involves determining the perpendicular **components** of a single vector, for example the *x- and y-*components, or the north-south and east-west components.

For example, we may know that the total displacement of a person walking in a city is 10.3 blocks in a direction 29.0° north of east and want to find out how many blocks east and north had to be walked. This method is called *finding the components (or parts)* of the displacement in the east and north directions, and it is the inverse of the process followed to find the total displacement. It is one example of finding the components of a vector. There are many applications in physics where this is a useful thing to do. We will see this soon in **Projectile Motion (https://legacy.cnx.org/content/m42042/latest/)** , and much more when we cover **forces** in **Dynamics: Newton's Laws of Motion (https://legacy.cnx.org/content/m42129/latest/)** . Most of these involve finding components along perpendicular axes (such as north and east), so that right triangles are involved. The analytical techniques presented in **Vector Addition and Subtraction: Analytical Methods (https://legacy.cnx.org/content/m42128/latest/)** are ideal for finding vector components.

PhET Explorations: Maze Game

Learn about position, velocity, and acceleration in the "Arena of Pain". Use the green arrow to move the ball. Add more walls to the arena to make the game more difficult. Try to make a goal as fast as you can.

PhET Interactive Simulation

Figure 3.25 Maze Game (http://legacy.cnx.org/content/m64160/1.2/maze-game_en.jar)

3.3 Vector Addition and Subtraction-Analytical Methods

UMASS AMHERST Instructor's Notes

Your Quiz would Cover

- Adding vectors by components. Don't focus too much on what it means to add vectors. Just learn the mechanics of how to do it. We will talk about the meaning in class.

Analytical methods of vector addition and subtraction employ geometry and simple trigonometry rather than the ruler and protractor of graphical methods. Part of the graphical technique is retained, because vectors are still represented by arrows for easy visualization. However, analytical methods are more concise, accurate, and precise than graphical methods, which are limited by the accuracy with which a drawing can be made. Analytical methods are limited only by the accuracy and precision with which physical quantities are known.

Resolving a Vector into Perpendicular Components

Analytical techniques and right triangles go hand-in-hand in physics because (among other things) motions along perpendicular directions are independent. We very often need to separate a vector into perpendicular components. For example, given a vector like $\mathbf{A}$ in Figure 3.26, we may wish to find which two perpendicular vectors, $\mathbf{A}_x$ and $\mathbf{A}_y$, add to produce it.

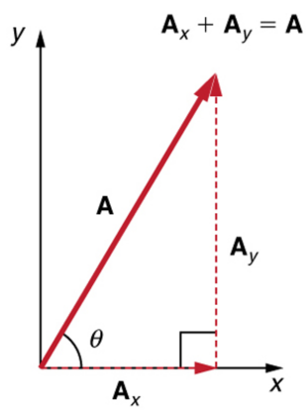


Figure 3.26 The vector $\mathbf{A}$, with its tail at the origin of an *x*, *y*-coordinate system, is shown together with its *x*- and *y*-components, $\mathbf{A}_x$ and $\mathbf{A}_y$. These vectors form a right triangle. The analytical relationships among these vectors are summarized below.

$\mathbf{A}_x$ and $\mathbf{A}_y$ are defined to be the components of $\mathbf{A}$ along the *x*- and *y*-axes. The three vectors $\mathbf{A}$, $\mathbf{A}_x$, and $\mathbf{A}_y$ form a right triangle:

$$\mathbf{A}_x + \mathbf{A_y} = \mathbf{A}. \tag{3.3}$$

Note that this relationship between vector components and the resultant vector holds only for vector quantities (which include both magnitude and direction). The relationship does not apply for the magnitudes alone. For example, if $\mathbf{A}_x = 3\text{ m}$ east, $\mathbf{A}_y = 4\text{ m}$ north, and $\mathbf{A} = 5\text{ m}$ north-east, then it is true that the vectors $\mathbf{A}_x + \mathbf{A_y} = \mathbf{A}$. However, it is *not* true that the sum of the magnitudes of the vectors is also equal. That is,

$$3\text{ m} + 4\text{ m} \neq 5\text{ m} \tag{3.4}$$

Thus,

$$A_x + A_y \neq A \tag{3.5}$$

If the vector $\mathbf{A}$ is known, then its magnitude A (its length) and its angle θ (its direction) are known. To find A_x and A_y, its *x*- and *y*-components, we use the following relationships for a right triangle.

$$A_x = A\cos\theta \tag{3.6}$$

and

$$A_y = A\sin\theta. \tag{3.7}$$

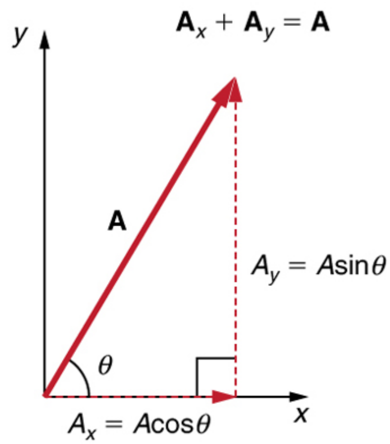


Figure 3.27 The magnitudes of the vector components $\mathbf{A}_x$ and $\mathbf{A}_y$ can be related to the resultant vector $\mathbf{A}$ and the angle θ with trigonometric identities. Here we see that $A_x = A\cos\theta$ and $A_y = A\sin\theta$.

Suppose, for example, that $\mathbf{A}$ is the vector representing the total displacement of the person walking in a city considered in **Kinematics in Two Dimensions: An Introduction (https://legacy.cnx.org/content/m42104/latest/)** and **Vector Addition and Subtraction: Graphical Methods (https://legacy.cnx.org/content/m42127/latest/)**.

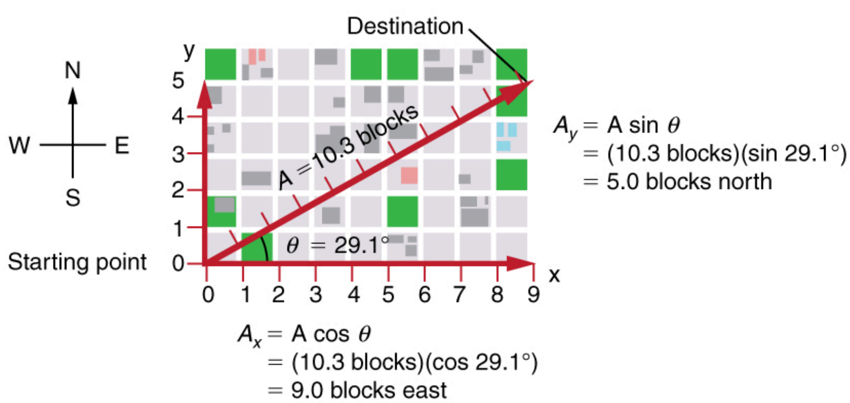


Figure 3.28 We can use the relationships $A_x = A\cos\theta$ and $A_y = A\sin\theta$ to determine the magnitude of the horizontal and vertical component vectors in this example.

Then $A = 10.3$ blocks and $\theta = 29.1^\circ$, so that

$$A_x = A\cos\theta = (10.3\ \text{blocks})(\cos 29.1^\circ) = 9.0\ \text{blocks} \tag{3.8}$$

$$A_y = A\sin\theta = (10.3\ \text{blocks})(\sin 29.1^\circ) = 5.0\ \text{blocks}. \tag{3.9}$$

Calculating a Resultant Vector

If the perpendicular components $\mathbf{A}_x$ and $\mathbf{A}_y$ of a vector $\mathbf{A}$ are known, then $\mathbf{A}$ can also be found analytically. To find the magnitude A and direction θ of a vector from its perpendicular components $\mathbf{A}_x$ and $\mathbf{A}_y$, we use the following relationships:

$$A = \sqrt{A_x{}^2 + A_y{}^2} \tag{3.10}$$

$$\theta = \tan^{-1}(A_y / A_x). \tag{3.11}$$

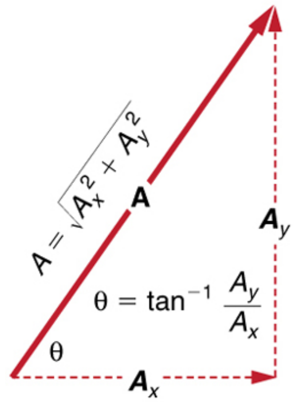


Figure 3.29 The magnitude and direction of the resultant vector can be determined once the horizontal and vertical components A_x and A_y have been determined.

Note that the equation $A = \sqrt{A_x^2 + A_y^2}$ is just the Pythagorean theorem relating the legs of a right triangle to the length of the hypotenuse. For example, if A_x and A_y are 9 and 5 blocks, respectively, then $A = \sqrt{9^2+5^2}=10.3$ blocks, again consistent with the example of the person walking in a city. Finally, the direction is $\theta = \tan^{-1}(5/9)=29.1^\circ$, as before.

Determining Vectors and Vector Components with Analytical Methods

Equations $A_x = A \cos \theta$ and $A_y = A \sin \theta$ are used to find the perpendicular components of a vector—that is, to go from A and θ to A_x and A_y. Equations $A = \sqrt{A_x^2 + A_y^2}$ and $\theta = \tan^{-1}(A_y / A_x)$ are used to find a vector from its perpendicular components—that is, to go from A_x and A_y to A and θ. Both processes are crucial to analytical methods of vector addition and subtraction.

UMASS AMHERST Instructor's Notes

Now that you know how to break down vectors into components, here's a procedure to adding vectors analytically. There's some trigonometry involved, so, again, if you're not familiar or comfortable with trigonometry, come see your instructor. You should be familiar with both methods. you should be able to add two vectors given their x and y components, and you should be able to draw the resulting vector of two added vectors. Also, we will go over how to use these to solve problems, so focus primarily on the methods of adding vectors.

Adding Vectors Using Analytical Methods

To see how to add vectors using perpendicular components, consider **Figure 3.30**, in which the vectors $\mathbf{A}$ and $\mathbf{B}$ are added to produce the resultant $\mathbf{R}$.

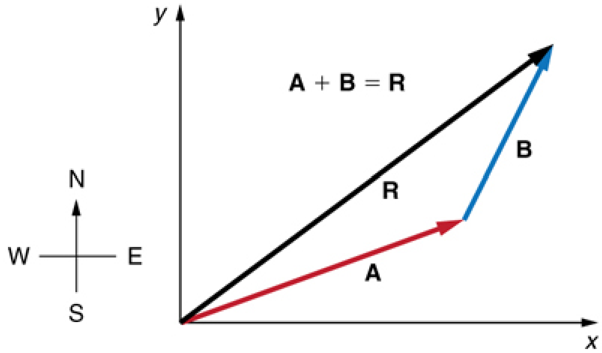


Figure 3.30 Vectors $\mathbf{A}$ and $\mathbf{B}$ are two legs of a walk, and $\mathbf{R}$ is the resultant or total displacement. You can use analytical methods to determine the magnitude and direction of $\mathbf{R}$.

If $\mathbf{A}$ and $\mathbf{B}$ represent two legs of a walk (two displacements), then $\mathbf{R}$ is the total displacement. The person taking the walk ends up at the tip of $\mathbf{R}.$ There are many ways to arrive at the same point. In particular, the person could have walked first in the *x*-direction and then in the *y*-direction. Those paths are the *x*- and *y*-components of the resultant, $\mathbf{R}_x$ and $\mathbf{R}_y$. If we know $\mathbf{R}_x$ and $\mathbf{R}_y$, we can find R and θ using the equations $A = \sqrt{A_x{}^2 + A_y{}^2}$ and $\theta = \tan^{-1}(A_y / A_x)$. When you use the analytical method of vector addition, you can determine the components or the magnitude and direction of a vector.

Step 1. *Identify the x- and y-axes that will be used in the problem. Then, find the components of each vector to be added along the chosen perpendicular axes.* Use the equations $A_x = A \cos \theta$ and $A_y = A \sin \theta$ to find the components. In **Figure 3.31**, these components are A_x , A_y , B_x , and B_y . The angles that vectors $\mathbf{A}$ and $\mathbf{B}$ make with the *x*-axis are θ_{A} and θ_{B} , respectively.

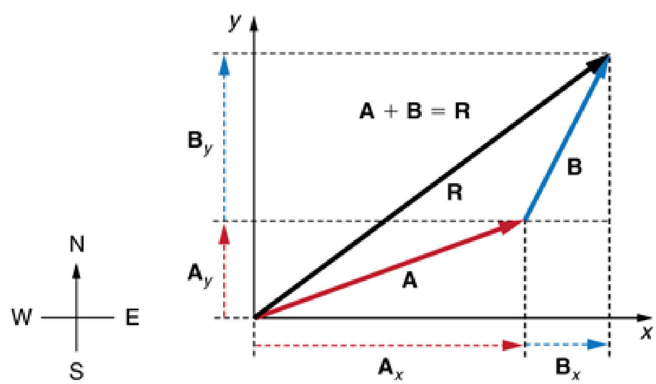


Figure 3.31 To add vectors $\mathbf{A}$ and $\mathbf{B}$, first determine the horizontal and vertical components of each vector. These are the dotted vectors $\mathbf{A}_x$, $\mathbf{A}_y$, $\mathbf{B}_x$ and $\mathbf{B}_y$ shown in the image.

Step 2. *Find the components of the resultant along each axis by adding the components of the individual vectors along that axis.* That is, as shown in **Figure 3.32**,

$$R_x = A_x + B_x \tag{3.12}$$

and

$$R_y = A_y + B_y. \tag{3.13}$$

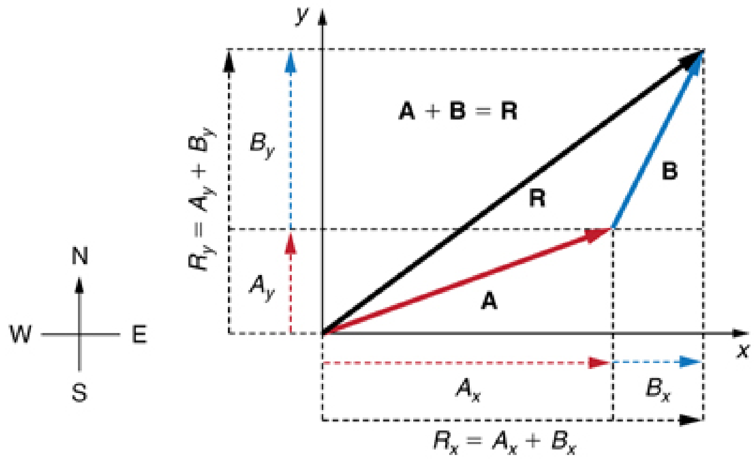


Figure 3.32 The magnitude of the vectors $\mathbf{A}_x$ and $\mathbf{B}_x$ add to give the magnitude R_x of the resultant vector in the horizontal direction. Similarly, the magnitudes of the vectors $\mathbf{A}_y$ and $\mathbf{B}_y$ add to give the magnitude R_y of the resultant vector in the vertical direction.

Components along the same axis, say the *x*-axis, are vectors along the same line and, thus, can be added to one another like ordinary numbers. The same is true for components along the *y*-axis. (For example, a 9-block eastward walk could be taken in two legs, the first 3 blocks east and the second 6 blocks east, for a total of 9, because they are along the same direction.) So resolving vectors into components along common axes makes it easier to add them. Now that the components of $\mathbf{R}$ are known, its magnitude and direction can be found.

Step 3. *To get the magnitude* R *of the resultant, use the Pythagorean theorem:*

$$R = \sqrt{R_x^2 + R_y^2}. \tag{3.14}$$

Step 4. *To get the direction of the resultant:*

$$\theta = \tan^{-1}(R_y / R_x). \tag{3.15}$$

The following example illustrates this technique for adding vectors using perpendicular components.

Example 3.3 Adding Vectors Using Analytical Methods

Add the vector $\mathbf{A}$ to the vector $\mathbf{B}$ shown in **Figure 3.33**, using perpendicular components along the *x*- and *y*-axes. The *x*- and *y*-axes are along the east–west and north–south directions, respectively. Vector $\mathbf{A}$ represents the first leg of a walk in which a person walks 53.0 m in a direction 20.0^o north of east. Vector $\mathbf{B}$ represents the second leg, a displacement of 34.0 m in a direction 63.0^o north of east.

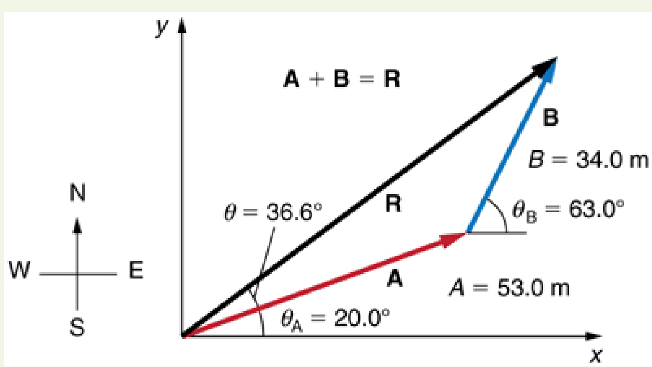


Figure 3.33 Vector $\mathbf{A}$ has magnitude 53.0 m and direction $20.0\,^\text{o}$ north of the *x*-axis. Vector $\mathbf{B}$ has magnitude 34.0 m and direction 63.0^o north of the *x*-axis. You can use analytical methods to determine the magnitude and direction of $\mathbf{R}$.

Strategy

The components of $\mathbf{A}$ and $\mathbf{B}$ along the *x*- and *y*-axes represent walking due east and due north to get to the same ending point. Once found, they are combined to produce the resultant.

Solution

Following the method outlined above, we first find the components of $\mathbf{A}$ and $\mathbf{B}$ along the *x*- and *y*-axes. Note that $A = 53.0 \text{ m}$, $\theta_\text{A} = 20.0^\circ$, $B = 34.0 \text{ m}$, and $\theta_\text{B} = 63.0^\circ$. We find the *x*-components by using $A_x = A \cos \theta$, which gives

$$\begin{aligned} A_x &= A \cos \theta_\text{A} = (53.0 \text{ m})(\cos 20.0^\circ) \\ &= (53.0 \text{ m})(0.940) = 49.8 \text{ m} \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} B_x &= B \cos \theta_\text{B} = (34.0 \text{ m})(\cos 63.0^\circ) \\ &= (34.0 \text{ m})(0.454) = 15.4 \text{ m}. \end{aligned} \tag{3.17}$$

Similarly, the *y*-components are found using $A_y = A \sin \theta_\text{A}$:

$$\begin{aligned} A_y &= A \sin \theta_\text{A} = (53.0 \text{ m})(\sin 20.0^\circ) \\ &= (53.0 \text{ m})(0.342) = 18.1 \text{ m} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} B_y &= B \sin \theta_\text{B} = (34.0 \text{ m})(\sin 63.0^\circ) \\ &= (34.0 \text{ m})(0.891) = 30.3 \text{ m}. \end{aligned} \tag{3.19}$$

The *x*- and *y*-components of the resultant are thus

$$R_x = A_x + B_x = 49.8 \text{ m} + 15.4 \text{ m} = 65.2 \text{ m} \tag{3.20}$$

and

$$R_y = A_y + B_y = 18.1 \text{ m} + 30.3 \text{ m} = 48.4 \text{ m}. \tag{3.21}$$

Now we can find the magnitude of the resultant by using the Pythagorean theorem:

$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(65.2)^2 + (48.4)^2} \text{ m} \tag{3.22}$$

so that

$$R = 81.2 \text{ m}. \tag{3.23}$$

Finally, we find the direction of the resultant:

$$\theta = \tan^{-1}(R_y / R_x) = +\tan^{-1}(48.4 / 65.2). \tag{3.24}$$

Thus,

$$\theta = \tan^{-1}(0.742) = 36.6^\circ. \tag{3.25}$$

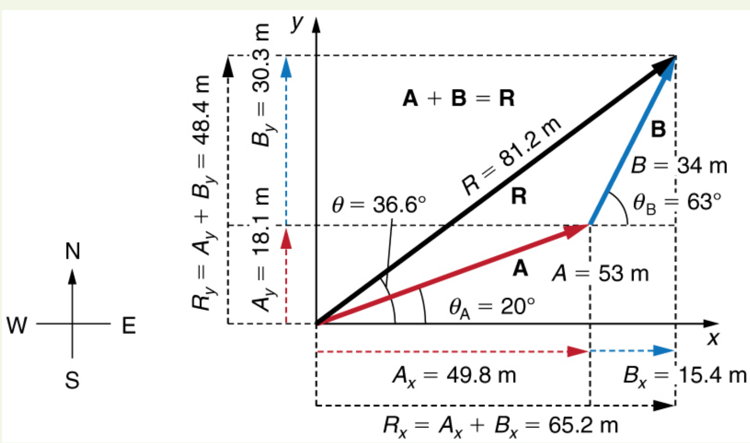


Figure 3.34 Using analytical methods, we see that the magnitude of $\mathbf{R}$ is 81.2 m and its direction is 36.6° north of east.

Discussion

This example illustrates the addition of vectors using perpendicular components. Vector subtraction using perpendicular components is very similar—it is just the addition of a negative vector.

Subtraction of vectors is accomplished by the addition of a negative vector. That is, $\mathbf{A} - \mathbf{B} \equiv \mathbf{A} + (-\mathbf{B})$. Thus, *the method for the subtraction of vectors using perpendicular components is identical to that for addition*. The components of $-\mathbf{B}$ are the negatives of the components of $\mathbf{B}$. The *x*- and *y*-components of the resultant $\mathbf{A} - \mathbf{B} = \mathbf{R}$ are thus

$$R_x = A_x + (-B_x) \tag{3.26}$$

and

$$R_y = A_y + \left(-B_y\right) \tag{3.27}$$

and the rest of the method outlined above is identical to that for addition. (See **Figure 3.35**.)

Analyzing vectors using perpendicular components is very useful in many areas of physics, because perpendicular quantities are often independent of one another. The next module, **Projectile Motion (https://legacy.cnx.org/content/m42042/latest/)** , is one of many in which using perpendicular components helps make the picture clear and simplifies the physics.

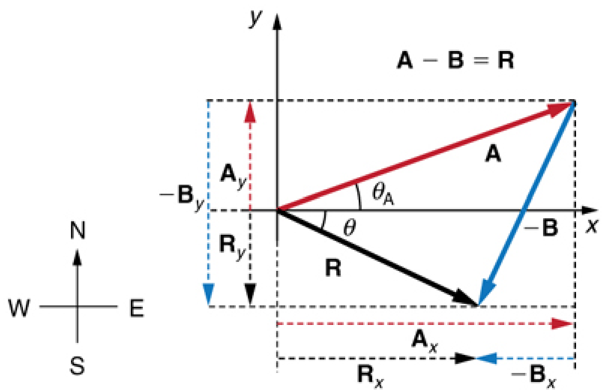


Figure 3.35 The subtraction of the two vectors shown in **Figure 3.30**. The components of $-\mathbf{B}$ are the negatives of the components of $\mathbf{B}$. The method of subtraction is the same as that for addition.

PhET Explorations: Vector Addition

Learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats.

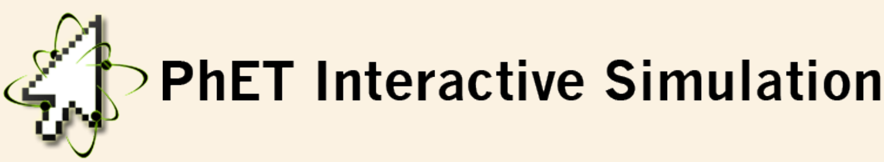


Figure 3.36 Vector Addition (http://legacy.cnx.org/content/m64164/1.3/vector-addition_en.jar)

3.4 Addition of Velocities

UMASS AMHERST Instructor's Notes

We will not being discussing this section in detail. This section is here primarily for your reference.

Relative Velocity

If a person rows a boat across a rapidly flowing river and tries to head directly for the other shore, the boat instead moves *diagonally* relative to the shore, as in **Figure 3.37**. The boat does not move in the direction in which it is pointed. The reason, of course, is that the river carries the boat downstream. Similarly, if a small airplane flies overhead in a strong crosswind, you can sometimes see that the plane is not moving in the direction in which it is pointed, as illustrated in **Figure 3.38**. The plane is moving straight ahead relative to the air, but the movement of the air mass relative to the ground carries it sideways.

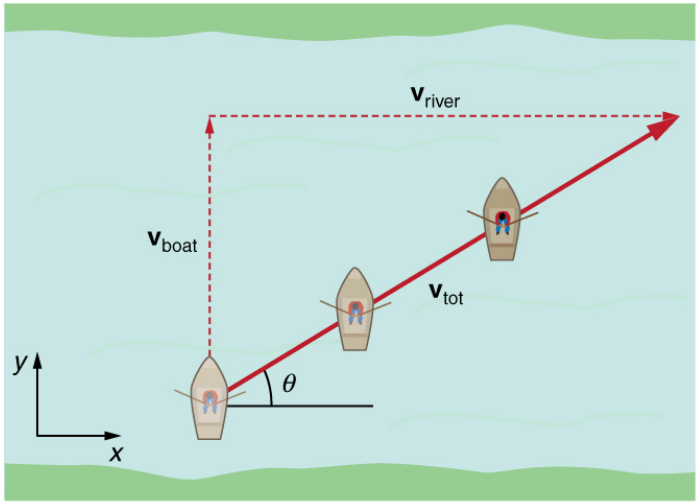


Figure 3.37 A boat trying to head straight across a river will actually move diagonally relative to the shore as shown. Its total velocity (solid arrow) relative to the shore is the sum of its velocity relative to the river plus the velocity of the river relative to the shore.

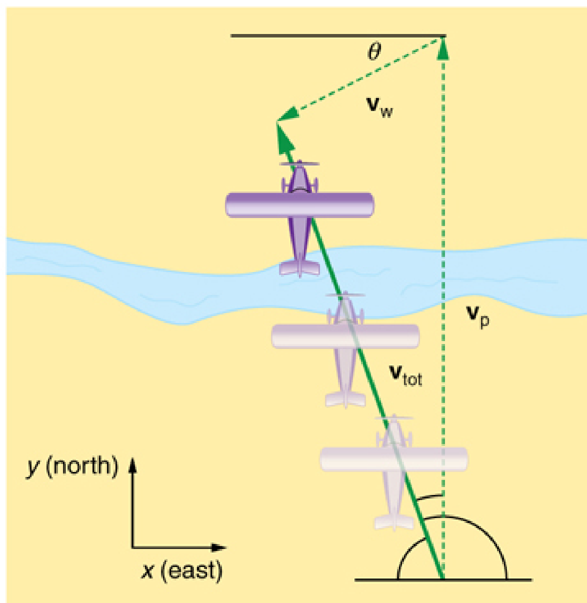


Figure 3.38 An airplane heading straight north is instead carried to the west and slowed down by wind. The plane does not move relative to the ground in the direction it points; rather, it moves in the direction of its total velocity (solid arrow).

In each of these situations, an object has a **velocity** relative to a medium (such as a river) and that medium has a velocity relative to an observer on solid ground. The velocity of the object *relative to the observer* is the sum of these velocity vectors, as indicated in Figure 3.37 and Figure 3.38. These situations are only two of many in which it is useful to add velocities. In this module, we first re-examine how to add velocities and then consider certain aspects of what relative velocity means.

How do we add velocities? Velocity is a vector (it has both magnitude and direction); the rules of **vector addition** discussed in Vector Addition and Subtraction: Graphical Methods (https://legacy.cnx.org/content/m42127/latest/) and Vector Addition and Subtraction: Analytical Methods (https://legacy.cnx.org/content/m42128/latest/) apply to the addition of velocities, just as they do for any other vectors. In one-dimensional motion, the addition of velocities is simple—they add like ordinary numbers. For example, if a field hockey player is moving at 5 m/s straight toward the goal and drives the ball in the same direction with a velocity of 30 m/s relative to her body, then the velocity of the ball is 35 m/s relative to the stationary, profusely sweating goalkeeper standing in front of the goal.

In two-dimensional motion, either graphical or analytical techniques can be used to add velocities. We will concentrate on analytical techniques. The following equations give the relationships between the magnitude and direction of velocity (v and θ) and its components (v_x and v_y) along the *x*- and *y*-axes of an appropriately chosen coordinate system:

$$v_x = v\cos\theta \tag{3.28}$$

$$v_y = v\sin\theta \tag{3.29}$$

$$v = \sqrt{v_x^2 + v_y^2} \tag{3.30}$$

$$\theta = \tan^{-1}(v_y / v_x). \tag{3.31}$$

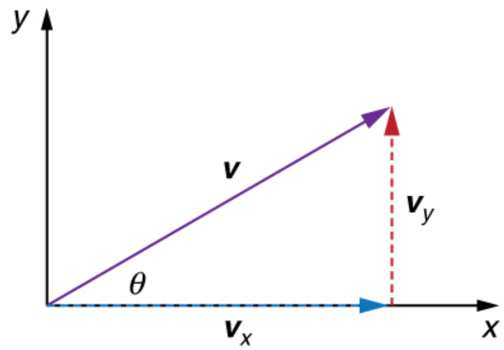


Figure 3.39 The velocity, v, of an object traveling at an angle θ to the horizontal axis is the sum of component vectors $\mathbf{v}_x$ and $\mathbf{v}_y$.

These equations are valid for any vectors and are adapted specifically for velocity. The first two equations are used to find the components of a velocity when its magnitude and direction are known. The last two are used to find the magnitude and direction of velocity when its components are known.

Take-Home Experiment: Relative Velocity of a Boat

Fill a bathtub half-full of water. Take a toy boat or some other object that floats in water. Unplug the drain so water starts to drain. Try pushing the boat from one side of the tub to the other and perpendicular to the flow of water. Which way do you need to push the boat so that it ends up immediately opposite? Compare the directions of the flow of water, heading of the boat, and actual velocity of the boat.

Example 3.4 Adding Velocities: A Boat on a River

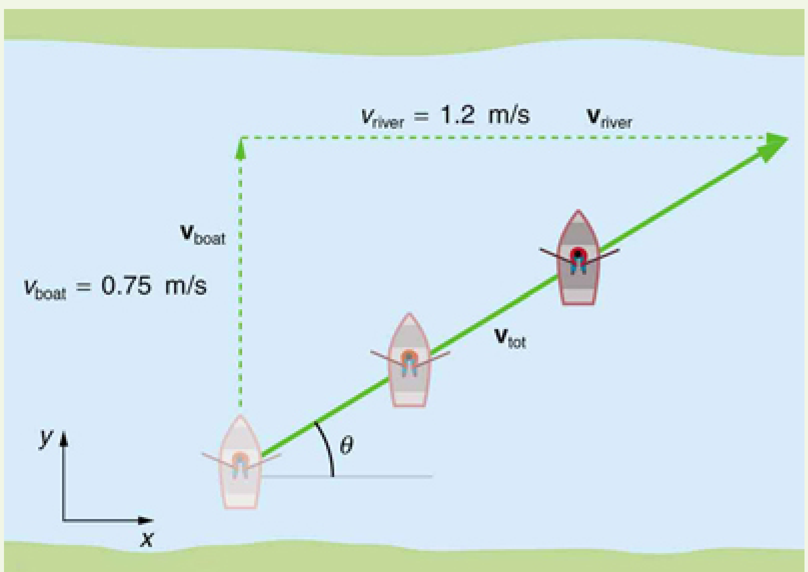


Figure 3.40 A boat attempts to travel straight across a river at a speed 0.75 m/s. The current in the river, however, flows at a speed of 1.20 m/s to the right. What is the total displacement of the boat relative to the shore?

Refer to **Figure 3.40**, which shows a boat trying to go straight across the river. Let us calculate the magnitude and direction of the boat's velocity relative to an observer on the shore, $\mathbf{v}_{\text{tot}}$. The velocity of the boat, $\mathbf{v}_{\text{boat}}$, is 0.75 m/s in the y-direction relative to the river and the velocity of the river, $\mathbf{v}_{\text{river}}$, is 1.20 m/s to the right.

Strategy

We start by choosing a coordinate system with its x-axis parallel to the velocity of the river, as shown in **Figure 3.40**. Because the boat is directed straight toward the other shore, its velocity relative to the water is parallel to the y-axis and perpendicular to the velocity of the river. Thus, we can add the two velocities by using the equations $v_{\text{tot}} = \sqrt{v_x^2 + v_y^2}$ and $\theta = \tan^{-1}(v_y / v_x)$ directly.

Solution

The magnitude of the total velocity is

$$v_{\text{tot}} = \sqrt{v_x^2 + v_y^2}, \tag{3.32}$$

where

$$v_x = v_{\text{river}} = 1.20 \text{ m/s} \tag{3.33}$$

and

$$v_y = v_{\text{boat}} = 0.750 \text{ m/s}. \tag{3.34}$$

Thus,

$$v_{\text{tot}} = \sqrt{(1.20 \text{ m/s})^2 + (0.750 \text{ m/s})^2} \tag{3.35}$$

yielding

$$v_{\text{tot}} = 1.42 \text{ m/s}. \tag{3.36}$$

The direction of the total velocity θ is given by:

$$\theta = \tan^{-1}(v_y / v_x) = \tan^{-1}(0.750 / 1.20). \tag{3.37}$$

This equation gives

$$\theta = 32.0^{\circ}. \tag{3.38}$$

Discussion

Both the magnitude v and the direction θ of the total velocity are consistent with **Figure 3.40**. Note that because the velocity of the river is large compared with the velocity of the boat, it is swept rapidly downstream. This result is evidenced by the small angle (only 32.0°) the total velocity has relative to the riverbank.

Example 3.5 Calculating Velocity: Wind Velocity Causes an Airplane to Drift

Calculate the wind velocity for the situation shown in **Figure 3.41**. The plane is known to be moving at 45.0 m/s due north relative to the air mass, while its velocity relative to the ground (its total velocity) is 38.0 m/s in a direction 20.0° west of north.

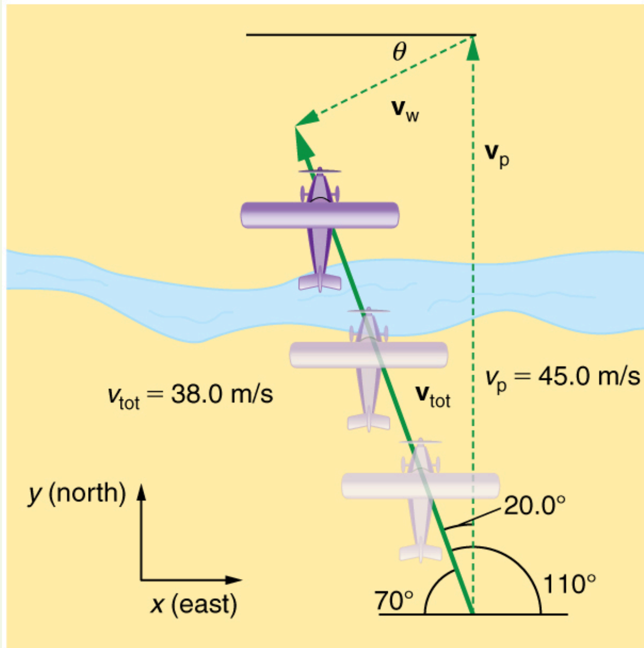


Figure 3.41 An airplane is known to be heading north at 45.0 m/s, though its velocity relative to the ground is 38.0 m/s at an angle west of north. What is the speed and direction of the wind?

Strategy

In this problem, somewhat different from the previous example, we know the total velocity $\mathbf{v}_{\text{tot}}$ and that it is the sum of two other velocities, $\mathbf{v}_{\text{w}}$ (the wind) and $\mathbf{v}_{\text{p}}$ (the plane relative to the air mass). The quantity $\mathbf{v}_{\text{p}}$ is known, and we are asked to find $\mathbf{v}_{\text{w}}$. None of the velocities are perpendicular, but it is possible to find their components along a common set of perpendicular axes. If we can find the components of $\mathbf{v}_{\text{w}}$, then we can combine them to solve for its magnitude and direction. As shown in Figure 3.41, we choose a coordinate system with its *x*-axis due east and its *y*-axis due north (parallel to $\mathbf{v}_{\text{p}}$). (You may wish to look back at the discussion of the addition of vectors using perpendicular components in **Vector Addition and Subtraction: Analytical Methods (https://legacy.cnx.org/content/m42128/latest/)** .)

Solution

Because $\mathbf{v}_{\text{tot}}$ is the vector sum of the $\mathbf{v}_{\text{w}}$ and $\mathbf{v}_{\text{p}}$, its *x*- and *y*-components are the sums of the *x*- and *y*-components of the wind and plane velocities. Note that the plane only has vertical component of velocity so $v_{\text{p}x} = 0$ and $v_{\text{p}y} = v_{\text{p}}$. That is,

$$v_{\text{tot}x} = v_{\text{w}x} \tag{3.39}$$

and

$$v_{\text{tot}y} = v_{\text{w}y} + v_{\text{p}}. \tag{3.40}$$

We can use the first of these two equations to find $v_{\text{w}x}$:

$$v_{\text{w}y} = v_{\text{tot}x} = v_{\text{tot}}\cos 110^{\circ}. \tag{3.41}$$

Because $v_{\text{tot}} = 38.0\ \text{m/s}$ and $\cos 110^{\circ} = -0.342$ we have

$$v_{\text{w}y} = (38.0\ \text{m/s})(–0.342) = –13\ \text{m/s}. \tag{3.42}$$

The minus sign indicates motion west which is consistent with the diagram.

Now, to find $v_{\text{w}y}$ we note that

$$v_{\text{tot}y} = v_{\text{w}y} + v_{\text{p}} \tag{3.43}$$

Here $v_{\text{tot}y} = v_{\text{tot}}\sin 110^{\circ}$; thus,

$$v_{\text{w}y} = (38.0\ \text{m/s})(0.940) - 45.0\ \text{m/s} = -9.29\ \text{m/s}. \tag{3.44}$$

This minus sign indicates motion south which is consistent with the diagram.

Now that the perpendicular components of the wind velocity v_{wx} and v_{wy} are known, we can find the magnitude and direction of $\mathbf{v}_w$. First, the magnitude is

$$\begin{aligned} v_w &= \sqrt{v_{wx}^2 + v_{wy}^2} \\ &= \sqrt{(-13.0 \text{ m/s})^2 + (-9.29 \text{ m/s})^2} \end{aligned} \tag{3.45}$$

so that

$$v_w = 16.0 \text{ m/s}. \tag{3.46}$$

The direction is:

$$\theta = \tan^{-1}(v_{wy} / v_{wx}) = \tan^{-1}(-9.29 / -13.0) \tag{3.47}$$

giving

$$\theta = 35.6^{\mathrm{o}}. \tag{3.48}$$

Discussion

The wind's speed and direction are consistent with the significant effect the wind has on the total velocity of the plane, as seen in **Figure 3.41**. Because the plane is fighting a strong combination of crosswind and head-wind, it ends up with a total velocity significantly less than its velocity relative to the air mass as well as heading in a different direction.

Note that in both of the last two examples, we were able to make the mathematics easier by choosing a coordinate system with one axis parallel to one of the velocities. We will repeatedly find that choosing an appropriate coordinate system makes problem solving easier. For example, in projectile motion we always use a coordinate system with one axis parallel to gravity.

Relative Velocities and Classical Relativity

When adding velocities, we have been careful to specify that the *velocity is relative to some reference frame*. These velocities are called **relative velocities**. For example, the velocity of an airplane relative to an air mass is different from its velocity relative to the ground. Both are quite different from the velocity of an airplane relative to its passengers (which should be close to zero). Relative velocities are one aspect of **relativity**, which is defined to be the study of how different observers moving relative to each other measure the same phenomenon.

Nearly everyone has heard of relativity and immediately associates it with Albert Einstein (1879–1955), the greatest physicist of the 20th century. Einstein revolutionized our view of nature with his *modern* theory of relativity, which we shall study in later chapters. The relative velocities in this section are actually aspects of classical relativity, first discussed correctly by Galileo and Isaac Newton. **Classical relativity** is limited to situations where speeds are less than about 1% of the speed of light—that is, less than $3,000 \text{ km/s}$. Most things we encounter in daily life move slower than this speed.

Let us consider an example of what two different observers see in a situation analyzed long ago by Galileo. Suppose a sailor at the top of a mast on a moving ship drops his binoculars. Where will it hit the deck? Will it hit at the base of the mast, or will it hit behind the mast because the ship is moving forward? The answer is that if air resistance is negligible, the binoculars will hit at the base of the mast at a point directly below its point of release. Now let us consider what two different observers see when the binoculars drop. One observer is on the ship and the other on shore. The binoculars have no horizontal velocity relative to the observer on the ship, and so he sees them fall straight down the mast. (See **Figure 3.42**.) To the observer on shore, the binoculars and the ship have the *same* horizontal velocity, so both move the same distance forward while the binoculars are falling. This observer sees the curved path shown in **Figure 3.42**. Although the paths look different to the different observers, each sees the same result—the binoculars hit at the base of the mast and not behind it. To get the correct description, it is crucial to correctly specify the velocities relative to the observer.

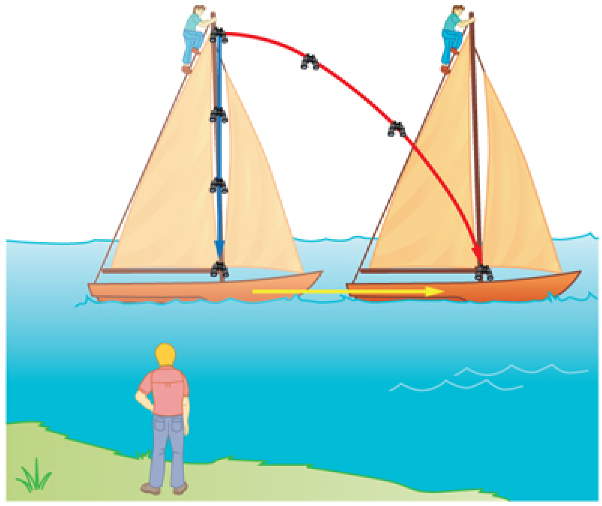


Figure 3.42 Classical relativity. The same motion as viewed by two different observers. An observer on the moving ship sees the binoculars dropped from the top of its mast fall straight down. An observer on shore sees the binoculars take the curved path, moving forward with the ship. Both observers see the binoculars strike the deck at the base of the mast. The initial horizontal velocity is different relative to the two observers. (The ship is shown moving rather fast to emphasize the effect.)

Example 3.6 Calculating Relative Velocity: An Airline Passenger Drops a Coin

An airline passenger drops a coin while the plane is moving at 260 m/s. What is the velocity of the coin when it strikes the floor 1.50 m below its point of release: (a) Measured relative to the plane? (b) Measured relative to the Earth?

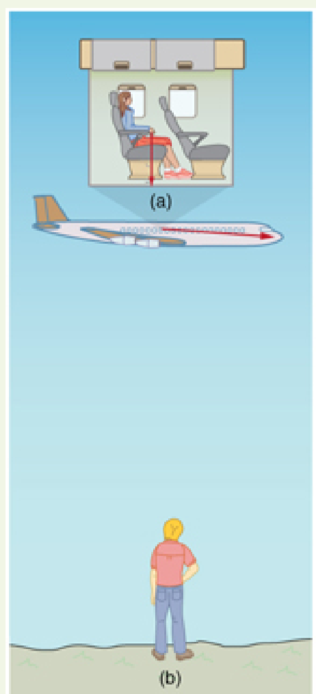


Figure 3.43 The motion of a coin dropped inside an airplane as viewed by two different observers. (a) An observer in the plane sees the coin fall straight down. (b) An observer on the ground sees the coin move almost horizontally.

Strategy

Both problems can be solved with the techniques for falling objects and projectiles. In part (a), the initial velocity of the coin is zero relative to the plane, so the motion is that of a falling object (one-dimensional). In part (b), the initial velocity is 260 m/s horizontal relative to the Earth and gravity is vertical, so this motion is a projectile motion. In both parts, it is best to use a coordinate system with vertical and horizontal axes.

Solution for (a)

Using the given information, we note that the initial velocity and position are zero, and the final position is 1.50 m. The final velocity can be found using the equation:

$$v_y{}^2 = v_{0y}{}^2 - 2g(y - y_0). \tag{3.49}$$

Substituting known values into the equation, we get

$$v_y{}^2 = 0^2 - 2(9.80 \text{ m/s}^2)(-1.50 \text{ m} - 0 \text{ m}) = 29.4 \text{ m}^2/\text{s}^2 \tag{3.50}$$

yielding

$$v_y = -5.42 \text{ m/s}. \tag{3.51}$$

We know that the square root of 29.4 has two roots: 5.42 and -5.42. We choose the negative root because we know that the velocity is directed downwards, and we have defined the positive direction to be upwards. There is no initial horizontal velocity relative to the plane and no horizontal acceleration, and so the motion is straight down relative to the plane.

Solution for (b)

Because the initial vertical velocity is zero relative to the ground and vertical motion is independent of horizontal motion, the final vertical velocity for the coin relative to the ground is $v_y = -5.42 \text{ m/s}$, the same as found in part (a). In contrast to part (a), there now is a horizontal component of the velocity. However, since there is no horizontal acceleration, the initial and final horizontal velocities are the same and $v_x = 260 \text{ m/s}$. The *x*- and *y*-components of velocity can be combined to find the magnitude of the final velocity:

$$v = \sqrt{v_x{}^2 + v_y{}^2}. \tag{3.52}$$

Thus,

$$v = \sqrt{(260 \text{ m/s})^2 + (-5.42 \text{ m/s})^2} \tag{3.53}$$

yielding

$$v = 260.06 \text{ m/s}. \tag{3.54}$$

The direction is given by:

$$\theta = \tan^{-1}(v_y / v_x) = \tan^{-1}(-5.42 / 260) \tag{3.55}$$

so that

$$\theta = \tan^{-1}(-0.0208) = -1.19^{\circ}. \tag{3.56}$$

Discussion

In part (a), the final velocity relative to the plane is the same as it would be if the coin were dropped from rest on the Earth and fell 1.50 m. This result fits our experience; objects in a plane fall the same way when the plane is flying horizontally as when it is at rest on the ground. This result is also true in moving cars. In part (b), an observer on the ground sees a much different motion for the coin. The plane is moving so fast horizontally to begin with that its final velocity is barely greater than the initial velocity. Once again, we see that in two dimensions, vectors do not add like ordinary numbers—the final velocity v in part (b) is *not* $(260 - 5.42) \text{ m/s}$; rather, it is 260.06 m/s. The velocity's magnitude had to be calculated to five digits to see any difference from that of the airplane. The motions as seen by different observers (one in the plane and one on the ground) in this example are analogous to those discussed for the binoculars dropped from the mast of a moving ship, except that the velocity of the plane is much larger, so that the two observers see *very* different paths. (See **Figure 3.43**.) In addition, both observers see the coin fall 1.50 m vertically, but the one on the ground also sees it move forward 144 m (this calculation is left for the reader). Thus, one observer sees a vertical path, the other a nearly horizontal path.

Making Connections: Relativity and Einstein

Because Einstein was able to clearly define how measurements are made (some involve light) and because the speed of light is the same for all observers, the outcomes are spectacularly unexpected. Time varies with observer, energy is stored as increased mass, and more surprises await.

PhET Explorations: Motion in 2D

Try the new "Ladybug Motion 2D" simulation for the latest updated version. Learn about position, velocity, and acceleration vectors. Move the ball with the mouse or let the simulation move the ball in four types of motion (2 types of linear, simple harmonic, circle).

PhET Interactive Simulation

Figure 3.44 Motion in 2D (http://legacy.cnx.org/content/m64143/1.3/motion-2d_en.jar)

Glossary

analytical method: the method of determining the magnitude and direction of a resultant vector using the Pythagorean theorem and trigonometric identities

classical relativity: the study of relative velocities in situations where speeds are less than about 1% of the speed of light—that is, less than 3000 km/s

commutative: refers to the interchangeability of order in a function; vector addition is commutative because the order in which vectors are added together does not affect the final sum

component (of a 2-d vector): a piece of a vector that points in either the vertical or the horizontal direction; every 2-d vector can be expressed as a sum of two vertical and horizontal vector components

direction (of a vector): the orientation of a vector in space

head (of a vector): the end point of a vector; the location of the tip of the vector's arrowhead; also referred to as the "tip"

head-to-tail method: a method of adding vectors in which the tail of each vector is placed at the head of the previous vector

magnitude (of a vector): the length or size of a vector; magnitude is a scalar quantity

relative velocity: the velocity of an object as observed from a particular reference frame

relativity: the study of how different observers moving relative to each other measure the same phenomenon

resultant: the sum of two or more vectors

resultant vector: the vector sum of two or more vectors

scalar: a quantity with magnitude but no direction

tail: the start point of a vector; opposite to the head or tip of the arrow

vector: a quantity that has both magnitude and direction; an arrow used to represent quantities with both magnitude and direction

vector addition: the rules that apply to adding vectors together

velocity: speed in a given direction

Section Summary

3.1 Kinematics in Two Dimensions-An introduction

- The shortest path between any two points is a straight line. In two dimensions, this path can be represented by a vector with horizontal and vertical components.
- The horizontal and vertical components of a vector are independent of one another. Motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

3.2 Vector Addition and Subtraction: Graphical Methods

- The **graphical method of adding vectors** $\mathbf{A}$ and $\mathbf{B}$ involves drawing vectors on a graph and adding them using the head-to-tail method. The resultant vector $\mathbf{R}$ is defined such that $\mathbf{A} + \mathbf{B} = \mathbf{R}$. The magnitude and direction of $\mathbf{R}$ are then determined with a ruler and protractor, respectively.
- The **graphical method of subtracting vector** $\mathbf{B}$ from $\mathbf{A}$ involves adding the opposite of vector $\mathbf{B}$, which is defined as $-\mathbf{B}$. In this case, $\mathbf{A} - \mathbf{B} = \mathbf{A} + (-\mathbf{B}) = \mathbf{R}$. Then, the head-to-tail method of addition is followed in the usual way to obtain the resultant vector $\mathbf{R}$.
- Addition of vectors is **commutative** such that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.

- The **head-to-tail method** of adding vectors involves drawing the first vector on a graph and then placing the tail of each subsequent vector at the head of the previous vector. The resultant vector is then drawn from the tail of the first vector to the head of the final vector.
- If a vector $\mathbf{A}$ is multiplied by a scalar quantity c, the magnitude of the product is given by cA. If c is positive, the direction of the product points in the same direction as $\mathbf{A}$; if c is negative, the direction of the product points in the opposite direction as $\mathbf{A}$.

3.3 Vector Addition and Subtraction-Analytical Methods

- The analytical method of vector addition and subtraction involves using the Pythagorean theorem and trigonometric identities to determine the magnitude and direction of a resultant vector.
- The steps to add vectors $\mathbf{A}$ and $\mathbf{B}$ using the analytical method are as follows:

  Step 1: Determine the coordinate system for the vectors. Then, determine the horizontal and vertical components of each vector using the equations

$$\begin{aligned} A_x &= A \cos \theta \\ B_x &= B \cos \theta \end{aligned}$$

  and

$$\begin{aligned} A_y &= A \sin \theta \\ B_y &= B \sin \theta. \end{aligned}$$

  Step 2: Add the horizontal and vertical components of each vector to determine the components R_x and R_y of the resultant vector, $\mathbf{R}$:

$$R_x = A_x + B_x$$

  and

$$R_y = A_y + B_y.$$

  Step 3: Use the Pythagorean theorem to determine the magnitude, R, of the resultant vector $\mathbf{R}$:

$$R = \sqrt{R_x^2 + R_y^2}.$$

  Step 4: Use a trigonometric identity to determine the direction, θ, of $\mathbf{R}$:

$$\theta = \tan^{-1}(R_y / R_x).$$

3.4 Addition of Velocities

- Velocities in two dimensions are added using the same analytical vector techniques, which are rewritten as

$$v_x = v \cos \theta$$

$$v_y = v \sin \theta$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$\theta = \tan^{-1}(v_y / v_x).$$

- Relative velocity is the velocity of an object as observed from a particular reference frame, and it varies dramatically with reference frame.
- **Relativity** is the study of how different observers measure the same phenomenon, particularly when the observers move relative to one another. **Classical relativity** is limited to situations where speed is less than about 1% of the speed of light (3000 km/s).

Conceptual Questions

3.2 Vector Addition and Subtraction: Graphical Methods

1. Which of the following is a vector: a person’s height, the altitude on Mt. Everest, the age of the Earth, the boiling point of water, the cost of this book, the Earth’s population, the acceleration of gravity?

2. Give a specific example of a vector, stating its magnitude, units, and direction.

3. What do vectors and scalars have in common? How do they differ?

4. Two campers in a national park hike from their cabin to the same spot on a lake, each taking a different path, as illustrated below. The total distance traveled along Path 1 is 7.5 km, and that along Path 2 is 8.2 km. What is the final displacement of each camper?

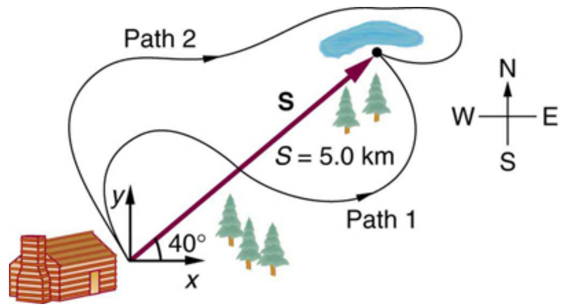


Figure 3.45

5. If an airplane pilot is told to fly 123 km in a straight line to get from San Francisco to Sacramento, explain why he could end up anywhere on the circle shown in **Figure 3.46**. What other information would he need to get to Sacramento?

Figure 3.46

6. Suppose you take two steps $\mathbf{A}$ and $\mathbf{B}$ (that is, two nonzero displacements). Under what circumstances can you end up at your starting point? More generally, under what circumstances can two nonzero vectors add to give zero? Is the maximum distance you can end up from the starting point $\mathbf{A}+\mathbf{B}$ the sum of the lengths of the two steps?

7. Explain why it is not possible to add a scalar to a vector.

8. If you take two steps of different sizes, can you end up at your starting point? More generally, can two vectors with different magnitudes ever add to zero? Can three or more?

3.3 Vector Addition and Subtraction-Analytical Methods

9. Suppose you add two vectors $\mathbf{A}$ and $\mathbf{B}$. What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?

10. Give an example of a nonzero vector that has a component of zero.

11. Explain why a vector cannot have a component greater than its own magnitude.

12. If the vectors $\mathbf{A}$ and $\mathbf{B}$ are perpendicular, what is the component of $\mathbf{A}$ along the direction of $\mathbf{B}$? What is the component of $\mathbf{B}$ along the direction of $\mathbf{A}$?

3.4 Addition of Velocities

13. What frame or frames of reference do you instinctively use when driving a car? When flying in a commercial jet airplane?

14. A basketball player dribbling down the court usually keeps his eyes fixed on the players around him. He is moving fast. Why doesn't he need to keep his eyes on the ball?

15. If someone is riding in the back of a pickup truck and throws a softball straight backward, is it possible for the ball to fall straight down as viewed by a person standing at the side of the road? Under what condition would this occur? How would the motion of the ball appear to the person who threw it?

16. The hat of a jogger running at constant velocity falls off the back of his head. Draw a sketch showing the path of the hat in the jogger's frame of reference. Draw its path as viewed by a stationary observer.

17. A clod of dirt falls from the bed of a moving truck. It strikes the ground directly below the end of the truck. What is the direction of its velocity relative to the truck just before it hits? Is this the same as the direction of its velocity relative to ground just before it hits? Explain your answers.

Problems & Exercises

3.2 Vector Addition and Subtraction: Graphical Methods

Use graphical methods to solve these problems. You may assume data taken from graphs is accurate to three digits.

1. Find the following for path A in Figure 3.47: (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

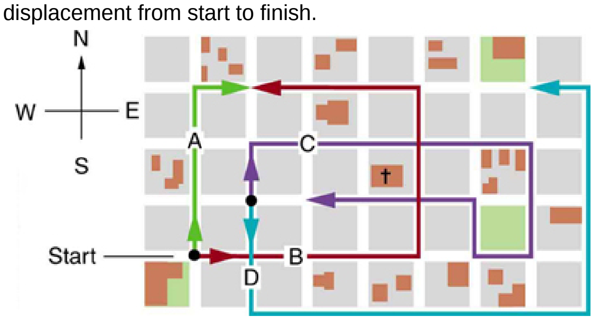


Figure 3.47 The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

2. Find the following for path B in Figure 3.47: (a) the total distance traveled, and (b) the magnitude and direction of the displacement from start to finish.

3. Find the north and east components of the displacement for the hikers shown in Figure 3.45.

4. Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $\mathbf{A}$ and $\mathbf{B}$, as in Figure 3.48, then this problem asks you to find their sum $\mathbf{R} = \mathbf{A} + \mathbf{B}$.)

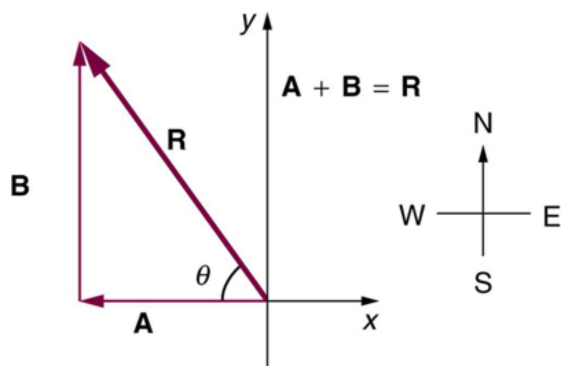


Figure 3.48 The two displacements $\mathbf{A}$ and $\mathbf{B}$ add to give a total displacement $\mathbf{R}$ having magnitude R and direction θ.

5. Suppose you first walk 12.0 m in a direction 20° west of north and then 20.0 m in a direction 40.0° south of west. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $\mathbf{A}$ and $\mathbf{B}$, as in Figure 3.49, then this problem finds their sum $\mathbf{R} = \mathbf{A} + \mathbf{B}$.)

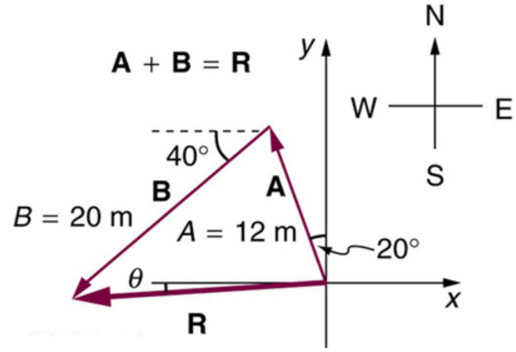


Figure 3.49

6. Repeat the problem above, but reverse the order of the two legs of the walk; show that you get the same final result. That is, you first walk leg $\mathbf{B}$, which is 20.0 m in a direction exactly 40° south of west, and then leg $\mathbf{A}$, which is 12.0 m in a direction exactly 20° west of north. (This problem shows that $\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$.)

7. (a) Repeat the problem two problems prior, but for the second leg you walk 20.0 m in a direction 40.0° north of east (which is equivalent to subtracting $\mathbf{B}$ from $\mathbf{A}$ —that is, to finding $\mathbf{R}' = \mathbf{A} - \mathbf{B}$). (b) Repeat the problem two problems prior, but now you first walk 20.0 m in a direction 40.0° south of west and then 12.0 m in a direction 20.0° east of south (which is equivalent to subtracting $\mathbf{A}$ from $\mathbf{B}$ —that is, to finding $\mathbf{R}'' = \mathbf{B} - \mathbf{A} = -\mathbf{R}'$). Show that this is the case.

8. Show that the *order* of addition of three vectors does not affect their sum. Show this property by choosing any three vectors $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$, all having different lengths and directions. Find the sum $\mathbf{A} + \mathbf{B} + \mathbf{C}$ then find their sum when added in a different order and show the result is the same. (There are five other orders in which $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ can be added; choose only one.)

9. Show that the sum of the vectors discussed in Example 3.2 gives the result shown in Figure 3.24.

10. Find the magnitudes of velocities v_{A} and v_{B} in **Figure 3.50**

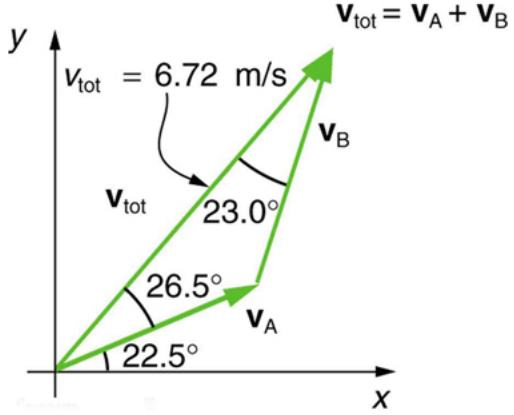


Figure 3.50 The two velocities $\mathbf{v}_{\text{A}}$ and $\mathbf{v}_{\text{B}}$ add to give a total $\mathbf{v}_{\text{tot}}$.

11. Find the components of v_{tot} along the *x*- and *y*-axes in **Figure 3.50**.

12. Find the components of v_{tot} along a set of perpendicular axes rotated 30° counterclockwise relative to those in **Figure 3.50**.

3.3 Vector Addition and Subtraction-Analytical Methods

13. Find the following for path C in **Figure 3.51**: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

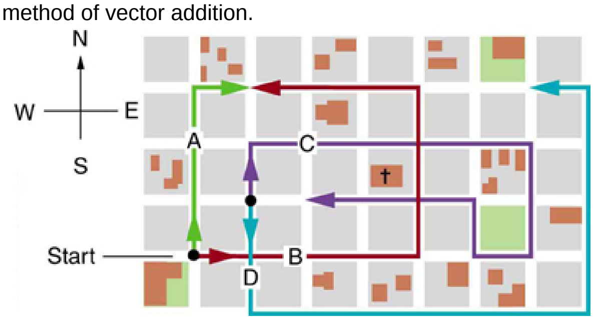


Figure 3.51 The various lines represent paths taken by different people walking in a city. All blocks are 120 m on a side.

14. Find the following for path D in **Figure 3.51**: (a) the total distance traveled and (b) the magnitude and direction of the displacement from start to finish. In this part of the problem, explicitly show how you follow the steps of the analytical method of vector addition.

15. Find the north and east components of the displacement from San Francisco to Sacramento shown in **Figure 3.52**.

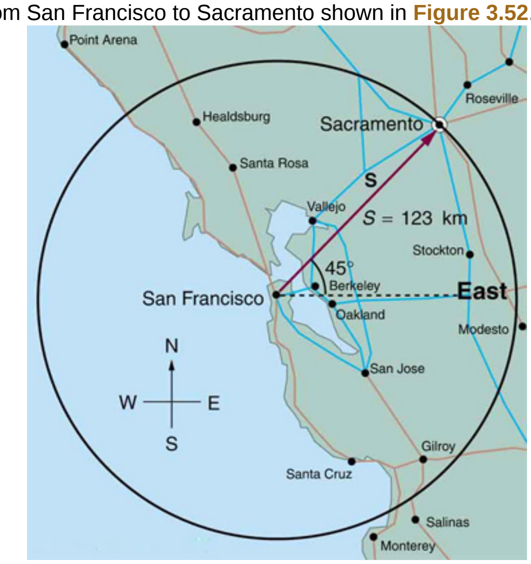


Figure 3.52

16. Solve the following problem using analytical techniques: Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point, and what is the compass direction of a line connecting your starting point to your final position? (If you represent the two legs of the walk as vector displacements $\mathbf{A}$ and $\mathbf{B}$, as in **Figure 3.53**, then this problem asks you to find their sum $\mathbf{R} = \mathbf{A} + \mathbf{B}$.)

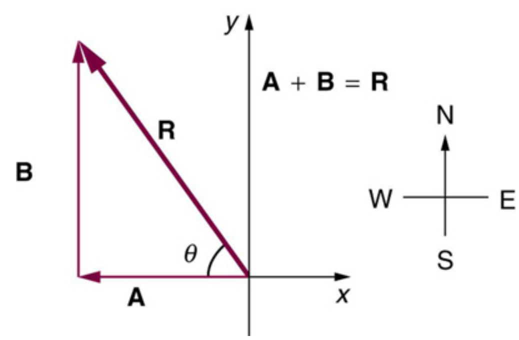


Figure 3.53 The two displacements $\mathbf{A}$ and $\mathbf{B}$ add to give a total displacement $\mathbf{R}$ having magnitude R and direction θ .

Note that you can also solve this graphically. Discuss why the analytical technique for solving this problem is potentially more accurate than the graphical technique.

17. Repeat **Exercise 3.16** using analytical techniques, but reverse the order of the two legs of the walk and show that you get the same final result. (This problem shows that adding them in reverse order gives the same result—that is, $\mathbf{B} + \mathbf{A} = \mathbf{A} + \mathbf{B}$.) Discuss how taking another path to reach the same point might help to overcome an obstacle blocking you other path.

18. You drive 7.50 km in a straight line in a direction 15° east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (This determination is equivalent to find the components of the displacement along the east and north directions.) (b) Show that you still arrive at the same point if the east and north legs are reversed in order.

19. Do **Exercise 3.16** again using analytical techniques and change the second leg of the walk to 25.0 m straight south. (This is equivalent to subtracting $\mathbf{B}$ from $\mathbf{A}$ —that is, finding $\mathbf{R}' = \mathbf{A} - \mathbf{B}$) (b) Repeat again, but now you first walk 25.0 m north and then 18.0 m east. (This is equivalent to subtract $\mathbf{A}$ from $\mathbf{B}$ —that is, to find $\mathbf{A} = \mathbf{B} + \mathbf{C}$. Is that consistent with your result?)

20. A new landowner has a triangular piece of flat land she wishes to fence. Starting at the west corner, she measures the first side to be 80.0 m long and the next to be 105 m. These sides are represented as displacement vectors $\mathbf{A}$ from $\mathbf{B}$ in **Figure 3.54**. She then correctly calculates the length and orientation of the third side $\mathbf{C}$. What is her result?

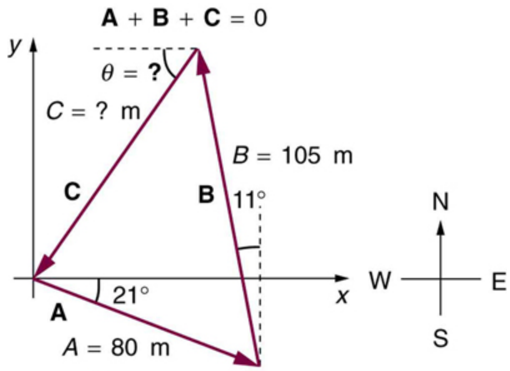


Figure 3.54

21. You fly 32.0 km in a straight line in still air in the direction 35.0° south of west. (a) Find the distances you would have to fly straight south and then straight west to arrive at the same point. (This determination is equivalent to finding the components of the displacement along the south and west directions.) (b) Find the distances you would have to fly first in a direction 45.0° south of west and then in a direction 45.0° west of north. These are the components of the displacement along a different set of axes—one rotated 45° .

22. A farmer wants to fence off his four-sided plot of flat land. He measures the first three sides, shown as $\mathbf{A},$ $\mathbf{B},$ and $\mathbf{C}$ in **Figure 3.55**, and then correctly calculates the length and orientation of the fourth side $\mathbf{D}$. What is his result?

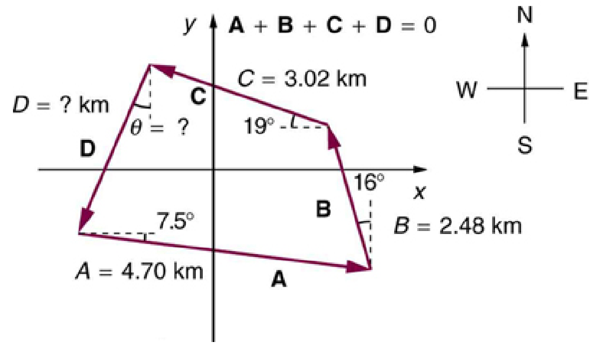


Figure 3.55

23. In an attempt to escape his island, Gilligan builds a raft and sets to sea. The wind shifts a great deal during the day, and he is blown along the following straight lines: 2.50 km 45.0° north of west; then 4.70 km 60.0° south of east; then 1.30 km 25.0° south of west; then 5.10 km straight east; then 1.70 km 5.00° east of north; then 7.20 km 55.0° south of west; and finally 2.80 km 10.0° north of east. What is his final position relative to the island?

24. Suppose a pilot flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east as shown in **Figure 3.56**. Find her total distance R from the starting point and the direction θ of the straight-line path to the final position. Discuss qualitatively how this flight would be altered by a wind from the north and how the effect of the wind would depend on both wind speed and the speed of the plane relative to the air mass.

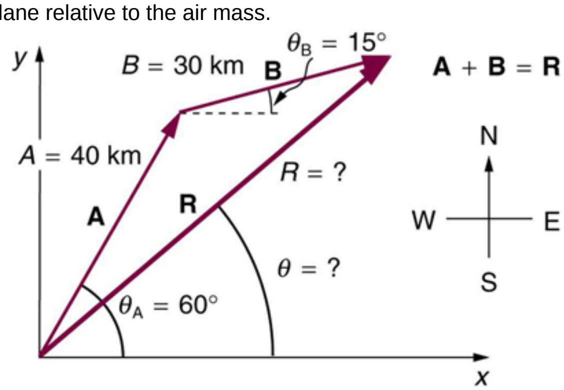


Figure 3.56

3.4 Addition of Velocities

25. Bryan Allen pedaled a human-powered aircraft across the English Channel from the cliffs of Dover to Cap Gris-Nez on June 12, 1979. (a) He flew for 169 min at an average velocity of 3.53 m/s in a direction 45° south of east. What was his total displacement? (b) Allen encountered a headwind averaging 2.00 m/s almost precisely in the opposite direction of his motion relative to the Earth. What was his average velocity relative to the air? (c) What was his total displacement relative to the air mass?

26. A seagull flies at a velocity of 9.00 m/s straight into the wind. (a) If it takes the bird 20.0 min to travel 6.00 km relative to the Earth, what is the velocity of the wind? (b) If the bird turns around and flies with the wind, how long will he take to return 6.00 km? (c) Discuss how the wind affects the total round-trip time compared to what it would be with no wind.

27. Near the end of a marathon race, the first two runners are separated by a distance of 45.0 m. The front runner has a velocity of 3.50 m/s, and the second a velocity of 4.20 m/s. (a) What is the velocity of the second runner relative to the first? (b) If the front runner is 250 m from the finish line, who will win the race, assuming they run at constant velocity? (c) What distance ahead will the winner be when she crosses the finish line?

28. Verify that the coin dropped by the airline passenger in the **Example 3.6** travels 144 m horizontally while falling 1.50 m in the frame of reference of the Earth.

29. A football quarterback is moving straight backward at a speed of 2.00 m/s when he throws a pass to a player 18.0 m straight downfield. The ball is thrown at an angle of 25.0° relative to the ground and is caught at the same height as it is released. What is the initial velocity of the ball *relative to the quarterback* ?

30. A ship sets sail from Rotterdam, The Netherlands, heading due north at 7.00 m/s relative to the water. The local ocean current is 1.50 m/s in a direction 40.0° north of east. What is the velocity of the ship relative to the Earth?

31. (a) A jet airplane flying from Darwin, Australia, has an air speed of 260 m/s in a direction 5.0° south of west. It is in the jet stream, which is blowing at 35.0 m/s in a direction 15° south of east. What is the velocity of the airplane relative to the Earth? (b) Discuss whether your answers are consistent with your expectations for the effect of the wind on the plane's path.

32. (a) In what direction would the ship in **Exercise 3.30** have to travel in order to have a velocity straight north relative to the Earth, assuming its speed relative to the water remains 7.00 m/s ? (b) What would its speed be relative to the Earth?

33. (a) Another airplane is flying in a jet stream that is blowing at 45.0 m/s in a direction 20° south of east (as in **Exercise 3.31**). Its direction of motion relative to the Earth is 45.0° south of west, while its direction of travel relative to the air is 5.00° south of west. What is the airplane's speed relative to the air mass? (b) What is the airplane's speed relative to the Earth?

34. A sandal is dropped from the top of a 15.0-m-high mast on a ship moving at 1.75 m/s due south. Calculate the velocity of the sandal when it hits the deck of the ship: (a) relative to the ship and (b) relative to a stationary observer on shore. (c) Discuss how the answers give a consistent result for the position at which the sandal hits the deck.

35. The velocity of the wind relative to the water is crucial to sailboats. Suppose a sailboat is in an ocean current that has a velocity of 2.20 m/s in a direction 30.0° east of north relative to the Earth. It encounters a wind that has a velocity of 4.50 m/s in a direction of 50.0° south of west relative to the Earth. What is the velocity of the wind relative to the water?

36. The great astronomer Edwin Hubble discovered that all distant galaxies are receding from our Milky Way Galaxy with velocities proportional to their distances. It appears to an observer on the Earth that we are at the center of an expanding universe. **Figure 3.57** illustrates this for five galaxies lying along a straight line, with the Milky Way Galaxy at the center. Using the data from the figure, calculate the velocities: (a) relative to galaxy 2 and (b) relative to galaxy 5. The results mean that observers on all galaxies will see themselves at the center of the expanding universe, and they would likely be aware of relative velocities, concluding that it is not possible to locate the center of expansion with the given information.

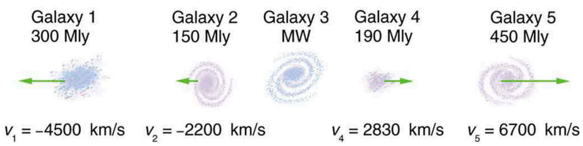


Figure 3.57 Five galaxies on a straight line, showing their distances and velocities relative to the Milky Way (MW) Galaxy. The distances are in millions of light years (Mly), where a light year is the distance light travels in one year. The velocities are nearly proportional to the distances. The sizes of the galaxies are greatly exaggerated; an average galaxy is about 0.1 Mly across.

37. (a) Use the distance and velocity data in **Figure 3.57** to find the rate of expansion as a function of distance.

(b) If you extrapolate back in time, how long ago would all of the galaxies have been at approximately the same position? The two parts of this problem give you some idea of how the Hubble constant for universal expansion and the time back to the Big Bang are determined, respectively.

38. An athlete crosses a 25-m-wide river by swimming perpendicular to the water current at a speed of 0.5 m/s relative to the water. He reaches the opposite side at a distance 40 m downstream from his starting point. How fast is the water in the river flowing with respect to the ground? What is the speed of the swimmer with respect to a friend at rest on the ground?

39. A ship sailing in the Gulf Stream is heading 25.0° west of north at a speed of 4.00 m/s relative to the water. Its velocity relative to the Earth is 4.80 m/s 5.00° west of north. What is the velocity of the Gulf Stream? (The velocity obtained is typical for the Gulf Stream a few hundred kilometers off the east coast of the United States.)

40. An ice hockey player is moving at 8.00 m/s when he hits the puck toward the goal. The speed of the puck relative to the player is 29.0 m/s. The line between the center of the goal and the player makes a 90.0° angle relative to his path as shown in **Figure 3.58**. What angle must the puck's velocity make relative to the player (in his frame of reference) to hit the center of the goal?

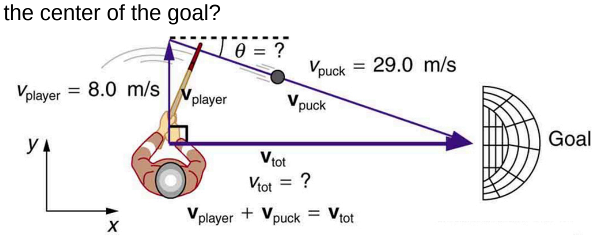


Figure 3.58 An ice hockey player moving across the rink must shoot backward to give the puck a velocity toward the goal.

41. Unreasonable Results Suppose you wish to shoot supplies straight up to astronauts in an orbit 36,000 km above the surface of the Earth. (a) At what velocity must the supplies be launched? (b) What is unreasonable about this velocity? (c) Is there a problem with the relative velocity between the supplies and the astronauts when the supplies reach their maximum height? (d) Is the premise unreasonable or is the available equation inapplicable? Explain your answer.

42. Unreasonable Results A commercial airplane has an air speed of 280 m/s due east and flies with a strong tailwind. It travels 3000 km in a direction 5^{o} south of east in 1.50 h. (a) What was the velocity of the plane relative to the ground? (b) Calculate the magnitude and direction of the tailwind's velocity. (c) What is unreasonable about both of these velocities? (d) Which premise is unreasonable?

43. Construct Your Own Problem Consider an airplane headed for a runway in a cross wind. Construct a problem in which you calculate the angle the airplane must fly relative to the air mass in order to have a velocity parallel to the runway. Among the things to consider are the direction of the runway, the wind speed and direction (its velocity) and the speed of the plane relative to the air mass. Also calculate the speed of the airplane relative to the ground. Discuss any last minute maneuvers the pilot might have to perform in order for the plane to land with its wheels pointing straight down the runway.

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Unit II
Forces

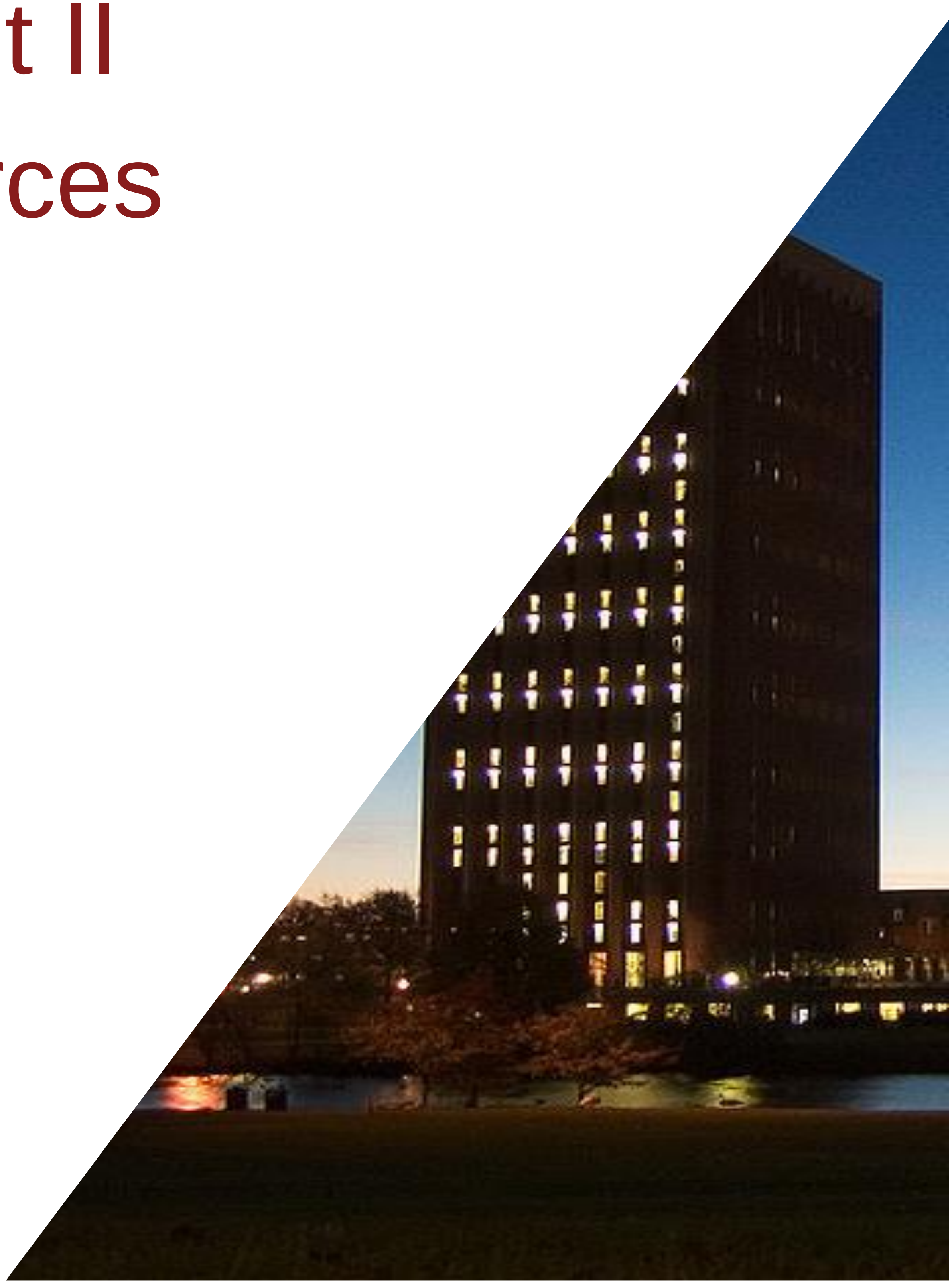

Photo credit: Wikipedia, Rhobite at English. *University of Massachusetts Amherst: Chapel and Library in the Evening*, November 3, 2004. Transferred from en.wikipedia to Commons. https://commons.wikimedia.org/wiki/File:Umass_Amherst_Chapel_%26_Library_in_the_evening.jpg.

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UNIT 2 OVERVIEW

This unit is centered on the idea of forces. So, what fundamentally makes this unit different from the first unit we talked about? In the first unit, we introduced the ideas of position, velocity, and acceleration, and we use these ideas to describe how things move. We are even able to use an iterative calculation to simulate the motion of an object moving under uniform acceleration. In this unit, we're going to be moving beyond these ideas, and building upon them to talk about the question "why does the motion of an object change?".

I want to draw your attention to two particular points: we're moving from describing how objects move to why objects move, and the question is why does motion change, not what causes motion. These are subtly different questions, and the difference between these questions is really at the core of the laws of Isaac Newton, that form the core of this course. So, we're switching from describing how objects move to why does motion change.

This is a more significant switch than it might first appear. Consider the case of a falling object. For millennia, people explained that objects fall using the logic of Aristotle. Aristotle posited that the natural state of an object is to be at rest on the surface of the earth. This explanation seemed to fit all observations at the time, but lacked any mechanism of why this was the case. In modern terms, we would call this a phenomenological description of what happens. It says, "things fall, it's the natural state of them to fall, we don't really know why, just that they fall". It's a phenomenological description without any description of mechanism on why do things fall. And without an understanding of mechanism, we can run into trouble.

For example, the New Horizons space probe that has just visited Pluto and is currently on its way out of the solar system is clearly not going to come to some natural state of rest on the surface of the earth. It's going to keep going forever. Moreover, this switch from description to mechanism is a huge part of the exciting developments in the life sciences that are taking place right now. A lot of the life sciences are really starting to move into mechanism, and it's leading to some interesting and exciting science. We'll look a little bit more at the difference between phenomenological and mechanistic descriptions in some readings from the University of Maryland, as well as in the introduction to chapter four in the OpenStax textbook.

So, why does motion change? In a word forces. Forces cause motion to change. This is one of the key points for this entire course. Now, this idea might be counter to your everyday experience. In our everyday experience, it seems that forces cause motion. For example, if the cabinet is sliding across the floor, I have to keep pushing to keep it motion, I have to keep applying a force or the cabinet will stop moving. So, in our everyday experience, it seems that forces cause motion, but it turns out that this is not true. Forces don't cause motion, forces cause motion to change, and this difference between our everyday experience and the real laws that govern the universe is because our world is very complicated. In the example of the cabinet, the friction between the cabinet and the floor is complicating and impeding our understanding. To get a true feel for what's going on, we need to remove all the complications of our real world. So, let's think about removing complications. This idea, which is explored more in OpenStax chapter 4.2, is critical to physics, and is becoming more of a feature in other sciences like biology. As these Sciences begin to look more and more at mechanistic explanations, the idea is to strip away all the complications from the world and think about the simplest possible world. A classic example is the world without any friction and without any type of air resistance. Then, thinking about this world, you figure out what laws apply, then once you've figured out what the fundamental laws are, you can add the complications back in.

So, while we'll spend a lot of time in this class talking about worlds without friction and air resistance, I want you to know that this idea has worked very well, and has developed a very strong set of fundamental physical laws, and these fundamental physical laws do translate to your other courses. The laws of Newton that we're going to study in this course are the fundamental laws that every other science course you ever will take must obey. Evolution is constrained by the laws of physics. Chemistry is constrained by the laws of physics. They're just these other complications that we strip away in this course, but get added back in, so learning to think in a way of removing complications and adding them back in is one of the key goals of this course. So, what do forces do? Forces cause motion to change. If you get nothing else from this class, I want you to get this idea that forces cause motion to change. In- class we will do some practical exercises to further develop this idea.

1 DYNAMICS: FORCE AND NEWTON'S LAWS

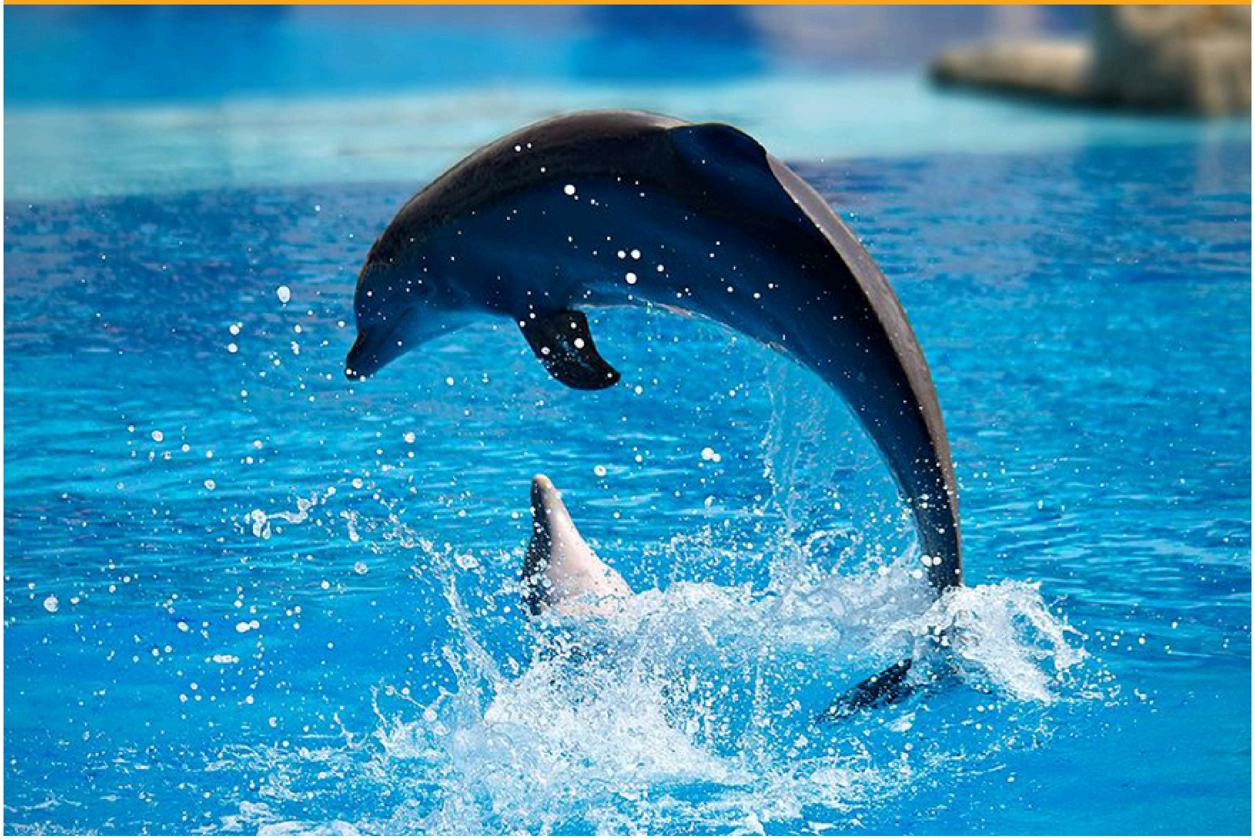


Figure 1.1 Newton’s laws of motion describe the motion of the dolphin’s path. (credit: Jin Jang)

Chapter Outline

Introduction to Dynamics - Newton's Laws of Motion

Motion draws our attention. Motion itself can be beautiful, causing us to marvel at the forces needed to achieve spectacular motion, such as that of a dolphin jumping out of the water, or a pole vaulter, or the flight of a bird, or the orbit of a satellite. The study of motion is kinematics, but kinematics only *describes* the way objects move—their velocity and their acceleration. **Dynamics** considers the forces that affect the motion of moving objects and systems. Newton’s laws of motion are the foundation of dynamics. These laws provide an example of the breadth and simplicity of principles under which nature functions. They are also universal laws in that they apply to similar situations on Earth as well as in space.

Isaac Newton’s (1642–1727) laws of motion were just one part of the monumental work that has made him legendary. The development of Newton’s laws marks the transition from the Renaissance into the modern era. This transition was characterized by a revolutionary change in the way people thought about the physical universe. For many centuries natural philosophers had debated the nature of the universe based largely on certain rules of logic with great weight given to the thoughts of earlier classical philosophers such as Aristotle (384–322 BC). Among the many great thinkers who contributed to this change were Newton and Galileo.

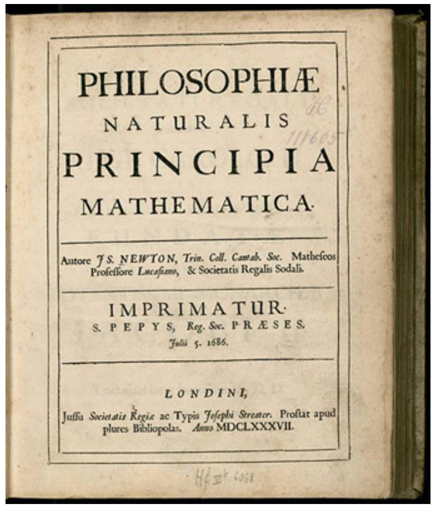


Figure 1.2 Isaac Newton’s monumental work, *Philosophiae Naturalis Principia Mathematica*, was published in 1687. It proposed scientific laws that are still used today to describe the motion of objects. (credit: Service commun de la documentation de l'Université de Strasbourg)

Galileo was instrumental in establishing *observation* as the absolute determinant of truth, rather than “logical” argument. Galileo’s use of the telescope was his most notable achievement in demonstrating the importance of observation. He discovered moons orbiting Jupiter and made other observations that were inconsistent with certain ancient ideas and religious dogma. For this reason, and because of the manner in which he dealt with those in authority, Galileo was tried by the Inquisition and punished. He spent the final years of his life under a form of house arrest. Because others before Galileo had also made discoveries by *observing* the nature of the universe, and because repeated observations verified those of Galileo, his work could not be suppressed or denied. After his death, his work was verified by others, and his ideas were eventually accepted by the church and scientific communities.

Galileo also contributed to the formation of what is now called Newton’s first law of motion. Newton made use of the work of his predecessors, which enabled him to develop laws of motion, discover the law of gravity, invent calculus, and make great contributions to the theories of light and color. It is amazing that many of these developments were made with Newton working alone, without the benefit of the usual interactions that take place among scientists today.

It was not until the advent of modern physics early in the 20th century that it was discovered that Newton’s laws of motion produce a good approximation to motion only when the objects are moving at speeds much, much less than the speed of light and when those objects are larger than the size of most molecules (about $10^{-9}\ \text{m}$ in diameter). These constraints define the realm of classical mechanics, as discussed in **Introduction to the Nature of Science and Physics (https://legacy.cnx.org/content/m42119/latest/)** . At the beginning of the 20^{th} century, Albert Einstein (1879–1955) developed the theory of relativity and, along with many other scientists, developed quantum theory. This theory does not have the constraints present in classical physics. All of the situations we consider in this chapter, and all those preceding the introduction of relativity in **Special Relativity (https://legacy.cnx.org/content/m42525/latest/)** , are in the realm of classical physics.

Making Connections: Past and Present Philosophy

The importance of observation and the concept of *cause and effect* were not always so entrenched in human thinking. This realization was a part of the evolution of modern physics from natural philosophy. The achievements of Galileo, Newton, Einstein, and others were key milestones in the history of scientific thought. Most of the scientific theories that are described in this book descended from the work of these scientists.

1.1 Phenomenology and Mechanism

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Explaining the difference between a phenomenological and mechanistic description

The following is based on:
umdberg / Phenomenology and mechanism. Available at: **http://umdberg.pbworks.com/w/page/54513653/Phenomenology%20and%20mechanism (http://umdberg.pbworks.com/w/page/54513653/Phenomenology%20and%20mechanism)** . (Accessed: 11th July 2017)

UMASS AMHERST Instructor's Notes

Both phenomenological and mechanistic explanations are important in modern science. In this unit, we will be looking at mechanistic explanations for motion while the last unit was purely phenomonological.

Two hundred years ago, biology was mostly about figuring out what kind of living organisms there were and how to describe and classify them. (cf. Linnaeus) But by 100 years ago, scientists like Darwin, Mendel, deVries, Fischer and others began to develop an understanding of how living beings worked and how they fit with their environments and with other organisms.

Today, and very likely for the next few decades, the most exciting developments are in the areas of figuring out how biological organisms work -- how they function in detail, through an understanding of their structure down to the biochemistry and the atomic and molecular level, and how they interact -- how populations and communities of organisms behave, through an understanding of ecology and evolution.

Biology remains highly complicated, and there are large sets of terms to learn. But the trends in modern biology means that a biologist or health-care professional who wants to keep up with developments needs to understand the difference between two kinds of scientific thinking: henomenology and mechanism

Phenomenology

The term *phenomenology* basically means the study of phenomenon -- what there is and what happens. It's largely descriptive.

Mechanism

The term mechanism means considering a phenomenon at a finer-grained level. What parts does it have? What are the relations of the parts to each other? What are the chain of event that lead to a transformation taking place? Mechanistic thinking is analytic -- it breaks things down. Mechanistic thinking is valuable not just in science. It helps you understand whether your plan to organize a party for your friends will work and whether a politicians plan for the country makes sense.

Any science involves both phenomenology and looking for mechanism -- description and analysis.

In biology, the phenomenology of photosynthesis might say that plants convert light into sugars and starches that can serve as food for animals. Understanding the mechanism of photosynthesis would require that we understand which light from the sun is effective (only certain very specific colors work), what chemicals exist in the plant that results in this transformation, and what is the pathway of chemical transformations that take place.

In physics, we can observe that when we hook a battery up to two identical bulbs connected in a row, the bulbs are dimmer than when we only hook up a single bulb. That's phenomenology. If we analyze the circuit by identifying the relevant properties of the battery, bulbs, and wire (voltage, resistance, and current) and figure out the relationships between them, we are exploring mechanism.

Note that analyzing mechanism can occur at many levels. With our batteries and bulbs, understanding currents, voltage, and resistance is a macroscopic mechanism. If we learn more and understand that currents in a battery and bulb circuits are electrons that are separated from their atoms and moving through the wires, we can explore a microscopic mechanism. In biology we can go up or down in scale. The mechanism of photosynthesis described above -- in terms of chemistry -- is a microscopic mechanism. But we might also consider how photosynthesis evolved in terms of the interaction of different organisms and in an ecological context -- a mechanism at a level above the functioning of a single organism.

Since physics "sets the rules", constraining how things can behave, physics is particularly important when trying to understand biological mechanisms.

1.2 Development of Force Concept

Instructor's Notes

Your Quiz will Cover

- Defining a force as a push or a pull
- Computing the net force by adding forces as vectors
- Describing what a free-body-diagram is

Instructor's Notes

As with most definitions, there's a general idea of what a force is, a push or a pull, but the scientific definition is a little more nuanced. Note that fores can also be represented as a vector; we will be working with forces as vectors through this course.

Dynamics is the study of the forces that cause objects and systems to move. To understand this, we need a working definition of force. Our intuitive definition of **force**—that is, a push or a pull—is a good place to start. We know that a push or pull has both magnitude and direction (therefore, it is a vector quantity) and can vary considerably in each regard. For example, a cannon exerts a strong force on a cannonball that is launched into the air. In contrast, Earth exerts only a tiny downward pull on a flea. Our everyday experiences also give us a good idea of how multiple forces add. If two people push in different directions on a third person, as illustrated in **Figure 1.3**, we might expect the total force to be in the direction shown. Since force is a vector, it adds just like other vectors, as illustrated in **Figure 1.3**(a) for two ice skaters. Forces, like other vectors, are represented by arrows and can be added using the familiar head-to-tail method or by trigonometric methods. These ideas were developed in **Two-Dimensional Kinematics (https://legacy.cnx.org/content/m42126/latest/)** .

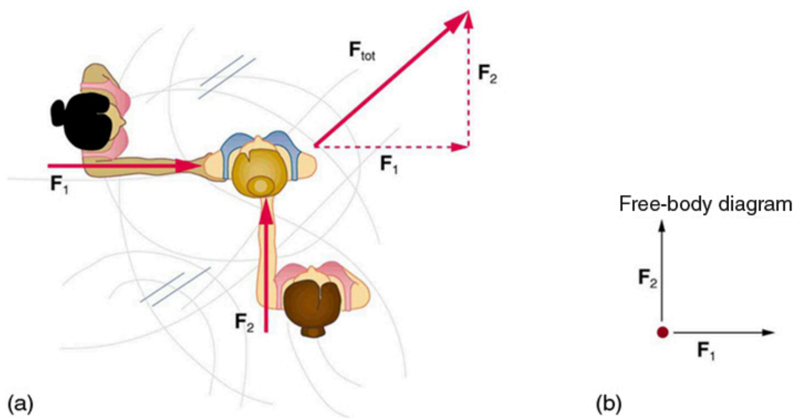


Figure 1.3 Part (a) shows an overhead view of two ice skaters pushing on a third. Forces are vectors and add like other vectors, so the total force on the third skater is in the direction shown. In part (b), we see a free-body diagram representing the forces acting on the third skater.

Instructor's Notes

Another goal here is to expose you to the idea of the free-body diagram. We will be practicing using free-body diagrams in class, so the focus is to just get an idea of what a free-body diagram is.

Figure 1.3(b) is our first example of a **free-body diagram**, which is a technique used to illustrate all the **external forces** acting on a body. The body is represented by a single isolated point (or free body), and only those forces acting *on* the body from the outside (external forces) are shown. (These forces are the only ones shown, because only external forces acting on the body affect its motion. We can ignore any internal forces within the body.) Free-body diagrams are very useful in analyzing forces acting on a system and are employed extensively in the study and application of Newton's laws of motion.

A more quantitative definition of force can be based on some standard force, just as distance is measured in units relative to a standard distance. One possibility is to stretch a spring a certain fixed distance, as illustrated in **Figure 1.4**, and use the force it exerts to pull itself back to its relaxed shape—called a *restoring force*—as a standard. The magnitude of all other forces can be stated as multiples of this standard unit of force. Many other possibilities exist for standard forces. (One that we will encounter in **Magnetism (https://legacy.cnx.org/content/m42365/latest/)** is the magnetic force between two wires carrying electric current.) Some alternative definitions of force will be given later in this chapter.

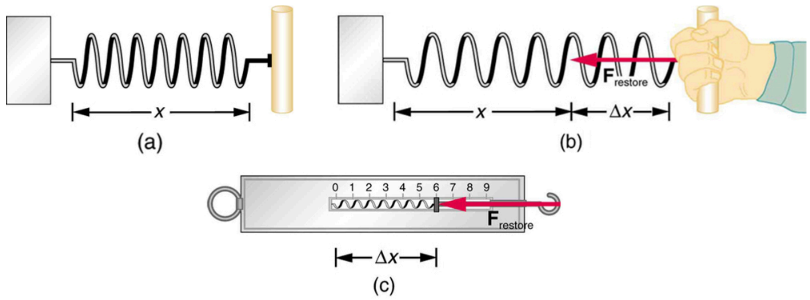


Figure 1.4 The force exerted by a stretched spring can be used as a standard unit of force. (a) This spring has a length x when undistorted. (b) When stretched a distance Δx, the spring exerts a restoring force, $\mathbf{F}_{\text{restore}}$, which is reproducible. (c) A spring scale is one device that uses a spring to measure force. The force $\mathbf{F}_{\text{restore}}$ is exerted on whatever is attached to the hook. Here $\mathbf{F}_{\text{restore}}$ has a magnitude of 6 units in the force standard being employed.

Take-Home Experiment: Force Standards

To investigate force standards and cause and effect, get two identical rubber bands. Hang one rubber band vertically on a hook. Find a small household item that could be attached to the rubber band using a paper clip, and use this item as a weight to investigate the stretch of the rubber band. Measure the amount of stretch produced in the rubber band with one, two, and four of these (identical) items suspended from the rubber band. What is the relationship between the number of items and the amount of stretch? How large a stretch would you expect for the same number of items suspended from two rubber bands? What happens to the amount of stretch of the rubber band (with the weights attached) if the weights are also pushed to the side with a pencil?

1.3 Object Egotism

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Defining the idea of object egotism

The following is based off of
umdberg / Object egotism. Available at: **http://umdberg.pbworks.com/w/page/45451187/Object%20egotism. (http://umdberg.pbworks.com/w/page/45451187/Object%20egotism)** (Accessed: 11th July 2017)

UMASS AMHERST Instructor's Notes

The essence of object egotism for forces is that the only forces that affect an object's acceleration are the forces acting on it at that instant, and the objects have no "memory" of the forces acting on it. This idea will play an important role when working with forces, so be sure you understand this concept before moving on.

The first thing we have to decide when we build Newton's theory of motion is this:

What object or collection of objects should we consider when we think about a motion?

After all, the world is a complicated place. Everything affects everything else in a huge tangled web of influences. To pull this web apart for laws of motion, we can succeed by going down to the simplest situation, understanding how it works, and then slowly building up principles that allow us to put together more complex situations.

The fact that this works is not preordained, and indeed, it's a bit surprising that it does. In biology the approach of reducing a complex system to the simplest case does not always get us very far and other methods have to be used (though now that we have the technology to take apart how living organisms function, it appears that it might provide some utility in biology as well) But in physics, the success of this approach in this channel on cat television might be what has established the entire "go for the simplest situation first" flavor that so colors the approaches most common to physics.

Thought experiment 1: Block on a table

Let's consider the simplest possible situation of motion: a block sitting on a table. What do we have to do to get it to move? Well, we might push it. But to follow our guiding star of starting with the simplest possible case, let's keep that push to a short time. Let's strike it quickly with a hammer. What happens? Well, the block jumps a bit and stops. It I hit it again, it jumps again and stops. I might hypothesize that a "tap" (what the hammer delivers to the block) produces a change in position.

But if we extend our experiments -- put some soapy water on the table or some sandpaper -- we will find that the same tap (and you will have to imagine for yourself how to create a system that can deliver identical taps) will produce different changes in position -- much more of a change if there is soapy water on the table, much less if the block is resting on a piece of taped down sandpaper.

What this shows is that "a block sitting on a table being hit by a hammer" is not the simplest situation we can conceive of. We have the sense that the hammer is what moves the block and the surface of the table is what stops it. So there seem to be two things going on -- the hammer starting the motion and the table stopping it. To understand what is going on we have to focus on the block.

Thinking like a block

To think about the block, it's best to try to put yourself in the position of the block itself. This is a non-trivial shift of perspective. If we are pushing a box along a concrete floor, we know we have to keep pushing if we want it to keep moving. We sense, from our view as pusher, that a single force is associated with a constant velocity. But if you imagine yourself to be the box rather than the person pushing the box, you will realize that you not only feel the person pushing on your shoulders, but the rough (and possibly painful) drag of the concrete floor on your bottom. You feel two important interactions, not just one.

I call this approach of "becoming the object" as physics by empathy or thinking inside the box.

[It is very much related to the principle of "method acting" practiced by such famous actors as Sean Penn, Robert De Niro, Paul Newman, Dustin Hoffman, and Marlon Brando. The method was invented in the 1920s by Stanislavski in Russia. The trick is to try to "become" the character -- to understand him or her and learn to think and behave like that character.]

When you think like a block, you realize that you have no concern for what you might be doing to anything else, but only respond to what you feel. And you have no memory of what happened earlier. I summarize this as a fundamental assumption of Newton's theoretical framework:

Objects respond only to influences acting upon them at the instant that those influences act.

I sometimes summarize this as object egotism - for objects it's "me, me, me, and right now!"

Now it's clear that while this makes sense for inert objects, for an active organism that has will and intent, it can do things to other objects -- and it may well interact with those other objects purposefully; like pushing forward on a wall when swimming laps to make yourself go backwards. While it is perfectly possible to formulate the theory so that it looks like this, it would be terrible for inert objects. You would have to say that the block moved when hit by the hammer because the block pushed back on the hammer!

Notice that in building our theories, we have choices which way to go. Here, we have decided to simplify as much as possible -- so as to start with pairs of objects rather than three -- and to consider inert objects as our typical example. We'll see that we'll be able to successfully and consistently handle complex interactions with three and more objects and even active organisms in our theory built initially for pairs of inert objects -- without having to add any new principles!

1.4 Newton's First Law - Inertia

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Explaining Newton’s Laws in your own words

Experience suggests that an object at rest will remain at rest if left alone, and that an object in motion tends to slow down and stop unless some effort is made to keep it moving. What **Newton’s first law of motion** states, however, is the following:

Newton’s First Law of Motion

A body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force.

UMASS AMHERST Instructor's Notes

Newton's first law will play an important role in this course, so pay close attention to this section.

Note the repeated use of the verb “remains.” We can think of this law as preserving the status quo of motion.

Rather than contradicting our experience, **Newton’s first law of motion** states that there must be a *cause* (which is a net external force) *for there to be any change in velocity (either a change in magnitude or direction)*. We will define *net external force* in the next section. An object sliding across a table or floor slows down due to the net force of friction acting on the object. If friction disappeared, would the object still slow down?

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface, we can imagine the object sliding in a straight line indefinitely. Friction is thus the *cause* of the slowing (consistent with Newton’s first law). The object would not slow down at all if friction were completely eliminated. Consider an air hockey table. When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we know enough about the friction, we can accurately predict how quickly the object will slow down. Friction is an external force.

Newton’s first law is completely general and can be applied to anything from an object sliding on a table to a satellite in orbit to blood pumped from the heart. Experiments have thoroughly verified that any change in velocity (speed or direction) must be caused by an external force. The idea of *generally applicable or universal laws* is important not only here—it is a basic feature of all laws of physics. Identifying these laws is like recognizing patterns in nature from which further patterns can be discovered. The genius of Galileo, who first developed the idea for the first law, and Newton, who clarified it, was to ask the fundamental question, “What is the cause?” Thinking in terms of cause and effect is a worldview fundamentally different from the typical ancient Greek approach when questions such as “Why does a tiger have stripes?” would have been answered in Aristotelian fashion, “That is the nature of the beast.” True perhaps, but not a useful insight.

UMASS AMHERST Instructor's Notes

Generally, mass is how much "stuff" is in something, as opposed to weight, which is how much the force of gravity is on something. This is all you need to know about mass for now; we will talk more about mass and how it differs from weight in class, so you can probably skip this section.

Mass

The property of a body to remain at rest or to remain in motion with constant velocity is called **inertia**. Newton’s first law is often called the **law of inertia**. As we know from experience, some objects have more inertia than others. It is obviously more difficult to change the motion of a large boulder than that of a basketball, for example. The inertia of an object is measured by its **mass**. Roughly speaking, mass is a measure of the amount of “stuff” (or matter) in something. The quantity or amount of matter in an object is determined by the numbers of atoms and molecules of various types it contains. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon. In practice, it is very difficult to count and identify all of the atoms and molecules in an object, so masses are not often determined in this manner. Operationally, the masses of objects are determined by comparison with the standard kilogram.

Check Your Understanding

Which has more mass: a kilogram of cotton balls or a kilogram of gold?

Solution

They are equal. A kilogram of one substance is equal in mass to a kilogram of another substance. The quantities that might differ between them are volume and density.

1.5 Newton's Second Law - Concept of a System

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Explaining Newton’s Laws in simple terms.
- Given all of the forces acting on an object, applying Newton’s Laws to determine the acceleration of an object at any given instant
- Given a net acceleration, predicting the direction and magnitude of the net force acting on an object

the most fundamental concepts in this entire course, so be sure to give this section a thorough read,

Newton’s second law of motion is closely related to Newton’s first law of motion. It mathematically states the cause and effect relationship between force and changes in motion. Newton’s second law of motion is more quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton’s second law as a simple equation giving the exact relationship of force, mass, and acceleration, we need to sharpen some ideas that have already been mentioned.

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is an **acceleration**. Newton’s first law says that a net external force causes a change in motion; thus, we see that a *net external force causes acceleration*.

Another question immediately arises. What do we mean by an external force? An intuitive notion of external is correct—an **external force** acts from outside the **system** of interest. For example, in **Figure 1.5**(a) the system of interest is the wagon plus the child in it. The two forces exerted by the other children are external forces. An internal force acts between elements of the system. Again looking at **Figure 1.5**(a), the force the child in the wagon exerts to hang onto the wagon is an internal force between elements of the system of interest. Only external forces affect the motion of a system, according to Newton’s first law. (The internal forces actually cancel, as we shall see in the next section.) *You must define the boundaries of the system before you can determine which forces are external*. Sometimes the system is obvious, whereas other times identifying the boundaries of a system is more subtle. The concept of a system is fundamental to many areas of physics, as is the correct application of Newton’s laws. This concept will be revisited many times on our journey through physics.

UMASS AMHERST Instructor's Notes

Note the free body diagrams here. Again, don't worry about being an expert on creating them and using them; we will go over that in class.

(a) A boy in a wagon is pushed by two girls toward the right. The force on the boy is represented by vector F one toward the right, and the force on the wagon is represented by vector F two in the same direction. Acceleration a is shown by a vector a toward the right and a friction force f is acting in the opposite direction, represented by a vector pointing toward the left. The weight W of the wagon is shown by a vector acting downward, and the normal force acting upward on the wagon is represented by a vector N. A free-body diagram is also shown, with F one and F two represented by arrows in the same direction toward the right and f represented by an arrow toward the left, so the resultant force F net is represented by an arrow toward the right. W is represented by an arrow downward and N is represented by an arrow upward; both the arrows have same length. (b) A boy in a wagon is pushed by a woman with a force F adult, represented by an arrow pointing toward the right. A vector a-prime, represented by an arrow, depicts acceleration toward the right. Friction force, represented by a vector f, acts toward the left. The weight of the wagon W is shown by a vector pointing downward, and the Normal force, represented by a vector N having same length as W, acts upward. A free-body diagram for this situation shows force F represented by an arrow pointing to the right having a large length; a friction force vector represented by an arrow f pointing left has a small length. The weight W is represented by an arrow pointing downward, and the normal force N, is represented by an arrow pointing upward, having the same length as W.

Figure 1.5 Different forces exerted on the same mass produce different accelerations. (a) Two children push a wagon with a child in it. Arrows representing all external forces are shown. The system of interest is the wagon and its rider. The weight $\mathbf{w}$ of the system and the support of the ground $\mathbf{N}$ are also shown for completeness and are assumed to cancel. The vector $\mathbf{f}$ represents the friction acting on the wagon, and it acts to the left, opposing the motion of the wagon. (b) All of the external forces acting on the system add together to produce a net force, $\mathbf{F}_{\text{net}}$. The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, we draw the vectors collinearly. (c) A larger net external force produces a larger acceleration ($\mathbf{a}' > \mathbf{a}$) when an adult pushes the child.

Now, it seems reasonable that acceleration should be directly proportional to and in the same direction as the net (total) external force acting on a system. This assumption has been verified experimentally and is illustrated in **Figure 1.5**. In part (a), a smaller force causes a smaller acceleration than the larger force illustrated in part (c). For completeness, the vertical forces are also shown; they are assumed to cancel since there is no acceleration in the vertical direction. The vertical forces are the weight $\mathbf{w}$ and the support of the ground $\mathbf{N}$, and the horizontal force $\mathbf{f}$ represents the force of friction. These will be discussed in more detail in later sections. For now, we will define **friction** as a force that opposes the motion past each other of objects that are touching. **Figure 1.5**(b) shows how vectors representing the external forces add together to produce a net force, $\mathbf{F}_{\text{net}}$.

To obtain an equation for Newton’s second law, we first write the relationship of acceleration and net external force as the proportionality

$$\mathbf{a} \propto \mathbf{F}_{\text{net}}, \tag{1.1}$$

where the symbol $\propto$ means “proportional to,” and $\mathbf{F}_{\text{net}}$ is the **net external force**. (The net external force is the vector sum of all external forces and can be determined graphically, using the head-to-tail method, or analytically, using components. The techniques are the same as for the addition of other vectors, and are covered in **Two-Dimensional Kinematics (https://legacy.cnx.org/content/m42126/latest/)** .) This proportionality states what we have said in words—*acceleration is directly proportional to the net external force*. Once the system of interest is chosen, it is important to identify the external forces and ignore the internal ones. It is a tremendous simplification not to have to consider the numerous internal forces acting between objects within the system, such as muscular forces within the child’s body, let alone the myriad of forces between atoms in the objects, but by doing so, we can easily solve some very complex problems with only minimal error due to our simplification

Now, it also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. And indeed, as illustrated in **Figure 1.6**, the same net external force applied to a car produces a much smaller acceleration than when applied to a basketball. The proportionality is written as

$$\mathbf{a} \propto \frac{1}{m} \tag{1.2}$$

where m is the mass of the system. Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is exactly linearly proportional to the net external force.

(a) A basketball player pushes the ball with the force shown by a vector F toward the right and an acceleration a-one represented by an arrow toward the right. M sub one is the mass of the ball. (b) The same basketball player is pushing a car with the same force, represented by the vector F towards the right, resulting in an acceleration shown by a vector a toward the right. The mass of the car is m sub two. The acceleration in the second case, a sub two, is represented by a shorter arrow than in the first case, a sub one.

Figure 1.6 The same force exerted on systems of different masses produces different accelerations. (a) A basketball player pushes on a basketball to make a pass. (The effect of gravity on the ball is ignored.) (b) The same player exerts an identical force on a stalled SUV and produces a far smaller acceleration (even if friction is negligible). (c) The free-body diagrams are identical, permitting direct comparison of the two situations. A series of patterns for the free-body diagram will emerge as you do more problems.

It has been found that the acceleration of an object depends *only* on the net external force and the mass of the object. Combining the two proportionalities just given yields Newton's second law of motion.

UMASS AMHERST Instructor's Notes

The block below is the statement of Newton's Second Law. It's a very simple looking law, but it has far reaching implications. Don't skip to this block; be sure to understand the reasoning behind the law.

Newton's Second Law of Motion

The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.

In equation form, Newton's second law of motion is

$$\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}. \tag{1.3}$$

This is often written in the more familiar form

$$\mathbf{F}_{\text{net}} = m\mathbf{a}. \tag{1.4}$$

When only the magnitude of force and acceleration are considered, this equation is simply

$$F_{\text{net}} = ma. \tag{1.5}$$

Although these last two equations are really the same, the first gives more insight into what Newton's second law means. The law is a *cause and effect relationship* among three quantities that is not simply based on their definitions. The validity of the second law is completely based on experimental verification.

Units of Force

$\mathbf{F}_{\text{net}} = m\mathbf{a}$ is used to define the units of force in terms of the three basic units for mass, length, and time. The SI unit of force is called the **newton** (abbreviated N) and is the force needed to accelerate a 1-kg system at the rate of 1m/s^2. That is, since $\mathbf{F}_{\text{net}} = m\mathbf{a}$,

$$1\text{ N} = 1\text{ kg}\cdot\text{m/s}^2. \tag{1.6}$$

While almost the entire world uses the newton for the unit of force, in the United States the most familiar unit of force is the pound (lb), where 1 N = 0.225 lb.

UMASS AMHERST Instructor's Notes

Feel free to skip the following section. We will cover gravity and weight in class. However, do take a look at examples 4.1 and 4.2 in this section. The problems in those examples are the kind of problems you are expected to be able to do in the homeworks and on the quizzes, so if you want to get a good sense of them, be sure to read them through.

Weight and the Gravitational Force

When an object is dropped, it accelerates toward the center of Earth. Newton's second law states that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its **weight** $\mathbf{w}$. Weight can be denoted as a vector $\mathbf{w}$ because it has a direction; *down* is, by definition, the direction of gravity, and hence weight is a downward force. The magnitude of weight is denoted as w. Galileo was instrumental in showing that, in the absence of air resistance, all objects fall with the same acceleration g. Using Galileo's result and Newton's second law, we can derive an equation for weight.

Consider an object with mass m falling downward toward Earth. It experiences only the downward force of gravity, which has magnitude w. Newton's second law states that the magnitude of the net external force on an object is $F_{\text{net}} = ma$.

Since the object experiences only the downward force of gravity, $F_{\text{net}} = w$. We know that the acceleration of an object due to gravity is g, or $a = g$. Substituting these into Newton's second law gives

Weight

This is the equation for *weight*—the gravitational force on a mass m:

$$w = mg. \tag{1.7}$$

Since $g = 9.80\text{ m/s}^2$ on Earth, the weight of a 1.0 kg object on Earth is 9.8 N, as we see:

$$w = mg = (1.0\text{ kg})(9.80\text{ m/s}^2) = 9.8\text{ N}. \tag{1.8}$$

Recall that g can take a positive or negative value, depending on the positive direction in the coordinate system. Be sure to take this into consideration when solving problems with weight.

When the net external force on an object is its weight, we say that it is in **free-fall**. That is, the only force acting on the object is the force of gravity. In the real world, when objects fall downward toward Earth, they are never truly in free-fall because there is always some upward force from the air acting on the object.

The acceleration due to gravity g varies slightly over the surface of Earth, so that the weight of an object depends on location and is not an intrinsic property of the object. Weight varies dramatically if one leaves Earth's surface. On the Moon, for example, the acceleration due to gravity is only 1.67 m/s^2. A 1.0-kg mass thus has a weight of 9.8 N on Earth and only about 1.7 N on the Moon.

The broadest definition of weight in this sense is that *the weight of an object is the gravitational force on it from the nearest large body*, such as Earth, the Moon, the Sun, and so on. This is the most common and useful definition of weight in physics. It differs dramatically, however, from the definition of weight used by NASA and the popular media in relation to space travel and exploration. When they speak of "weightlessness" and "microgravity," they are really referring to the phenomenon we call "free-fall" in physics. We shall use the above definition of weight, and we will make careful distinctions between free-fall and actual weightlessness.

It is important to be aware that weight and mass are very different physical quantities, although they are closely related. Mass is the quantity of matter (how much "stuff") and does not vary in classical physics, whereas weight is the gravitational force and does vary depending on gravity. It is tempting to equate the two, since most of our examples take place on Earth, where the weight of an object only varies a little with the location of the object. Furthermore, the terms *mass* and *weight* are used interchangeably in everyday language; for example, our medical records often show our "weight" in kilograms, but never in the correct units of newtons.

Common Misconceptions: Mass vs. Weight

Mass and weight are often used interchangeably in everyday language. However, in science, these terms are distinctly different from one another. Mass is a measure of how much matter is in an object. The typical measure of mass is the kilogram (or the "slug" in English units). Weight, on the other hand, is a measure of the force of gravity acting on an object. Weight is equal to the mass of an object (m) multiplied by the acceleration due to gravity (g). Like any other force, weight is measured in terms of newtons (or pounds in English units).

Assuming the mass of an object is kept intact, it will remain the same, regardless of its location. However, because weight depends on the acceleration due to gravity, the weight of an object *can change* when the object enters into a region with stronger or weaker gravity. For example, the acceleration due to gravity on the Moon is 1.67 m/s^2 (which is much less than the acceleration due to gravity on Earth, 9.80 m/s^2). If you measured your weight on Earth and then measured your weight on the Moon, you would find that you "weigh" much less, even though you do not look any skinnier. This is because the force of gravity is weaker on the Moon. In fact, when people say that they are "losing weight," they really mean that they are losing "mass" (which in turn causes them to weigh less).

Take-Home Experiment: Mass and Weight

What do bathroom scales measure? When you stand on a bathroom scale, what happens to the scale? It depresses slightly. The scale contains springs that compress in proportion to your weight—similar to rubber bands expanding when pulled. The springs provide a measure of your weight (for an object which is not accelerating). This is a force in newtons (or pounds). In most countries, the measurement is divided by 9.80 to give a reading in mass units of kilograms. The scale measures weight but is calibrated to provide information about mass. While standing on a bathroom scale, push down on a table next to you. What happens to the reading? Why? Would your scale measure the same "mass" on Earth as on the Moon?

Example 1.1 What Acceleration Can a Person Produce when Pushing a Lawn Mower?

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb) parallel to the ground. The mass of the mower is 24 kg. What is its acceleration?

A man pushing a lawnmower to the right. A red vector above the lawnmower is pointing to the right and labeled F sub net.

Figure 1.7 The net force on a lawn mower is 51 N to the right. At what rate does the lawn mower accelerate to the right?

Strategy

Since $\mathbf{F}_{\text{net}}$ and m are given, the acceleration can be calculated directly from Newton's second law as stated in $\mathbf{F}_{\text{net}} = m\mathbf{a}$.

Solution

The magnitude of the acceleration a is $a = \frac{F_{\text{net}}}{m}$. Entering known values gives

$$a = \frac{51\ \text{N}}{24\ \text{kg}} \tag{1.9}$$

Substituting the units $\text{kg} \cdot \text{m/s}^2$ for N yields

$$a = \frac{51\ \text{kg} \cdot \text{m/s}^2}{24\ \text{kg}} = 2.1\ \text{m/s}^2. \tag{1.10}$$

Discussion

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moves forward), and the vertical forces must cancel if there is to be no acceleration in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long because the person's top speed would soon be reached.

Example 1.2 What Rocket Thrust Accelerates This Sled?

Prior to manned space flights, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets. Calculate the magnitude of force exerted by each rocket, called its thrust $\mathbf{T}$, for the four-rocket propulsion system shown in **Figure 1.8**. The sled's initial acceleration is $49\ \text{m/s}^2,$ the mass of the system is 2100 kg, and the force of friction opposing the motion is known to be 650 N.

A sled is shown with four rockets, each producing the same thrust, represented by equal length arrows labeled as vector T pushing the sled toward the right. Friction force is represented by an arrow labeled as vector f pointing toward the left on the sled. The weight of the sled is represented by an arrow labeled as vector W, shown pointing downward, and the normal force is represented by an arrow labeled as vector N having the same length as W acting upward on the sled. A free-body diagram is also shown for the situation. Four arrows of equal length representing vector T point toward the right, a vector f represented by a smaller arrow points left, vector N is an arrow pointing upward, and the weight W is an arrow of equal length pointing downward.

Figure 1.8 A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust $\mathbf{T}$. As in other situations where there is only horizontal acceleration, the vertical forces cancel. The ground exerts an upward force $\mathbf{N}$ on the system that is equal in magnitude and opposite in direction to its weight, $\mathbf{w}$. The system here is the sled, its rockets, and rider, so none of the forces *between* these objects are considered. The arrow representing friction ($\mathbf{f}$) is drawn larger than scale.

Strategy

Although there are forces acting vertically and horizontally, we assume the vertical forces cancel since there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem. Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in the figure.

Solution

Since acceleration, mass, and the force of friction are given, we start with Newton’s second law and look for ways to find the thrust of the engines. Since we have defined the direction of the force and acceleration as acting “to the right,” we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with

$$F_{\text{net}} = ma, \tag{1.11}$$

where F_{net} is the net force along the horizontal direction. We can see from **Figure 1.8** that the engine thrusts add, while friction opposes the thrust. In equation form, the net external force is

$$F_{\text{net}} = 4T - f. \tag{1.12}$$

Substituting this into Newton’s second law gives

$$F_{\text{net}} = ma = 4T - f. \tag{1.13}$$

Using a little algebra, we solve for the total thrust 4*T*:

$$4T = ma + f. \tag{1.14}$$

Substituting known values yields

$$4T = ma + f = (2100\ \text{kg})(49\ \text{m/s}^2) + 650\ \text{N}. \tag{1.15}$$

So the total thrust is

$$4T = 1.0\times10^5\ \text{N}, \tag{1.16}$$

and the individual thrusts are

$$T = \frac{1.0\times10^5\ \text{N}}{4} = 2.6\times10^4\ \text{N}. \tag{1.17}$$

Discussion

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance and the setup designed to protect human subjects in jet fighter emergency ejections. Speeds of 1000 km/h were obtained, with accelerations of 45 g 's. (Recall that g , the acceleration due to gravity, is $9.80\ \text{m/s}^2$. When we say that an acceleration is 45 g 's, it is $45\times9.80\ \text{m/s}^2$, which is approximately $440\ \text{m/s}^2$.) While living subjects are not used any more, land speeds of 10,000 km/h have been obtained with rocket sleds. In this example, as in the preceding one, the system of interest is obvious. We will see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton’s second law of motion is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature. The next section introduces the third and final law of motion.

1.6 Simulations: Iterative Force Calculations

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Using iterative methods to determine the motion of an object acted on by a force

Links

- **Iterative Force Calcluations (https://www.youtube.com/watch?v=QFhYJrsDIP0)**
- **Iterative Force Calculations - A more complex example (https://www.youtube.com/watch?v=6v4kVX_8VF8)**

Iterative Force Calculations Example

In the last unit, we have looked at describing motion using iterative methods given an acceleration. We can solve for the position and the velocity step by step. Now, we want to add the idea of this unit, forces which are the causes of acceleration. A key principle in solving Newton's second law problems iteratively is the idea of object egoism, discussed earlier in this preparation. Object egoism can be summarized through the idea of "me me me" and right now. This can be restated as that only the forces that are acting at any given instant on an object are relevant. What happened before, what happened in the future don't really matter. The easiest way to learn how to solve Newton's second law problems through iterative methods is probably through an example.

So, let's begin with our first example. Say we have a 1000-kilogram car stopped at stoplight. When the light turns green, the engine of the car will begin to apply a constant force of 5000 Newtons to the car. We want to model the motion of this car for the first 0.02 seconds using iterative methods with a .01 second step. We begin by constructing our table, where we have the usual columns of time, position, velocity, and acceleration.

Table 1.1

Time (s)	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Force (N)
0				
0.01				
0.02				

We've now added a new column force with the units Newtons, and we have our times 0.01, because we're working at a 0.01 second step, and then .02 seconds, which is as far as we care to go for this particular problem. Let's begin with t=0, thinking about one instant at a time. What is happening with the car at t=0? Well, we can define the stoplight to be at position equals 0, so we'll do so. We also know that the car is stopped, which tells us that the initial velocity of the car is also 0. What else do we know at t=0? Well we know that the engine is providing a net force of 5000 newtons to the car we can now use this net force to solve for the acceleration of the car using Newton's second law, F=ma, or rearranged, the acceleration is the force applied divided by the mass of the car. In this particular case, the force applied is 5,000 newtons, and the mass is 1,000 kilograms, giving us an acceleration of 5 meters per second squared.

Table 1.2

Time (s)	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Force (N)
0	0	0	5	5000
0.01				
0.02				

Now let's move on to the next instant in time, .01 seconds. Again, we know that the force that the engine is applying to the car is a constant 5,000 newtons, so we can just put in 5,000 newtons for the force applied to the car. We can solve for the acceleration of the car in the same way, using F=ma, which again we have 5,000 newtons divided by 1,000 kilograms, giving us again an acceleration of 5 m/s^2. The next thing we might be interested in is the velocity of the car. Now we think back to how we solve problems iteratively, a key principle that one instant predicts the next. So, in this case we're going to use t=0 to predict t=0.01. We're going to use the fundamental definition of acceleration as $\Delta v/\Delta t$. We can explode the Δv into v_{final}-$v_{initial}$ and rearrange the equation into this form:$v_{final}=v_{initial}-a\Delta t$Our initial velocity is 0, the acceleration is 5 m/s^2, from the table above, and our Δt is 0.01, the time step. Substituting in these values gives a final velocity of 0.05 m/s.

The next thing we might be interested in is to solve for the position. Again, one instant predicts the next, so we're going to use t=0 to predict t=0.01. This time, we're going to use the fundamental definition of velocity as $\Delta x/\Delta t$. We expand the Δx into x_{final}-$x_{initial}$, and do some algebraic manipulation to get this familiar form.$x_{final}=x_{initial}-v\Delta t$We repeat the same process as above.

Initial position is 0, velocity is 0, and our Δt is 0.01, and substituting these values in gives us a final position of 0.

Table 1.3

Time (s)	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Force (N)
0	0	0	5	5000
0.01	0	0.05	5	5000
0.02				

Let’s think about the general procedure. First, we identify what is the force at any given instant. Second, we think about translating that force to the acceleration using Newton’s second law. Third, we move into solving for the velocity. In this case, we use one instant to predict the next, and the definition of acceleration. Finally, we move into calculating the velocity, the position, where again, one instant predicts the next, and we use the fundamental definition of velocity. Repeating this process, we can finish the table:

Table 1.4

Time (s)	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Force (N)
0	0	0	5	5000
0.01	0	0.05	5	5000
0.02	0.0005	0.1	5	5000

A More Complex Example

Let’s have a look at a second more complex example.

An object of mass five kilograms being acted upon by the empirical force law, F=-kx, where x is the position of the object, and k, equal to 50 N/m, is a constant measured from the data. The object begins at 0.1 meters with a speed of 2 m/s. Solve for the motion of the object at 0.01s iteratively in 0.01s steps.

We want to solve for the motion of the object iteratively, for 0.01 seconds using a step of 0.01 seconds, so our table will only have two rows, t=0 and t=0.01 with the usual columns, time, position, velocity, acceleration, and force.

Table 1.5

Time (s)	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Force (N)
0				
0.01				

Let’s begin with t=0. What do we know? Well, we know that the object is initially at 0.1 meters, so we can substitute in that value, and we know its initial speed is 2 m/s, so we can substitute in that value. Now we move on to the force in this problem. The force is not a simple number; it’s a function F=-kx. We know what K is, it’s 50. It’s given to us in the problem. But now we got to think about x: which x should we use? Well, the idea of object egoism tells us “me me me” and right now, so I need to think about what’s going on with the object right now. Right now, the object is at 0.1 meters, and so we substitute 0.1 meters in for x. Solving the problem, we get a force of -5 N. Now we can move on to solving for the acceleration using Newton’s second law, F=ma. A -5 N force divided by 5 kilograms gives us an acceleration of -1 m/s^2.

Table 1.6

Time (s)	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Force (N)
0	0.1	2	-1	-5
0.01				

So, our force and our acceleration are in opposite directions. From our velocity, our velocity is positive, acceleration is negative. Thus, from our Unit 1 knowledge, we can predict that the object should probably slow down. When we go from 0.00 to 0.01. Now, in the last problem, we started with force, so let’s try that again. Our force law is still -kx. We still know that k is equal to 50, but we don’t know what x is. We need to use “me me me” and right now. We don’t know the position of the object right now. Sure, we knew where it was, but that’s not what matters in physics, what matters is what’s going on to the object right now. Objects are stupid for the most part; they don’t remember, so we need to think what’s going on right now, and we don’t know, so consequently we should probably solve for position first, using the definition of velocity expanded into this usual form. We start looking at plugging in the numbers. The initial position is 0.1 m, the initial velocity is 2 m/s, and the Δt from 0 to 0.01 is 0.01, which comes out to 0.12 meters. Now that we have a position, we can use this position in our force law to solve for the force, and get a force of -6 N. Now we can continue in our more usual way of using F=ma to solve for the acceleration, -6 newtons divided by 5 kilograms will give us an acceleration of -1.2 m/s^2. Finally, we have to deal with the velocity, and we use the definition of acceleration, expanded into this typical algebraic form, and we look at substituting our numbers. The initial velocity over this interval is 2 so we substitute 2 m/s. Our initial acceleration is - 1 m/s^2, and our Δt is 0.01 s. Turning the crank on these numbers, we get a velocity of 1.99 m/s, so our object has slowed down in agreement with our expectations.

Table 1.7

Time (s)	Position (m)	Velocity (m/s)	Acceleration (m/s^2)	Force (N)
0	0.1	2	-1	-5
0.01	0.1	1.99	-1.2	-6

Our object went from 2 m/s to 1.99 m/s, which is what we expect, given that at t=0, our velocity and our acceleration are in opposite directions. So, let's conclude. Many of the procedures that we've discussed in this video are like the iterative calculations we have already discussed in unit one. Remember to think about one instant at a time, and to use one instant to predict the next. The new part introduced in this video is that the acceleration at a given instant is determined by the force at that same instant so we use the force at t=0.003 to solve for the acceleration. This is in line with our object egoism of "me me me" and right now. Similarly, if the force depends upon other variables such as position or velocity then I need to use the values for the same instant for which I want to calculate the force. So, if I want to calculate the force at 0.004 seconds and it depends upon position, then I need to use the position at 0.004 s. Again, "me me me" and right now.

1.7 Newton's Third Law - Symmetry in Forces

UMASS AMHERST Instructor's Notes

This section is here for your reference; we will be going over this material in class, so this reading is not required.

There is a passage in the musical *Man of la Mancha* that relates to Newton's third law of motion. Sancho, in describing a fight with his wife to Don Quixote, says, "Of course I hit her back, Your Grace, but she's a lot harder than me and you know what they say, 'Whether the stone hits the pitcher or the pitcher hits the stone, it's going to be bad for the pitcher.'" This is exactly what happens whenever one body exerts a force on another—the first also experiences a force (equal in magnitude and opposite in direction). Numerous common experiences, such as stubbing a toe or throwing a ball, confirm this. It is precisely stated in **Newton's third law of motion**.

Newton's Third Law of Motion

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts.

This law represents a certain *symmetry in nature*: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as "action-reaction," where the force exerted is the action and the force experienced as a consequence is the reaction. Newton's third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system.

We can readily see Newton's third law at work by taking a look at how people move about. Consider a swimmer pushing off from the side of a pool, as illustrated in **Figure 1.9**. She pushes against the pool wall with her feet and accelerates in the direction *opposite* to that of her push. The wall has exerted an equal and opposite force back on the swimmer. You might think that two equal and opposite forces would cancel, but they do not *because they act on different systems*. In this case, there are two systems that we could investigate: the swimmer or the wall. If we select the swimmer to be the system of interest, as in the figure, then $\mathbf{F}_{\text{wall on feet}}$ is an external force on this system and affects its motion. The swimmer moves in the direction of $\mathbf{F}_{\text{wall on feet}}$. In contrast, the force $\mathbf{F}_{\text{feet on wall}}$ acts on the wall and not on our system of interest. Thus $\mathbf{F}_{\text{feet on wall}}$ does not directly affect the motion of the system and does not cancel $\mathbf{F}_{\text{wall on feet}}$. Note that the swimmer pushes in the direction opposite to that in which she wishes to move. The reaction to her push is thus in the desired direction.

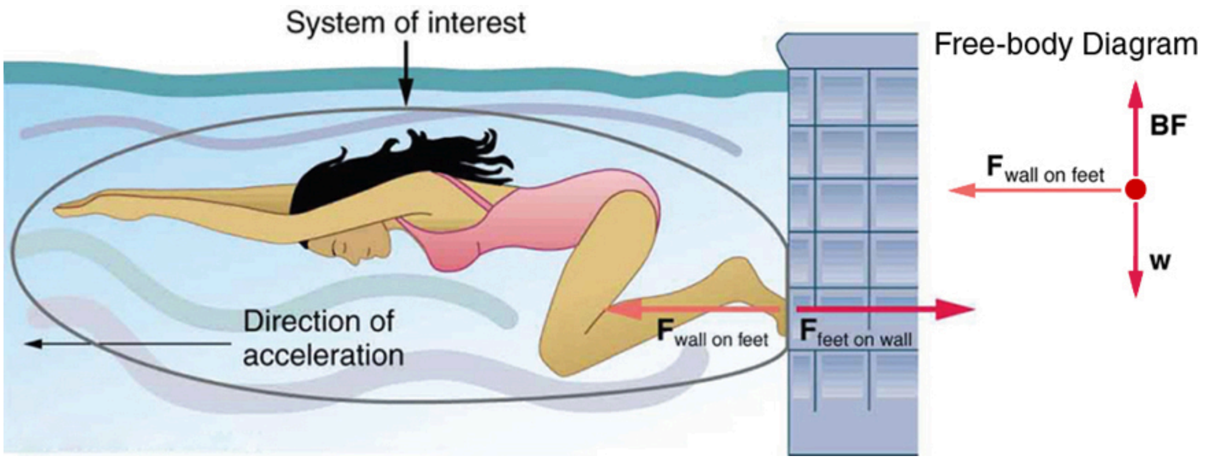


Figure 1.9 When the swimmer exerts a force $\mathbf{F}_{\text{feet on wall}}$ on the wall, she accelerates in the direction opposite to that of her push. This means the net external force on her is in the direction opposite to $\mathbf{F}_{\text{feet on wall}}$. This opposition occurs because, in accordance with Newton's third law of motion, the wall exerts a force $\mathbf{F}_{\text{wall on feet}}$ on her, equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Note that $\mathbf{F}_{\text{feet on wall}}$ does not act on this system (the swimmer) and, thus, does not cancel $\mathbf{F}_{\text{wall on feet}}$. Thus the free-body diagram shows only $\mathbf{F}_{\text{wall on feet}}$, $\mathbf{w}$, the gravitational force, and $\mathbf{BF}$, the buoyant force of the water supporting the swimmer's weight. The vertical forces $\mathbf{w}$ and $\mathbf{BF}$ cancel since there is no vertical motion.

Other examples of Newton's third law are easy to find. As a professor paces in front of a whiteboard, she exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes her to accelerate forward. Similarly, a car accelerates because the ground pushes forward on the drive wheels in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw rocks backward. In another example, rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber, and the gas therefore exerts a large reaction force forward on the rocket. This reaction force is called **thrust**. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases. Helicopters similarly create lift by pushing air down, thereby experiencing an upward reaction force. Birds and airplanes also fly by exerting force on air in a direction opposite to that of whatever force they need. For example, the wings of a bird force air downward and backward in order to get lift and move forward. An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski. In a situation similar to Sancho's, professional cage fighters experience reaction forces when they punch, sometimes breaking their hand by hitting an opponent's body.

Example 1.3 Getting Up To Speed: Choosing the Correct System

A physics professor pushes a cart of demonstration equipment to a lecture hall, as seen in **Figure 1.10**. Her mass is 65.0 kg, the cart's is 12.0 kg, and the equipment's is 7.0 kg. Calculate the acceleration produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart's wheels and air resistance, total 24.0 N.

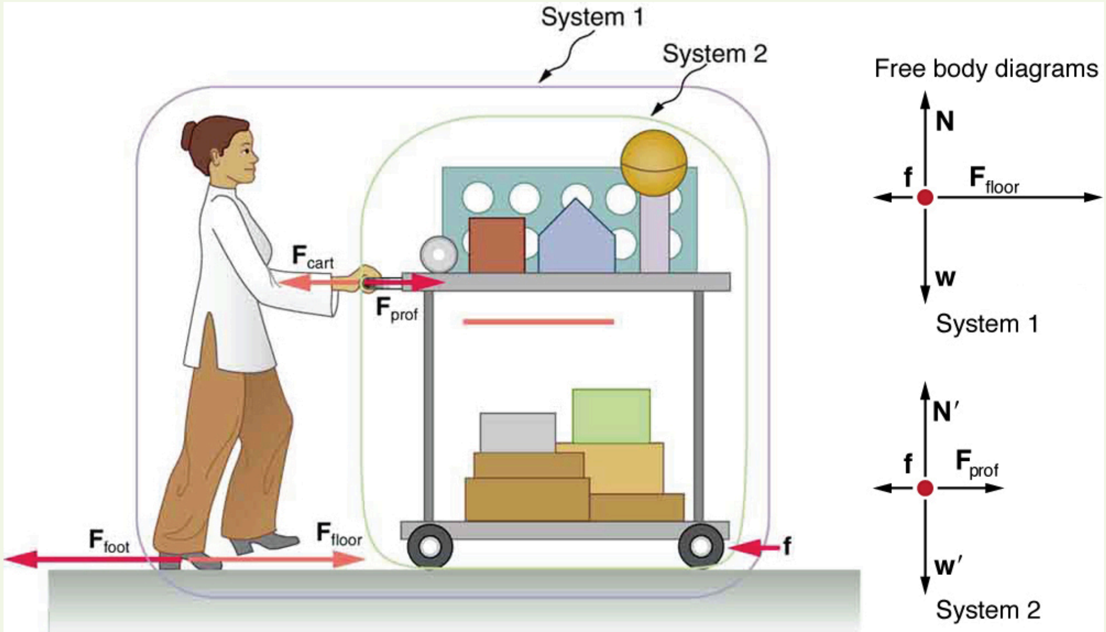


Figure 1.10 A professor pushes a cart of demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for $\mathbf{f}$, since it is too small to draw to scale). Different questions are asked in each example; thus, the system of interest must be defined differently for each. System 1 is appropriate for **Example 1.4**, since it asks for the acceleration of the entire group of objects. Only $\mathbf{F}_{\text{floor}}$ and $\mathbf{f}$ are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for this example so that $\mathbf{F}_{\text{prof}}$ will be an external force and enter into Newton's second law. Note that the free-body diagrams, which allow us to apply Newton's second law, vary with the system chosen.

Strategy

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in **Figure 1.10**. The professor pushes backward with a force $\mathbf{F}_{\text{foot}}$ of 150 N. According to Newton's third law, the floor exerts a forward reaction force $\mathbf{F}_{\text{floor}}$ of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. The problem is therefore one-dimensional along the horizontal direction. As noted, $\mathbf{f}$ opposes the motion and is thus in the opposite direction of $\mathbf{F}_{\text{floor}}$. Note that we do not include the forces $\mathbf{F}_{\text{prof}}$ or $\mathbf{F}_{\text{cart}}$ because these are internal forces, and we do not include $\mathbf{F}_{\text{foot}}$ because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton's second law to find the acceleration as requested. See the free-body diagram in the figure.

Solution

Newton's second law is given by

$$a = \frac{F_{\text{net}}}{m}. \tag{1.18}$$

The net external force on System 1 is deduced from **Figure 1.10** and the discussion above to be

$$F_{\text{net}} = F_{\text{floor}} - f = 150\text{ N} - 24.0\text{ N} = 126\text{ N}. \tag{1.19}$$

The mass of System 1 is

$$m = (65.0 + 12.0 + 7.0)\text{ kg} = 84\text{ kg}. \tag{1.20}$$

These values of F_{net} and m produce an acceleration of

$$\begin{aligned} a &= \frac{F_{\text{net}}}{m}, \\ a &= \frac{126\text{ N}}{84\text{ kg}} = 1.5\text{ m/s}^2. \end{aligned} \tag{1.21}$$

Discussion

None of the forces between components of System 1, such as between the professor's hands and the cart, contribute to the net external force because they are internal to System 1. Another way to look at this is to note that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the professor on the cart results in an equal and opposite force back on her. In this case both forces act on the same system and, therefore, cancel. Thus internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

Example 1.4 Force on the Cart—Choosing a New System

Calculate the force the professor exerts on the cart in **Figure 1.10** using data from the previous example if needed.

Strategy

If we now define the system of interest to be the cart plus equipment (System 2 in **Figure 1.10**), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart, $\mathbf{F}_{\text{prof}}$, is an external force acting on System 2. $\mathbf{F}_{\text{prof}}$ was internal to System 1, but it is external to System 2 and will enter Newton's second law for System 2.

Solution

Newton's second law can be used to find $\mathbf{F}_{\text{prof}}$. Starting with

$$a = \frac{F_{\text{net}}}{m} \tag{1.22}$$

and noting that the magnitude of the net external force on System 2 is

$$F_{\text{net}} = F_{\text{prof}} - f, \tag{1.23}$$

we solve for F_{prof}, the desired quantity:

$$F_{\text{prof}} = F_{\text{net}} + f. \tag{1.24}$$

The value of f is given, so we must calculate net F_{net}. That can be done since both the acceleration and mass of System 2 are known. Using Newton's second law we see that

$$F_{\text{net}} = ma, \tag{1.25}$$

where the mass of System 2 is 19.0 kg (m = 12.0 kg + 7.0 kg) and its acceleration was found to be $a = 1.5\ \text{m/s}^2$ in the previous example. Thus,

$$F_{\text{net}} = ma, \tag{1.26}$$

$$F_{\text{net}} = (19.0\ \text{kg})(1.5\ \text{m/s}^2) = 29\ \text{N}. \tag{1.27}$$

Now we can find the desired force:

$$F_{\text{prof}} = F_{\text{net}} + f, \tag{1.28}$$

$$F_{\text{prof}} = 29\ \text{N}+24.0\ \text{N} = 53\ \text{N}. \tag{1.29}$$

Discussion

It is interesting that this force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor.

The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which is not necessarily the same thing).

PhET Explorations: Gravity Force Lab

Visualize the gravitational force that two objects exert on each other. Change properties of the objects in order to see how it changes the gravity force.

PhET Interactive Simulation

Figure 1.11 Gravity Force Lab (http://legacy.cnx.org/content/m64297/1.1/gravity-force-lab_en.jar)

Glossary

acceleration: the rate at which an object's velocity changes over a period of time

dynamics: the study of how forces affect the motion of objects and systems

external force: a force acting on an object or system that originates outside of the object or system

force: a push or pull on an object with a specific magnitude and direction; can be represented by vectors; can be expressed as a multiple of a standard force

free-body diagram: a sketch showing all of the external forces acting on an object or system; the system is represented by a dot, and the forces are represented by vectors extending outward from the dot

free-fall: a situation in which the only force acting on an object is the force due to gravity

friction: a force past each other of objects that are touching; examples include rough surfaces and air resistance

inertia: the tendency of an object to remain at rest or remain in motion

law of inertia: see Newton's first law of motion

mass: the quantity of matter in a substance; measured in kilograms

net external force: the vector sum of all external forces acting on an object or system; causes a mass to accelerate

Newton's first law of motion: a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force; also known as the law of inertia

Newton's second law of motion: the net external force $\mathbf{F}_{\text{net}}$ on an object with mass m is proportional to and in the same direction as the acceleration of the object, $\mathbf{a}$, and inversely proportional to the mass; defined mathematically as $\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$

Newton's third law of motion: whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts

system: defined by the boundaries of an object or collection of objects being observed; all forces originating from outside of the system are considered external forces

thrust: a reaction force that pushes a body forward in response to a backward force; rockets, airplanes, and cars are pushed forward by a thrust reaction force

weight: the force $\mathbf{w}$ due to gravity acting on an object of mass m; defined mathematically as: $\boldsymbol{w} = m\boldsymbol{g}$, where $\mathbf{g}$ is the magnitude and direction of the acceleration due to gravity

Section Summary

1.2 Development of Force Concept

- **Dynamics** is the study of how forces affect the motion of objects.
- **Force** is a push or pull that can be defined in terms of various standards, and it is a vector having both magnitude and direction.
- **External forces** are any outside forces that act on a body. A **free-body diagram** is a drawing of all external forces acting on a body.

1.4 Newton's First Law - Inertia

- **Newton's first law of motion** states that a body at rest remains at rest, or, if in motion, remains in motion at a constant velocity unless acted on by a net external force. This is also known as the **law of inertia**.

- **Inertia** is the tendency of an object to remain at rest or remain in motion. Inertia is related to an object's mass.
- **Mass** is the quantity of matter in a substance.

1.5 Newton's Second Law - Concept of a System

- Acceleration, $\mathbf{a}$, is defined as a change in velocity, meaning a change in its magnitude or direction, or both.
- An external force is one acting on a system from outside the system, as opposed to internal forces, which act between components within the system.
- Newton's second law of motion states that the acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system, and inversely proportional to its mass.
- In equation form, Newton's second law of motion is $\mathbf{a} = \frac{\mathbf{F}_{\text{net}}}{m}$.
- This is often written in the more familiar form: $\mathbf{F}_{\text{net}} = m\mathbf{a}$.
- The weight $\mathbf{w}$ of an object is defined as the force of gravity acting on an object of mass m . The object experiences an acceleration due to gravity $\mathbf{g}$:

$$\mathbf{w} = m\mathbf{g}.$$

- If the only force acting on an object is due to gravity, the object is in free fall.
- Friction is a force that opposes the motion past each other of objects that are touching.

1.7 Newton's Third Law - Symmetry in Forces

- **Newton's third law of motion** represents a basic symmetry in nature. It states: Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that the first body exerts.
- A **thrust** is a reaction force that pushes a body forward in response to a backward force. Rockets, airplanes, and cars are pushed forward by a thrust reaction force.

Conceptual Questions

1.2 Development of Force Concept

1. Propose a force standard different from the example of a stretched spring discussed in the text. Your standard must be capable of producing the same force repeatedly.

2. What properties do forces have that allow us to classify them as vectors?

1.4 Newton's First Law - Inertia

3. How are inertia and mass related?

4. What is the relationship between weight and mass? Which is an intrinsic, unchanging property of a body?

1.5 Newton's Second Law - Concept of a System

5. Which statement is correct? (a) Net force causes motion. (b) Net force causes change in motion. Explain your answer and give an example.

6. Why can we neglect forces such as those holding a body together when we apply Newton's second law of motion?

7. Explain how the choice of the "system of interest" affects which forces must be considered when applying Newton's second law of motion.

8. Describe a situation in which the net external force on a system is not zero, yet its speed remains constant.

9. A system can have a nonzero velocity while the net external force on it *is* zero. Describe such a situation.

10. A rock is thrown straight up. What is the net external force acting on the rock when it is at the top of its trajectory?

11. (a) Give an example of different net external forces acting on the same system to produce different accelerations. (b) Give an example of the same net external force acting on systems of different masses, producing different accelerations. (c) What law accurately describes both effects? State it in words and as an equation.

12. If the acceleration of a system is zero, are no external forces acting on it? What about internal forces? Explain your answers.

13. If a constant, nonzero force is applied to an object, what can you say about the velocity and acceleration of the object?

14. The gravitational force on the basketball in Figure 1.6 is ignored. When gravity *is* taken into account, what is the direction of the net external force on the basketball—above horizontal, below horizontal, or still horizontal?

1.7 Newton's Third Law - Symmetry in Forces

15. When you take off in a jet aircraft, there is a sensation of being pushed back into the seat. Explain why you move backward in the seat—is there really a force backward on you? (The same reasoning explains whiplash injuries, in which the head is apparently thrown backward.)

16. A device used since the 1940s to measure the kick or recoil of the body due to heart beats is the “ballistocardiograph.” What physics principle(s) are involved here to measure the force of cardiac contraction? How might we construct such a device?

17. Describe a situation in which one system exerts a force on another and, as a consequence, experiences a force that is equal in magnitude and opposite in direction. Which of Newton’s laws of motion apply?

18. Why does an ordinary rifle recoil (kick backward) when fired? The barrel of a recoilless rifle is open at both ends. Describe how Newton’s third law applies when one is fired. Can you safely stand close behind one when it is fired?

19. An American football lineman reasons that it is senseless to try to out-push the opposing player, since no matter how hard he pushes he will experience an equal and opposite force from the other player. Use Newton’s laws and draw a free-body diagram of an appropriate system to explain how he can still out-push the opposition if he is strong enough.

20. Newton’s third law of motion tells us that forces always occur in pairs of equal and opposite magnitude. Explain how the choice of the “system of interest” affects whether one such pair of forces cancels.

Problems & Exercises

1.5 Newton's Second Law - Concept of a System

You may assume data taken from illustrations is accurate to three digits.

1. A 63.0-kg sprinter starts a race with an acceleration of $4.20\ \text{m/s}^2$. What is the net external force on him?

2. If the sprinter from the previous problem accelerates at that rate for 20 m, and then maintains that velocity for the remainder of the 100-m dash, what will be his time for the race?

3. A cleaner pushes a 4.50-kg laundry cart in such a way that the net external force on it is 60.0 N. Calculate the magnitude of its acceleration.

4. Since astronauts in orbit are apparently weightless, a clever method of measuring their masses is needed to monitor their mass gains or losses to adjust diets. One way to do this is to exert a known force on an astronaut and measure the acceleration produced. Suppose a net external force of 50.0 N is exerted and the astronaut's acceleration is measured to be $0.893\ \text{m/s}^2$. (a) Calculate her mass. (b) By exerting a force on the astronaut, the vehicle in which they orbit experiences an equal and opposite force. Discuss how this would affect the measurement of the astronaut's acceleration. Propose a method in which recoil of the vehicle is avoided.

5. In **Figure 1.7**, the net external force on the 24-kg mower is stated to be 51 N. If the force of friction opposing the motion is 24 N, what force F (in newtons) is the person exerting on the mower? Suppose the mower is moving at 1.5 m/s when the force F is removed. How far will the mower go before stopping?

6. The same rocket sled drawn in **Figure 1.12** is decelerated at a rate of $196\ \text{m/s}^2$. What force is necessary to produce this deceleration? Assume that the rockets are off. The mass of the system is 2100 kg.

A sled is shown with four rockets. Friction force is represented by an arrow labeled as vector f pointing toward the left on the sled. Weight of the sled is represented by an arrow labeled as vector W, shown pointing downward, and normal force is represented by an arrow labeled as vector N having the same length as W acting upward on the sled.

Figure 1.12

7. (a) If the rocket sled shown in **Figure 1.13** starts with only one rocket burning, what is the magnitude of its acceleration? Assume that the mass of the system is 2100 kg, the thrust T is 2.4×10^4 N, and the force of friction opposing the motion is known to be 650 N. (b) Why is the acceleration not one-fourth of what it is with all rockets burning?

A sled is shown with thrust represented by a vector T pushing the sled toward the right. Friction force is represented by an arrow labeled as vector f pointing toward the left on the sled. The weight of the sled is represented by an arrow labeled as vector W, shown pointing downward, and the normal force is represented by an arrow labeled as vector N having the same length as W acting upward on the sled.

Figure 1.13

8. What is the deceleration of the rocket sled if it comes to rest in 1.1 s from a speed of 1000 km/h? (Such deceleration caused one test subject to black out and have temporary blindness.)

9. Suppose two children push horizontally, but in exactly opposite directions, on a third child in a wagon. The first child exerts a force of 75.0 N, the second a force of 90.0 N, friction is 12.0 N, and the mass of the third child plus wagon is 23.0 kg. (a) What is the system of interest if the acceleration of the child in the wagon is to be calculated? (b) Draw a free-body diagram, including all forces acting on the system. (c) Calculate the acceleration. (d) What would the acceleration be if friction were 15.0 N?

10. A powerful motorcycle can produce an acceleration of $3.50\ \text{m/s}^2$ while traveling at 90.0 km/h. At that speed the forces resisting motion, including friction and air resistance, total 400 N. (Air resistance is analogous to air friction. It always opposes the motion of an object.) What is the magnitude of the force the motorcycle exerts backward on the ground to produce its acceleration if the mass of the motorcycle with rider is 245 kg?

11. The rocket sled shown in **Figure 1.14** accelerates at a rate of $49.0\ \text{m/s}^2$. Its passenger has a mass of 75.0 kg. (a) Calculate the horizontal component of the force the seat exerts against his body. Compare this with his weight by using a ratio. (b) Calculate the direction and magnitude of the total force the seat exerts against his body.

A sled is shown with four rockets. Friction force is represented by an arrow labeled as vector f pointing toward the left on the sled. The weight of the sled is represented by an arrow labeled as vector W, shown pointing downward, and the normal force is represented by an arrow labeled as vector N having the same length as W acting upward on the sled.

Figure 1.14

12. Repeat the previous problem for the situation in which the rocket sled decelerates at a rate of $201\ \text{m/s}^2$. In this problem, the forces are exerted by the seat and restraining belts.

13. The weight of an astronaut plus his space suit on the Moon is only 250 N. How much do they weigh on Earth? What is the mass on the Moon? On Earth?

14. Suppose the mass of a fully loaded module in which astronauts take off from the Moon is 10,000 kg. The thrust of its engines is 30,000 N. (a) Calculate its the magnitude of acceleration in a vertical takeoff from the Moon. (b) Could it lift off from Earth? If not, why not? If it could, calculate the magnitude of its acceleration.

1.7 Newton's Third Law - Symmetry in Forces

15. What net external force is exerted on a 1100-kg artillery shell fired from a battleship if the shell is accelerated at $2.40\times10^4\ \text{m/s}^2$? What is the magnitude of the force exerted on the ship by the artillery shell?

16. A brave but inadequate rugby player is being pushed backward by an opposing player who is exerting a force of 800 N on him. The mass of the losing player plus equipment is 90.0 kg, and he is accelerating at $1.20\ \text{m/s}^2$ backward. (a) What is the force of friction between the losing player's feet and the grass? (b) What force does the winning player exert on the ground to move forward if his mass plus equipment is 110 kg? (c) Draw a sketch of the situation showing the system of interest used to solve each part. For this situation, draw a free-body diagram and write the net force equation.

2 KINDS OF FORCES

2.1 Introduction

In this chapter, we will looking at the different kinds of forces that we will be talking about in this class. If you look around you, it may seem like there's a huge number of forces in the world around us. Frictions, pushes pulls, air resistance, whatever force causes you to float, these different forces. The goal of this chapter is to acquaint you with the different types of forces that we will be dealing with explicitly in this class.

Fundamentally, there are only four forces, so all the various forces that we see around us are, at the microscopic level, a manifestation of one of these four fundamental forces. In order of strength, the four fundamental forces are the strong nuclear force, which is responsible for holding the protons and neutrons in nuclei together, the electricity and magnetism forces, which you may have some familiarity with from the idea that opposite charges attract, like charges repel, and from playing with magnets, there is also the weak nuclear force which is responsible for radioactive decay, and then the weakest of the four fundamental forces is gravity, which you have some experience with as its the force that holds you to the earth, and holds the earth in orbit around the Sun.

We will not be exploring the strong nuclear force and the weak nuclear force at all in this class. While these forces are very important, their range is limited to sizes smaller than an atomic nucleus, and so don't have visible measurable effects at our everyday scales. Gravity on the other hand, we will talk about in some level of detail. Electricity and magnetism, on the other hand, is primarily dealt with in Physics 132. However, I do you expect you to have the basic understanding that opposite charges attract and like charges repel, because this is fundamentally the origin of the non-fundamental forces that we will discuss in this class.

So, what non-fundamental forces will we discuss in this class? Non-fundamental forces are forces that at the microscopic scale can be explained in terms of electrical forces, but at the macroscopic scale, we just average over all the atoms and call it a new type of force. For example, the normal force the normal force is probably best understood by setting a book on top of a table. The gravitational force pulls the book down; why doesn't the book just fall through the table? Well, there is a normal force from the table on the book to counter this force of gravity. At the microscopic scale, this normal force arises from the propulsion of electrons in the book to the electrons within the table. So, at the microscopic scale, this force is electrical, however, at our macroscopic scale that we deal with in our everyday world, we're averaging over these different atoms and just calling their net effect a normal force. One characteristic of the normal force is that it's perpendicular. It's always perpendicular. In fact, the word "normal" means perpendicular in mathematic-ese, so in mathematics, the word normal and the word perpendicular are just synonyms. This can help you remember the directions of the normal force. Another non-fundamental force we will discuss in this class is tension. Tension really arises when you start to have ropes and chains and that kind of a thing.

Consider a box hanging from a rope. Again, the force of gravity is pulling the box down. What keeps the box from falling? There is a tension in the rope that is countering the weight of the box holding it up. Again, at the microscopic scale, the tension force, arises from electricity, as the atomic bonds which are electrical in nature between one molecule of rope and the next are responsible for this force of tension. We'll also discuss forces involved with springs such as big metal coils that you might have had some experience with.

When I compress a spring, the spring exerts a force back outward as it tries to re-expand. You can imagine your hand compressing the spring, the spring would be pushing outward in the direction of this blue arrow when it is compressed. Conversely, if I stretch the spring, the direction of the spring on your hand would then be in the opposite direction, as the spring tries to pull itself back to its rest length.

The final set of non-fundamental forces we will discuss are frictional forces. These are the forces that come when you have rough surfaces in contact and are fundamentally electrical, and arise from Van der Waals interactions in hydrogen bonds between surfaces. There are two different kinds of friction. One is static friction, this is what happens when objects are not moving relative to each other, and then there is kinetic friction, which occurs when objects are sliding past each other. The directions of frictional forces can sometimes be somewhat tricky, and we'll have a lab in class to directly deal with them.

In summary, there are only four fundamental forces: the strong nuclear force, the electric and magnetic forces, weak nuclear force, and gravity. The only fundamental force will deal with in this glass is the weakest of the four, the gravitational force. We will also deal with five non-fundamental forces that are just electrical in nature. The normal force, which is what prevents objects from passing through to each other. It's due to electrical repulsion and is always perpendicular to the surfaces between objects. We'll also talk about tension forces, which come into play when you're dealing with ropes, chains, and the like. These are due to molecular bonds and therefore also electrical, and the direction of tension forces is always along the direction of the rope. We'll talk about spring forces, which of course deal, we'll talk about spring forces which of course come into play when we're talking about springs, and the direction of spring forces depends upon if the spring is either being stretched or compressed. Finally, we'll talk about friction forces, which at the microscopic level are due to van der Waals interactions in hydrogen bonds. We'll talk about both kinds of friction. Static friction, which occurs when objects are not moving relative to each other, this is the force you need to overcome to get an object to move, and we'll discuss kinetic friction which is the friction that occurs when objects are sliding past each other. This is the force that you need to overcome to keep an object moving across the rough surface.

2.2 The Fundamental Forces

One of the most remarkable simplifications in physics is that only four distinct forces account for all known phenomena. In fact, nearly all of the forces we experience directly are due to only one basic force, called the electromagnetic force. (The gravitational force is the only force we experience directly that is not electromagnetic.) This is a tremendous simplification of the myriad of *apparently* different forces we can list, only a few of which were discussed in the previous section. As we will see, the basic forces are all thought to act through the exchange of microscopic carrier particles, and the characteristics of the basic forces are determined by the types of particles exchanged. Action at a distance, such as the gravitational force of Earth on the Moon, is explained by the existence of a **force field** rather than by "physical contact."

The *four basic forces* are the gravitational force, the electromagnetic force, the weak nuclear force, and the strong nuclear force. Their properties are summarized in **Table 2.1**. Since the weak and strong nuclear forces act over an extremely short range, the size of a nucleus or less, we do not experience them directly, although they are crucial to the very structure of matter. These forces determine which nuclei are stable and which decay, and they are the basis of the release of energy in certain nuclear reactions. Nuclear forces determine not only the stability of nuclei, but also the relative abundance of elements in nature. The properties of the nucleus of an atom determine the number of electrons it has and, thus, indirectly determine the chemistry of the atom. More will be said of all of these topics in later chapters.

Concept Connections: The Four Basic Forces

The four basic forces will be encountered in more detail as you progress through the text. The gravitational force is defined in **Uniform Circular Motion and Gravitation (https://legacy.cnx.org/content/m42140/latest/)** , electric force in **Electric Charge and Electric Field (https://legacy.cnx.org/content/m42299/latest/)** , magnetic force in **Magnetism (https://legacy.cnx.org/content/m42365/latest/)** , and nuclear forces in **Radioactivity and Nuclear Physics (https://legacy.cnx.org/content/m42620/latest/)** . On a macroscopic scale, electromagnetism and gravity are the basis for all forces. The nuclear forces are vital to the substructure of matter, but they are not directly experienced on the macroscopic scale.

Table 2.1 Properties of the Four Basic Forces[1]

Force	Approximate Relative Strengths	Range	Attraction/Repulsion	Carrier Particle
Gravitational	10^{-38}	∞	attractive only	Graviton
Electromagnetic	10^{-2}	∞	attractive and repulsive	Photon
Weak nuclear	10^{-13}	$< 10^{-18}\text{ m}$	attractive and repulsive	W^+, W^-, Z^0
Strong nuclear	1	$< 10^{-15}\text{ m}$	attractive and repulsive	gluons

The gravitational force is surprisingly weak—it is only because gravity is always attractive that we notice it at all. Our weight is the gravitational force due to the *entire* Earth acting on us. On the very large scale, as in astronomical systems, the gravitational force is the dominant force determining the motions of moons, planets, stars, and galaxies. The gravitational force also affects the nature of space and time. As we shall see later in the study of general relativity, space is curved in the vicinity of very massive bodies, such as the Sun, and time actually slows down near massive bodies.

Electromagnetic forces can be either attractive or repulsive. They are long-range forces, which act over extremely large distances, and they nearly cancel for macroscopic objects. (Remember that it is the *net* external force that is important.) If they did not cancel, electromagnetic forces would completely overwhelm the gravitational force. The electromagnetic force is a combination of electrical forces (such as those that cause static electricity) and magnetic forces (such as those that affect a compass needle). These two forces were thought to be quite distinct until early in the 19th century, when scientists began to discover that they are different manifestations of the same force. This discovery is a classical case of the *unification of forces*. Similarly, friction, tension, and all of the other classes of forces we experience directly (except gravity, of course) are due to electromagnetic interactions of atoms and molecules. It is still convenient to consider these forces separately in specific applications, however, because of the ways they manifest themselves.

Concept Connections: Unifying Forces

Attempts to unify the four basic forces are discussed in relation to elementary particles later in this text. By "unify" we mean finding connections between the forces that show that they are different manifestations of a single force. Even if such unification is achieved, the forces will retain their separate characteristics on the macroscopic scale and may be identical only under extreme conditions such as those existing in the early universe.

Physicists are now exploring whether the four basic forces are in some way related. Attempts to unify all forces into one come under the rubric of Grand Unified Theories (GUTs), with which there has been some success in recent years. It is now known that under conditions of extremely high density and temperature, such as existed in the early universe, the electromagnetic and weak nuclear forces are indistinguishable. They can now be considered to be different manifestations of one force, called the

1. The graviton is a proposed particle, though it has not yet been observed by scientists. See the discussion of gravitational waves later in this section. The particles W^+, W^-, and Z^0 are called vector bosons; these were predicted by theory and first observed in 1983. There are eight types of gluons proposed by scientists, and their existence is indicated by meson exchange in the nuclei of atoms.

electroweak force. So the list of four has been reduced in a sense to only three. Further progress in unifying all forces is proving difficult—especially the inclusion of the gravitational force, which has the special characteristics of affecting the space and time in which the other forces exist.

While the unification of forces will not affect how we discuss forces in this text, it is fascinating that such underlying simplicity exists in the face of the overt complexity of the universe. There is no reason that nature must be simple—it simply is.

Action at a Distance: Concept of a Field

All forces act at a distance. This is obvious for the gravitational force. Earth and the Moon, for example, interact without coming into contact. It is also true for all other forces. Friction, for example, is an electromagnetic force between atoms that may not actually touch. What is it that carries forces between objects? One way to answer this question is to imagine that a **force field** surrounds whatever object creates the force. A second object (often called a *test object*) placed in this field will experience a force that is a function of location and other variables. The field itself is the "thing" that carries the force from one object to another. The field is defined so as to be a characteristic of the object creating it; the field does not depend on the test object placed in it. Earth's gravitational field, for example, is a function of the mass of Earth and the distance from its center, independent of the presence of other masses. The concept of a field is useful because equations can be written for force fields surrounding objects (for gravity, this yields $w = mg$ at Earth's surface), and motions can be calculated from these equations.

(See **Figure 2.1**.)

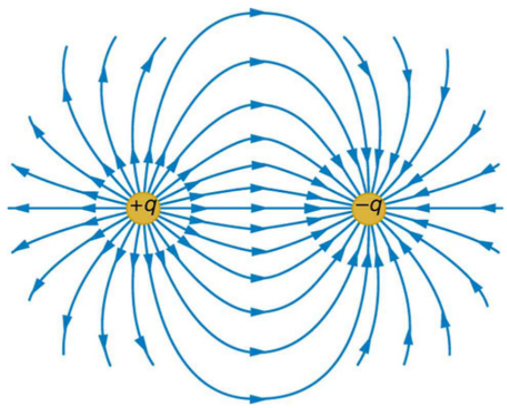


Figure 2.1 The electric force field between a positively charged particle and a negatively charged particle. When a positive test charge is placed in the field, the charge will experience a force in the direction of the force field lines.

Concept Connections: Force Fields

The concept of a *force field* is also used in connection with electric charge and is presented in **Electric Charge and Electric Field (https://legacy.cnx.org/content/m42299/latest/)** . It is also a useful idea for all the basic forces, as will be seen in **Particle Physics (https://legacy.cnx.org/content/m42667/latest/)** . Fields help us to visualize forces and how they are transmitted, as well as to describe them with precision and to link forces with subatomic carrier particles.

The field concept has been applied very successfully; we can calculate motions and describe nature to high precision using field equations. As useful as the field concept is, however, it leaves unanswered the question of what carries the force. It has been proposed in recent decades, starting in 1935 with Hideki Yukawa's (1907–1981) work on the strong nuclear force, that all forces are transmitted by the exchange of elementary particles. We can visualize particle exchange as analogous to macroscopic phenomena such as two people passing a basketball back and forth, thereby exerting a repulsive force without touching one another. (See **Figure 2.2**.)

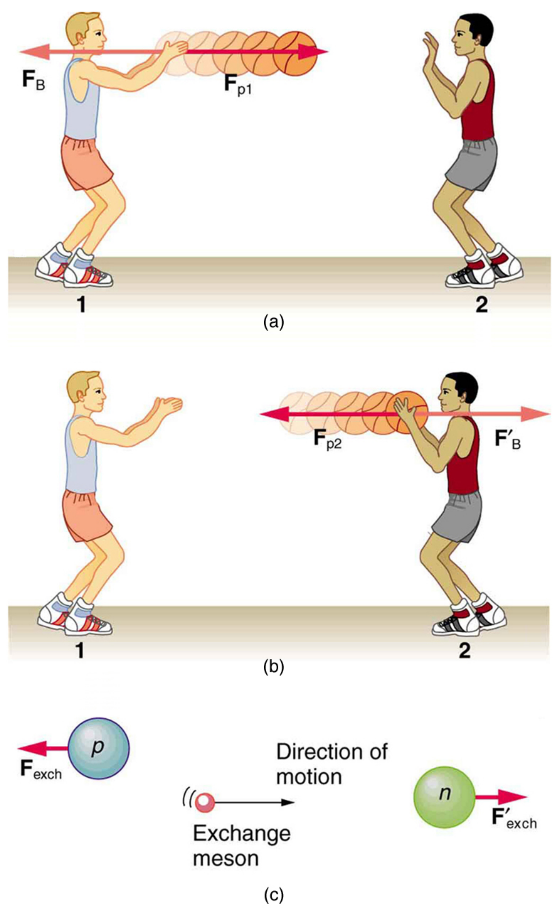


Figure 2.2 The exchange of masses resulting in repulsive forces. (a) The person throwing the basketball exerts a force $\mathbf{F}_{\text{p1}}$ on it toward the other person and feels a reaction force $\mathbf{F}_{\text{B}}$ away from the second person. (b) The person catching the basketball exerts a force $\mathbf{F}_{\text{p2}}$ on it to stop the ball and feels a reaction force $\mathbf{F}'_{\text{B}}$ away from the first person. (c) The analogous exchange of a meson between a proton and a neutron carries the strong nuclear forces $\mathbf{F}_{\text{exch}}$ and $\mathbf{F}'_{\text{exch}}$ between them. An attractive force can also be exerted by the exchange of a mass—if person 2 pulled the basketball away from the first person as he tried to retain it, then the force between them would be attractive.

This idea of particle exchange deepens rather than contradicts field concepts. It is more satisfying philosophically to think of something physical actually moving between objects acting at a distance. **Table 2.1** lists the exchange or **carrier particles**, both observed and proposed, that carry the four forces. But the real fruit of the particle-exchange proposal is that searches for Yukawa's proposed particle found it *and* a number of others that were completely unexpected, stimulating yet more research. All of this research eventually led to the proposal of quarks as the underlying substructure of matter, which is a basic tenet of GUTs. If successful, these theories will explain not only forces, but also the structure of matter itself. Yet physics is an experimental science, so the test of these theories must lie in the domain of the real world. As of this writing, scientists at the CERN laboratory in Switzerland are starting to test these theories using the world's largest particle accelerator: the Large Hadron Collider. This accelerator (27 km in circumference) allows two high-energy proton beams, traveling in opposite directions, to collide. An energy of 14 trillion electron volts will be available. It is anticipated that some new particles, possibly force carrier particles, will be found. (See **Figure 2.3**.) One of the force carriers of high interest that researchers hope to detect is the Higgs boson. The observation of its properties might tell us why different particles have different masses.

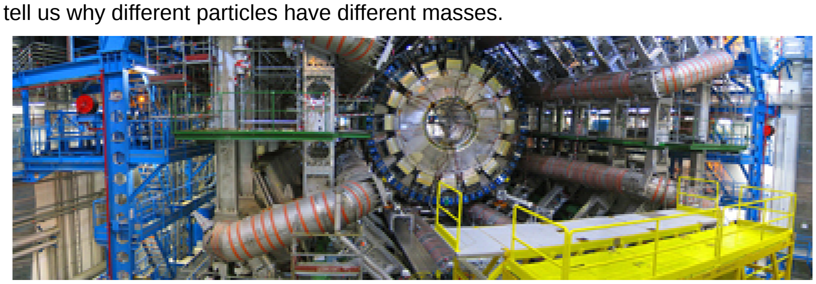


Figure 2.3 The world's largest particle accelerator spans the border between Switzerland and France. Two beams, traveling in opposite directions close to the speed of light, collide in a tube similar to the central tube shown here. External magnets determine the beam's path. Special detectors will analyze particles created in these collisions. Questions as broad as what is the origin of mass and what was matter like the first few seconds of our universe will be explored. This accelerator began preliminary operation in 2008. (credit: Frank Hommes)

Tiny particles also have wave-like behavior, something we will explore more in a later chapter. To better understand force-carrier particles from another perspective, let us consider gravity. The search for gravitational waves has been going on for a number of years. Almost 100 years ago, Einstein predicted the existence of these waves as part of his general theory of relativity. Gravitational waves are created during the collision of massive stars, in black holes, or in supernova explosions—like shock waves. These gravitational waves will travel through space from such sites much like a pebble dropped into a pond sends out ripples—except these waves move at the speed of light. A detector apparatus has been built in the U.S., consisting of two large installations nearly 3000 km apart—one in Washington state and one in Louisiana! The facility is called the Laser Interferometer Gravitational-Wave Observatory (LIGO). Each installation is designed to use optical lasers to examine any slight shift in the relative positions of two masses due to the effect of gravity waves. The two sites allow simultaneous measurements of these small effects to be separated from other natural phenomena, such as earthquakes. Initial operation of the detectors began in 2002, and work is proceeding on increasing their sensitivity. Similar installations have been built in Italy (VIRGO), Germany (GEO600), and Japan (TAMA300) to provide a worldwide network of gravitational wave detectors.

International collaboration in this area is moving into space with the joint EU/US project LISA (Laser Interferometer Space Antenna). Earthquakes and other Earthly noises will be no problem for these monitoring spacecraft. LISA will complement LIGO by looking at much more massive black holes through the observation of gravitational-wave sources emitting much larger wavelengths. Three satellites will be placed in space above Earth in an equilateral triangle (with 5,000,000-km sides) (**Figure 2.4**). The system will measure the relative positions of each satellite to detect passing gravitational waves. Accuracy to within 10% of the size of an atom will be needed to detect any waves. The launch of this project might be as early as 2018.

"I'm sure LIGO will tell us something about the universe that we didn't know before. The history of science tells us that any time you go where you haven't been before, you usually find something that really shakes the scientific paradigms of the day. Whether gravitational wave astrophysics will do that, only time will tell." —David Reitze, LIGO Input Optics Manager, University of Florida

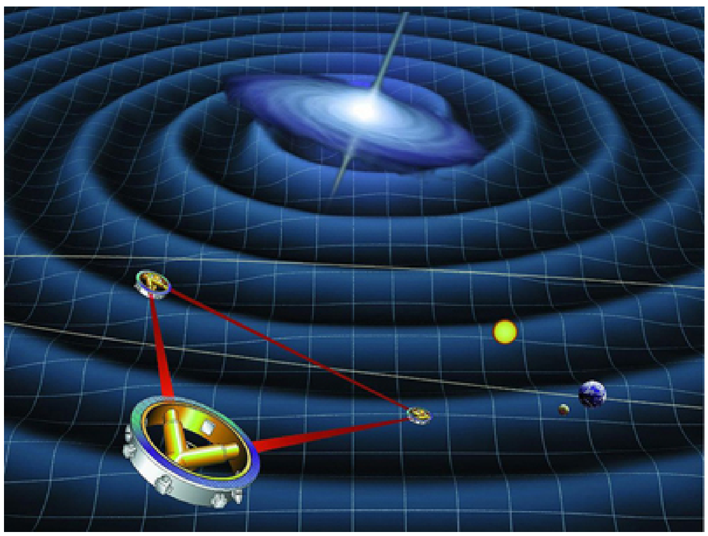


Figure 2.4 Space-based future experiments for the measurement of gravitational waves. Shown here is a drawing of LISA's orbit. Each satellite of LISA will consist of a laser source and a mass. The lasers will transmit a signal to measure the distance between each satellite's test mass. The relative motion of these masses will provide information about passing gravitational waves. (credit: NASA)

The ideas presented in this section are but a glimpse into topics of modern physics that will be covered in much greater depth in later chapters.

2.3 Weight and Gravity

UMASS AMHERST Instructor's Notes

This section is also available as a video, available here: **https://www.youtube.com/watch?v=5RiNj5IiTbg (https://www.youtube.com/watch?v=5RiNj5IiTbg)**

Gravity is one of the fundamental forces. We're going to explain gravitational interactions in terms of the idea of the gravitational field, which we indicate by the little letter g. We'll discover that this field is a vector, hence the little vector symbol above the variable. We're going to describe when the so-called flat earth gravity is a valid approximation.

So, let's think a little bit about the history of physics' understanding of the force of gravity. The first real description of the force of gravity comes from Isaac Newton in 1687. Isaac Newton was the first person to think about the fact that the same force that causes an apple to fall from a tree keeps the moon in orbit around the Earth. Both are consequences of the force of gravity. But, you might say to yourself, the apple falls straight down, while everyone knows that the moon goes around and around and around. These seem like fundamentally different motions.

Here's an applet that can help us think about gravity like Isaac Newton did: **http://waowen.screaming.net/revision/force&motion/ncananim.htm (http://waowen.screaming.net/revision/force&motion/ncananim.htm)** (if you cannot get the applet to work, the YouTube version of this section has a run through of this applet). Play around with the velocities; try launching the ball at 2000 m/s, 3000 m/s, and 4500 m/s. For the 2000 m/s and 3000 m/s launch, the cannonball shoots out, goes some distance, and then falls to the earth. However, the cannonball at 4500 m/s acts differently. The cannonball never actually hits the earth. It keeps falling around and around and around and around, without ever hitting, and this is the crux of what an orbit is. It's falling and then missing. The reason you miss is because the earth falls away faster than you fall towards the center, and this was Isaac Newton's realization, that the same force of attraction that pulls an apple to the ground holds the moon in orbit around the earth.

This was a very revolutionary idea for the time because back in the 17th century it was thought that different physical laws operated on earth then operated in "the heavens", as they called it at that time. So, apple falling and moon orbiting related by the fundamental idea of gravity. This is one of the most powerful and first examples of a single fundamental idea explaining a variety of different phenomena, and really shows you the power of fundamental ideas in physics.

So, let's stop and think for a second. How does the moon, or if you prefer, the apple know that the earth is there? I mean, the moon is very far away from the earth. It's not touching the earth; how does the moon know that the earth is there? Well, Isaac Newton himself could not come up with a particularly good answer to this question. He called it "action at a distance" and sort of left it at that. Now, the way we modern physicists envision this is we say that the earth generates what's known as a gravitational field, and this gravitational field is an invisible field that extends out from the earth in all directions. This is the gravitational field that we indicate in this class by little g, and it has a direction so it's a vector, hence the little vector symbol above the g. The moon does touch the gravitational field, because this gravitational field, these are just some sample lines, the gravitational field goes everywhere, so the moon does touch the gravitational field of the earth, and it responds to this gravitational field by feeling a force, and the magnitude of the moon's force, or the magnitude of the force from the earth on the moon, is the magnitude of the gravitational field from the earth at the spot of the moon multiplied by the mass of the moon.

This is the fundamental idea of the field. This field concept will be used at great more length in Physics 132 in the context of the electric field. Looking at this definition of force in terms of gravitational field, we can see the units of gravitational field. The units of force are newtons, the units of mass are kilograms, and therefore the units of the gravitational field must be newtons over kilograms, or people will say it newtons per kilogram.

So, let's review the fundamental features of this gravitational field. Every object with mass, not just planets but every object with mass, including yourself, generates a gravitational field, little g. Now, planets are the only things that generate big enough gravitational fields to matter, but everything generates a gravitational field. Every object with mass interacts with all the fields around it by feeling a force.

So, in the case of our apple the earth generates a gravitational field down, and the apple interacts with that field by feeling a force towards the Earth. You might ask yourself, "well, doesn't the apple generate its own field?". Yes, the apple does generate its own field, albeit a very tiny one, so the apple will also generate a very tiny field towards it, and the earth will respond to that tiny field by feeling a force upwards. We'll talk a little bit more about this seeming paradox, we don't see the earth move, in class, but it is true. So, every object interacts with all the other fields by feeling a force, m times g. It's important to keep in mind that objects don't or interact with their own field, they only interact with the surrounding fields. So, the apple interacts with the field of the earth and the earth interacts with the field from the apple. The earth doesn't interact with its own field. We've already talked about the fact that the units of g are newtons per kilogram, and the consequence of this is that every object in the universe with mass attracts each other. So, any two objects with mass in the entire universe attract each other, which might lead you to the question, "why doesn't the whole universe just collapse?". If everything with mass is attracting each other, it seems like everything we just fall into one big giant heap. Well the answer is that the field, and therefore the force, since the force is equal to the mass of an object times the field, gets smaller with distance. So, if the field gets smaller, the force will get smaller, and this gets smaller with distance from the center of the object. That's the relevant quantity not distance from the surface, but distance from the center. This has important implications for our class.

So, how are we going to deal with gravity in this class? Remember. the gravitational field gets weaker as the distance from the center of the object. in our case. the earth. because we're all on the earth. increases. In this course, we're going to be dealing with everyday heights that we all can experience. These are all very small compared to the radius of the Earth. The radius of the Earth is 10^6 meters: 6 million meters. Even if you were to go to the top of Mount Everest, which is the highest mountain on Earth, as you probably, know that's still an only an extra 8,000 meters, and you've only increased your distance from the center of the earth by a very tiny amount. Thus, for this course we are always essentially the radius of the earth from the center. This is called the "flat-earth approximation" now this might seem like a very strange idea; you're in a university physics class and we're talking about the earth being flat. Well, we're not really. We're making an approximation at the earth is flat, and the approximation is in other words that the earth is very big compared to anything else we're dealing with. Even compared to Mount Everest the earth is huge, and so we can treat it as very big, in which case it is essentially flat. We don't have to worry about the fact that the earth is round so thus it's called the "flat-earth approximation". Now, if you were to go to, say, the moon, which is many times further away than the size of the earth, this approximation would no longer be valid, but if you're close to the surface of the earth and this approximation is good, then the gravitational field is going to be essentially constant. We're only moving very tiny amounts relative to the radius of the earth, so the gravitational field as far as we are ever going to experience is not really going to change. It's going to be constant. This constant value has been measured to be 9.8 newtons per kilogram. Therefore, in this class, we will say that the force of gravity from the earth on an object, whatever it is, and forces are vectors, will be the mass of the object, say an apple, times g, where this g is 9.8 newtons per kilogram.

We will also occasionally speak of this force of gravity as the weight force. This is just equivalent terminology, two different names for the same thing. The weight force is often indicated by a w. Again, it's a vector, so again, it would be, the weight force would be the mass of the object times g, where this g is still the 9.8 newtons per kilogram. This is how we will deal with weight force and gravitational forces in this class, but I thought it relevant to bring up why the moon goes around the earth, how this is connected to falling objects, because it demonstrates the power of fundamental ideas.

2.4 Normal Force

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Defining normal force in your own terms
- Identifying that normal force is a constraint force, i.e. has no formula

The goal of this section is to introduce you to some of the basic properties of the normal force. So, what is the normal force? The normal force is the force that keeps objects from passing through each other. A common example is that of a book sitting on a table. The force of gravity tries to pull the book down through the table; clearly, the book does not go through the table, and so we are forced to conclude that there must be some type of force from the table on the book pushing upwards to balance this. We call this the **normal force**. In general, a normal force can be thought of as any time one object pushes on another. In the example of the book on the table, the table is pushing on the book upward, keeping it from falling through.

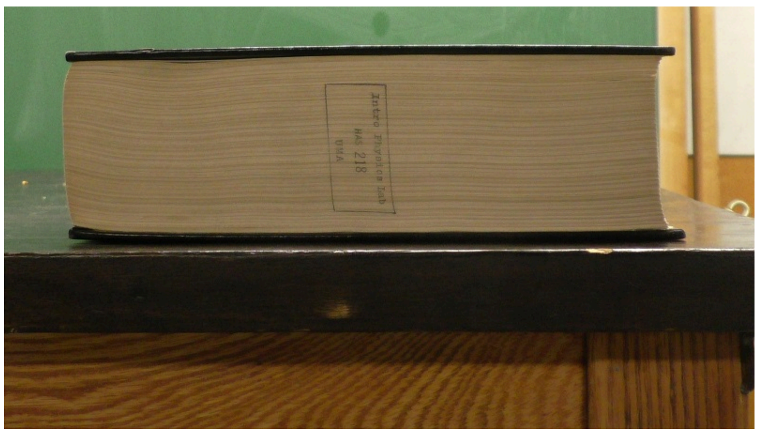


Figure 2.5 A book sitting on top of a table. The normal force from the table on the book balances the force of gravity from the Earth on the book, preventing the book from falling through the table.

Another example with an active agent is a person pushing on a box. What is the force on the box? It’s a normal force from the hand on the box at the molecular level. The normal force arises from the electrons in one surface repelling the electrons in the other surface, but we overlook this microscopic level detail and just call the net effect the normal force.

It’s worth taking a few minutes to talk about the connections between the normal force and a scale. The question essentially is, what do scales actually read? Well, you might think that a scale just reads the amount of weight put on it. In the picture, a 500-gram weight is placed on a scale, and the scale reads 500 grams. However, that’s not all scales can measure. When the scale is pushed down on, you can see that the number goes up. So, what does this scale actually measure? It measures the amount of force being applied to the scale. In essence, this scale reads the normal force.

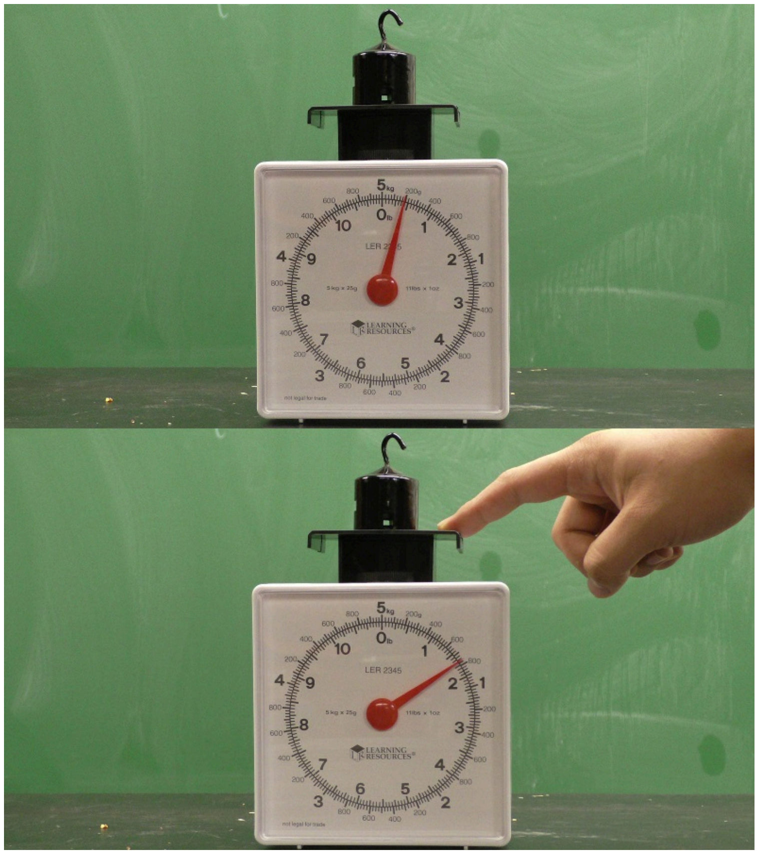


Figure 2.6 A 500g weight resting on top of a scale. The scale reads the force exerted on it, which is the force of the weight and the force exerted by the finger.

in summary, scales measure the force with which you press on them. They measure the force with which one object, my finger and this weight, push on another the platform of the scale. Scales measure the normal force. This is an important fact to remember as we will be using a variant of a scale known as a force plate in class.

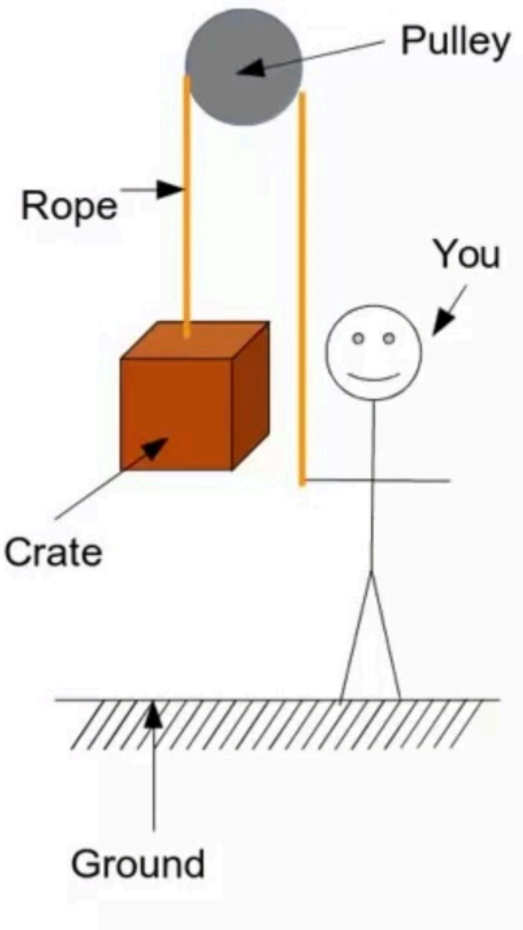
So, let’s summarize the characteristics of the normal force. The normal force is a contact force. The two objects must be in contact for the normal force to be present. The normal force is also a constraint force. This means that there’s no formula for the magnitude of the normal force; it takes on whatever value is needed to keep Newton’s second law true. It’s also important to remember that the normal force is always, well, normal. Normal means perpendicular in mathematic-ese, and the normal force is always perpendicular to the surface. Finally, it’s important to remember that scales measure the normal force.

2.5 Tension

UMASS AMHERST Instructor's Notes

This section has two parts. The first part is an example with a pulley to help you develop an understanding of tension from an everyday example, and the second part is the text from the OpenStax book.

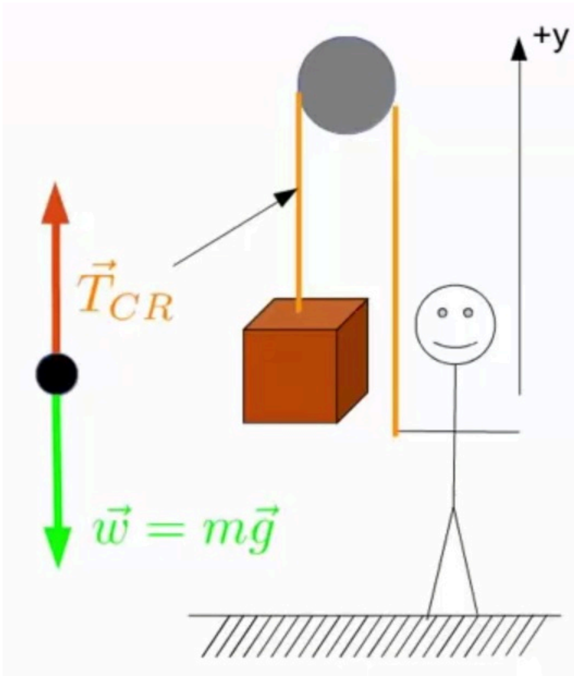
To explore this tension, let’s consider a crate suspended above the ground, held up by a rope.



This rope goes over a pulley to you at the other end of the rope. Now, it may be obvious to you the answer to the question “what holds up the stationary crate?”. Well, the rope does, but we want to explore this situation using our physics language.

Let’s consider this situation from a physics perspective. What do we know about the crate? Well, we know that the crate is stationary. The fact that the crate is stationary means that the velocity isn’t changing with respect to time. In other words, there is no net acceleration. By Newton’s second law, no net acceleration implies that there is zero net force, a fact which we can write mathematically like this.

When considering forces in physics, it’s often helpful to draw free body diagrams. Here’s one for this example:



We'll model the crate as a black dot. What else do we know about this problem? Well, we've already determined that the net force has to be zero, which means that there must be some other force pointing in opposition to the force of the weight. By vector addition, if the weight points down and the net force is zero, then this other force must point up. This force is due to the tension within the rope, and so we call it the tension force, and it acts on the crate from the rope. Now let's apply Newton's second law to this situation. Remember, the crate is stationary, so there's no acceleration. We'll start by just stating Newton's second law mathematically, the sum of the forces is equal to the mass times the acceleration. The forces in this case include the tension on the crate from the rope and the weight, which we've already decided to model as mg. We've already determined that the acceleration must be zero because the crate is stationary. Now, both the tension and the weight force are vectors. Since we're adding them up, we need to break them into components.

In order to break them into components, we need to establish a coordinate system. We define the y-direction to be positive going up. With this convention, the tension on the crate from the rope is positive, and the weight is negative. Doing the algebra gives us the tension on the crate from the rope is equal to the mass of the crate multiplied by little G which, on earth recall, is 9.8 meters per second squared. Would this equality between the tension and the rope and the weight of the box be true if the box were accelerating? No. Why not? Because this equality between the tension in the way only arose because we set the acceleration equal to 0 here in the second step. If that was not true then the tension would not be equal to the weight. What about you on the other end of the rope? How hard do you have to pull down? One property of ropes that we will make extensive use of in this class is that the tension in a rope is the same everywhere. What do I mean by this? Well, we solve for the tension over here where the crate is connected to the rope. By analyzing this part of the system, we can conclude that the tension on the crate from the rope was equal to the weight of the crate.

The fact that the tension must be the same everywhere in the rope means that the tension in the rope where it meets the hand is also equal to mg. Therefore, the rope is pulling up on my hand with the tension force equal to the weight of the crate, mg, which means that if I want everything to stay stationary I have to pull down with a force equal to the weight of the crate, mg, to keep everything in place. This should be in conjunction with your everyday experience, where to lift a box using any type of pulley, you've got to pull down with at least the weight of the box. The pulley makes it easier because you're pulling down using your weight to help lift the box as opposed lifting it.

Tension

A **tension** is a force along the length of a medium, especially a force carried by a flexible medium, such as a rope or cable. The word "tension" comes from a Latin word meaning "to stretch." Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called *tendons*. Any flexible connector, such as a string, rope, chain, wire, or cable, can exert pulls only parallel to its length; thus, a force carried by a flexible connector is a tension with direction parallel to the connector. It is important to understand that tension is a pull in a connector. In contrast, consider the phrase: "You can't push a rope." The tension force pulls outward along the two ends of a rope.

Consider a person holding a mass on a rope as shown in **Figure 2.7**.

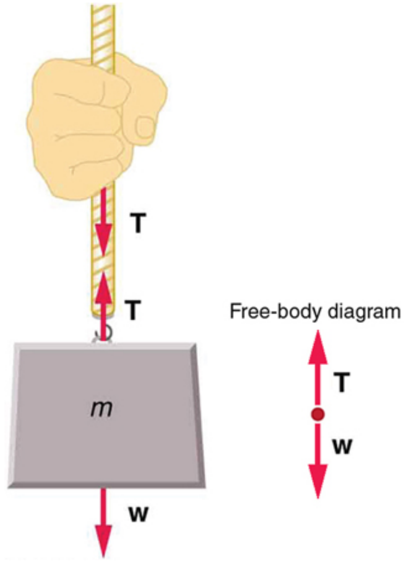


Figure 2.7 When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force $\mathbf{T}$, that force must be parallel to the length of the rope, as shown. The pull such a flexible connector exerts is a tension. Note that the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). This is an example of Newton's third law. The rope is the medium that carries the equal and opposite forces between the two objects. The tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.

Tension in the rope must equal the weight of the supported mass, as we can prove using Newton's second law. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero, and thus $\mathbf{F}_{\text{net}} = 0$. The only external forces acting on the mass are its weight $\mathbf{w}$ and the tension $\mathbf{T}$ supplied by the rope. Thus,

$$F_{\text{net}} = T - w = 0, \tag{2.1}$$

where T and w are the magnitudes of the tension and weight and their signs indicate direction, with up being positive here. Thus, just as you would expect, the tension equals the weight of the supported mass:

$$T = w = mg. \tag{2.2}$$

For a 5.00-kg mass, then (neglecting the mass of the rope) we see that

$$T = mg = (5.00\text{ kg})(9.80\text{ m/s}^2) = 49.0\text{ N}. \tag{2.3}$$

If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a finger joint, or a bicycle brake cable. If there is no friction, the tension is transmitted undiminished. Only its direction changes, and it is always parallel to the flexible connector. This is illustrated in **Figure 2.8** (a) and (b).

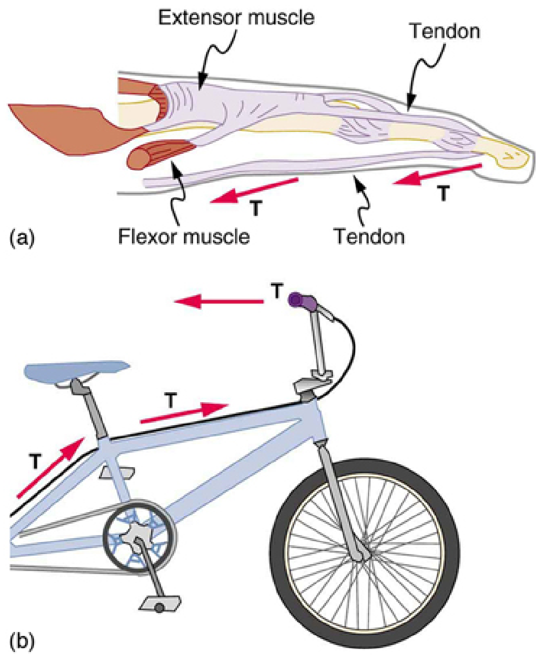


Figure 2.8 (a) Tendons in the finger carry force $\mathbf{T}$ from the muscles to other parts of the finger, usually changing the force's direction, but not its magnitude (the tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension $\mathbf{T}$ from the handlebars to the brake mechanism. Again, the direction but not the magnitude of $\mathbf{T}$ is changed.

Example 2.1 What Is the Tension in a Tightrope?

Calculate the tension in the wire supporting the 70.0-kg tightrope walker shown in **Figure 2.9**.

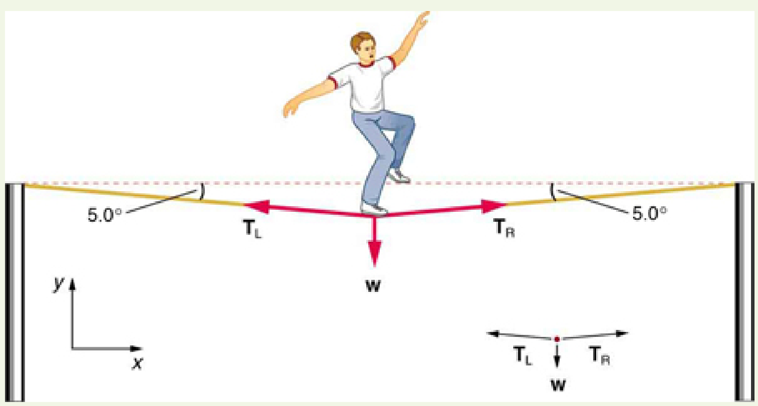


Figure 2.9 The weight of a tightrope walker causes a wire to sag by 5.0 degrees. The system of interest here is the point in the wire at which the tightrope walker is standing.

Strategy

As you can see in the figure, the wire is not perfectly horizontal (it cannot be!), but is bent under the person's weight. Thus, the tension on either side of the person has an upward component that can support his weight. As usual, forces are vectors represented pictorially by arrows having the same directions as the forces and lengths proportional to their magnitudes. The system is the tightrope walker, and the only external forces acting on him are his weight $\mathbf{w}$ and the two tensions $\mathbf{T}_\mathrm{L}$ (left tension) and $\mathbf{T}_\mathrm{R}$ (right tension), as illustrated. It is reasonable to neglect the weight of the wire itself. The net external force is zero since the system is stationary. A little trigonometry can now be used to find the tensions. One conclusion is possible at the outset—we can see from part (b) of the figure that the magnitudes of the tensions T_L and T_R must be equal. This is because there is no horizontal acceleration in the rope, and the only forces acting to the left and right are T_L and T_R. Thus, the magnitude of those forces must be equal so that they cancel each other out.

Whenever we have two-dimensional vector problems in which no two vectors are parallel, the easiest method of solution is to pick a convenient coordinate system and project the vectors onto its axes. In this case the best coordinate system has one axis horizontal and the other vertical. We call the horizontal the x-axis and the vertical the y-axis.

Solution

First, we need to resolve the tension vectors into their horizontal and vertical components. It helps to draw a new free-body diagram showing all of the horizontal and vertical components of each force acting on the system.

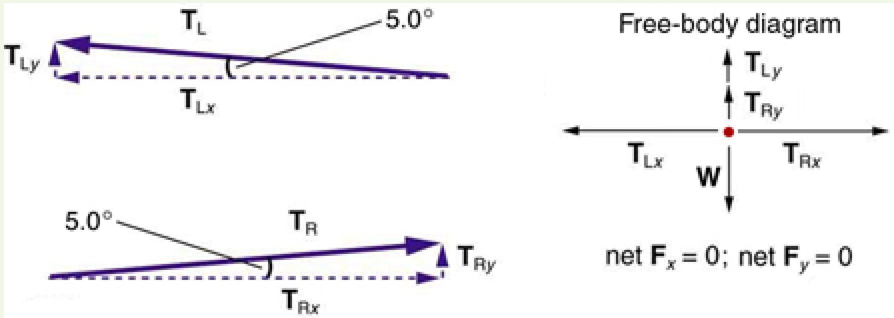


Figure 2.10 When the vectors are projected onto vertical and horizontal axes, their components along those axes must add to zero, since the tightrope walker is stationary. The small angle results in T being much greater than w .

Consider the horizontal components of the forces (denoted with a subscript x):

$$F_{\text{net}x} = T_{\text{L}x} - T_{\text{R}x}. \tag{2.4}$$

The net external horizontal force $F_{\text{net}x} = 0$, since the person is stationary. Thus,

$$\begin{aligned} F_{\text{net}x} = 0 &= T_{\text{L}x} - T_{\text{R}x} \\ T_{\text{L}x} &= T_{\text{R}x}. \end{aligned} \tag{2.5}$$

Now, observe **Figure 2.10**. You can use trigonometry to determine the magnitude of T_L and T_R . Notice that:

$$\begin{aligned} \cos(5.0^\circ) &= \frac{T_{\text{L}x}}{T_\text{L}} \\ T_{\text{L}x} &= T_\text{L}\cos(5.0^\circ) \\ \cos(5.0^\circ) &= \frac{T_{\text{R}x}}{T_\text{R}} \\ T_{\text{R}x} &= T_\text{R}\cos(5.0^\circ). \end{aligned} \tag{2.6}$$

Equating $T_{\text{L}x}$ and $T_{\text{R}x}$:

$$T_\text{L}\cos(5.0^\circ) = T_\text{R}\cos(5.0^\circ). \tag{2.7}$$

Thus,

$$T_\text{L} = T_\text{R} = T, \tag{2.8}$$

as predicted. Now, considering the vertical components (denoted by a subscript y), we can solve for T . Again, since the person is stationary, Newton's second law implies that net $F_y = 0$. Thus, as illustrated in the free-body diagram in **Figure 2.10**,

$$F_{\text{net}y} = T_{\text{L}y} + T_{\text{R}y} - w = 0. \tag{2.9}$$

Observing **Figure 2.10**, we can use trigonometry to determine the relationship between $T_{\text{L}y}$, $T_{\text{R}y}$, and T . As we determined from the analysis in the horizontal direction, $T_\text{L} = T_\text{R} = T$:

$$\begin{aligned} \sin(5.0^\circ) &= \frac{T_{\text{L}y}}{T_\text{L}} \\ T_{\text{L}y} = T_\text{L}\sin(5.0^\circ) &= T\sin(5.0^\circ) \\ \sin(5.0^\circ) &= \frac{T_{\text{R}y}}{T_\text{R}} \\ T_{\text{R}y} = T_\text{R}\sin(5.0^\circ) &= T\sin(5.0^\circ). \end{aligned} \tag{2.10}$$

Now, we can substitute the values for $T_{\mathrm{L}y}$ and $T_{\mathrm{R}y}$, into the net force equation in the vertical direction:

$$\begin{aligned} F_{\mathrm{net}y} &= T_{\mathrm{L}y} + T_{\mathrm{R}y} - w = 0 \\ F_{\mathrm{net}y} &= T\sin(5.0^\circ) + T\sin(5.0^\circ) - w = 0 \\ 2\,T\sin(5.0^\circ) - w &= 0 \\ 2\,T\sin(5.0^\circ) &= w \end{aligned} \tag{2.11}$$

and

$$T = \frac{w}{2\sin(5.0^\circ)} = \frac{mg}{2\sin(5.0^\circ)}, \tag{2.12}$$

so that

$$T = \frac{(70.0\text{ kg})(9.80\text{ m/s}^2)}{2(0.0872)}, \tag{2.13}$$

and the tension is

$$T = 3900\text{ N}. \tag{2.14}$$

Discussion

Note that the vertical tension in the wire acts as a normal force that supports the weight of the tightrope walker. The tension is almost six times the 686-N weight of the tightrope walker. Since the wire is nearly horizontal, the vertical component of its tension is only a small fraction of the tension in the wire. The large horizontal components are in opposite directions and cancel, and so most of the tension in the wire is not used to support the weight of the tightrope walker.

If we wish to *create* a very large tension, all we have to do is exert a force perpendicular to a flexible connector, as illustrated in **Figure 2.11**. As we saw in the last example, the weight of the tightrope walker acted as a force perpendicular to the rope. We saw that the tension in the roped related to the weight of the tightrope walker in the following way:

$$T = \frac{w}{2\sin(\theta)}. \tag{2.15}$$

We can extend this expression to describe the tension T created when a perpendicular force ($\mathbf{F}_\perp$) is exerted at the middle of a flexible connector:

$$T = \frac{F_\perp}{2\sin(\theta)}. \tag{2.16}$$

Note that θ is the angle between the horizontal and the bent connector. In this case, T becomes very large as θ approaches zero. Even the relatively small weight of any flexible connector will cause it to sag, since an infinite tension would result if it were horizontal (i.e., $\theta = 0$ and $\sin\theta = 0$). (See **Figure 2.11**.)

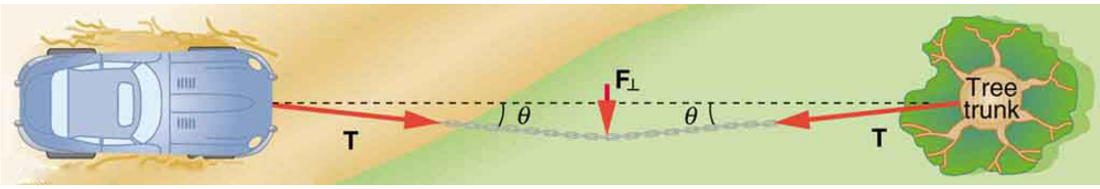


Figure 2.11 We can create a very large tension in the chain by pushing on it perpendicular to its length, as shown. Suppose we wish to pull a car out of the mud when no tow truck is available. Each time the car moves forward, the chain is tightened to keep it as nearly straight as possible. The tension in the chain is given by $T = \frac{F_\perp}{2\sin(\theta)}$; since θ is small, T is very large. This situation is analogous to the tightrope walker shown in **Figure 2.9**, except that the tensions shown here are those transmitted to the car and the tree rather than those acting at the point where $\mathbf{F}_\perp$ is applied.

Figure 2.12 Unless an infinite tension is exerted, any flexible connector—such as the chain at the bottom of the picture—will sag under its own weight, giving a characteristic curve when the weight is evenly distributed along the length. Suspension bridges—such as the Golden Gate Bridge shown in this image—are essentially very heavy flexible connectors. The weight of the bridge is evenly distributed along the length of flexible connectors, usually cables, which take on the characteristic shape. (credit: Leaflet, Wikimedia Commons)

2.6 Friction

UMASS AMHERST Instructor's Notes

This section will be discussed in class, so reading this section is not required.

Friction is a force that is around us all the time that opposes relative motion between systems in contact but also allows us to move (which you have discovered if you have ever tried to walk on ice). While a common force, the behavior of friction is actually very complicated and is still not completely understood. We have to rely heavily on observations for whatever understandings we can gain. However, we can still deal with its more elementary general characteristics and understand the circumstances in which it behaves.

Friction

Friction is a force that opposes relative motion between systems in contact.

One of the simpler characteristics of friction is that it is parallel to the contact surface between systems and always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called **kinetic friction**. For example, friction slows a hockey puck sliding on ice. But when objects are stationary, **static friction** can act between them; the static friction is usually greater than the kinetic friction between the objects.

Kinetic Friction

If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction.

Imagine, for example, trying to slide a heavy crate across a concrete floor—you may push harder and harder on the crate and not move it at all. This means that the static friction responds to what you do—it increases to be equal to and in the opposite direction of your push. But if you finally push hard enough, the crate seems to slip suddenly and starts to move. Once in motion it is easier to keep it in motion than it was to get it started, indicating that the kinetic friction force is less than the static friction force. If you add mass to the crate, say by placing a box on top of it, you need to push even harder to get it started and also to keep it moving. Furthermore, if you oiled the concrete you would find it to be easier to get the crate started and keep it going (as you might expect).

Figure 2.13 is a crude pictorial representation of how friction occurs at the interface between two objects. Close-up inspection of these surfaces shows them to be rough. So when you push to get an object moving (in this case, a crate), you must raise the object until it can skip along with just the tips of the surface hitting, break off the points, or do both. A considerable force can be resisted by friction with no apparent motion. The harder the surfaces are pushed together (such as if another box is placed on the crate), the more force is needed to move them. Part of the friction is due to adhesive forces between the surface molecules of the two objects, which explain the dependence of friction on the nature of the substances. Adhesion varies with substances in contact and is a complicated aspect of surface physics. Once an object is moving, there are fewer points of contact (fewer molecules adhering), so less force is required to keep the object moving. At small but nonzero speeds, friction is nearly independent of speed.

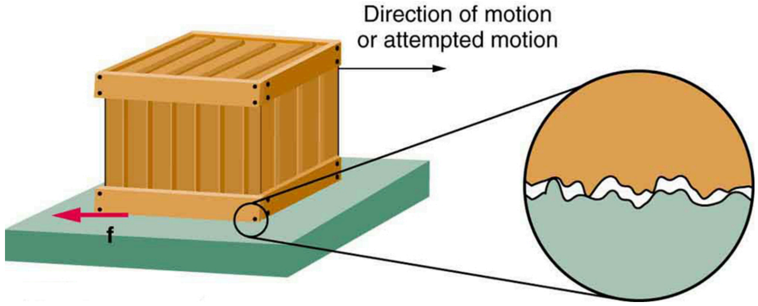


Figure 2.13 Frictional forces, such as f, always oppose motion or attempted motion between objects in contact. Friction arises in part because of the roughness of the surfaces in contact, as seen in the expanded view. In order for the object to move, it must rise to where the peaks can skip along the bottom surface. Thus a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. Such adhesive forces also depend on the substances the surfaces are made of, explaining, for example, why rubber-soled shoes slip less than those with leather soles.

The magnitude of the frictional force has two forms: one for static situations (static friction), the other for when there is motion (kinetic friction).

When there is no motion between the objects, the **magnitude of static friction $\mathbf{f_s}$** is

$$f_s \leq \mu_s N, \tag{2.17}$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force (the force perpendicular to the surface).

Magnitude of Static Friction

Magnitude of static friction f_s is

$$f_s \leq \mu_s N, \tag{2.18}$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force.

The symbol $\leq$ means *less than or equal to*, implying that static friction can have a minimum and a maximum value of $\mu_s N$. Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds $f_{s(max)}$, the object will move. Thus

$$f_{s(max)} = \mu_s N. \tag{2.19}$$

Once an object is moving, the **magnitude of kinetic friction $\mathbf{f_k}$** is given by

$$f_k = \mu_k N, \tag{2.20}$$

where μ_k is the coefficient of kinetic friction. A system in which $f_k = \mu_k N$ is described as a system in which *friction behaves simply*.

Magnitude of Kinetic Friction

The magnitude of kinetic friction f_k is given by

$$f_k = \mu_k N, \tag{2.21}$$

where μ_k is the coefficient of kinetic friction.

As seen in **Table 2.2**, the coefficients of kinetic friction are less than their static counterparts. That values of μ in **Table 2.2** are stated to only one or, at most, two digits is an indication of the approximate description of friction given by the above two equations.

Table 2.2 Coefficients of Static and Kinetic Friction

System	Static friction μ_s	Kinetic friction μ_k
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.7	0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.04	0.02

The equations given earlier include the dependence of friction on materials and the normal force. The direction of friction is always opposite that of motion, parallel to the surface between objects, and perpendicular to the normal force. For example, if the crate you try to push (with a force parallel to the floor) has a mass of 100 kg, then the normal force would be equal to its weight, $W = mg = (100\ \text{kg})(9.80\ \text{m/s}^2) = 980\ \text{N}$, perpendicular to the floor. If the coefficient of static friction is 0.45, you would have to exert a force parallel to the floor greater than $f_{\text{s(max)}} = \mu_\text{s}N = (0.45)(980\ \text{N}) = 440\ \text{N}$ to move the crate. Once there is motion, friction is less and the coefficient of kinetic friction might be 0.30, so that a force of only 290 N ($f_\text{k} = \mu_\text{k}N = (0.30)(980\ \text{N}) = 290\ \text{N}$) would keep it moving at a constant speed. If the floor is lubricated, both coefficients are considerably less than they would be without lubrication. Coefficient of friction is a unit less quantity with a magnitude usually between 0 and 1.0. The coefficient of the friction depends on the two surfaces that are in contact.

Take-Home Experiment

Find a small plastic object (such as a food container) and slide it on a kitchen table by giving it a gentle tap. Now spray water on the table, simulating a light shower of rain. What happens now when you give the object the same-sized tap? Now add a few drops of (vegetable or olive) oil on the surface of the water and give the same tap. What happens now? This latter situation is particularly important for drivers to note, especially after a light rain shower. Why?

Many people have experienced the slipperiness of walking on ice. However, many parts of the body, especially the joints, have much smaller coefficients of friction—often three or four times less than ice. A joint is formed by the ends of two bones, which are connected by thick tissues. The knee joint is formed by the lower leg bone (the tibia) and the thighbone (the femur). The hip is a ball (at the end of the femur) and socket (part of the pelvis) joint. The ends of the bones in the joint are covered by cartilage, which provides a smooth, almost glassy surface. The joints also produce a fluid (synovial fluid) that reduces friction and wear. A damaged or arthritic joint can be replaced by an artificial joint (**Figure 2.14**). These replacements can be made of metals (stainless steel or titanium) or plastic (polyethylene), also with very small coefficients of friction.

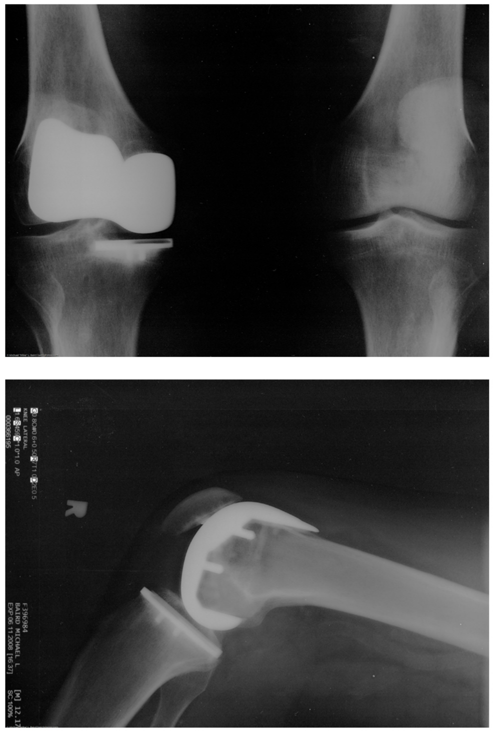


Figure 2.14 Artificial knee replacement is a procedure that has been performed for more than 20 years. In this figure, we see the post-op x rays of the right knee joint replacement. (credit: Mike Baird, Flickr)

Other natural lubricants include saliva produced in our mouths to aid in the swallowing process, and the slippery mucus found between organs in the body, allowing them to move freely past each other during heartbeats, during breathing, and when a person moves. Artificial lubricants are also common in hospitals and doctor’s clinics. For example, when ultrasonic imaging is carried out, the gel that couples the transducer to the skin also serves to to lubricate the surface between the transducer and the skin—thereby reducing the coefficient of friction between the two surfaces. This allows the transducer to mover freely over the skin.

Example 2.2 Skiing Exercise

A skier with a mass of 62 kg is sliding down a snowy slope. Find the coefficient of kinetic friction for the skier if friction is known to be 45.0 N.

Strategy

The magnitude of kinetic friction was given in to be 45.0 N. Kinetic friction is related to the normal force N as $f_{\mathrm{k}} = \mu_{\mathrm{k}} N$; thus, the coefficient of kinetic friction can be found if we can find the normal force of the skier on a slope. The normal force is always perpendicular to the surface, and since there is no motion perpendicular to the surface, the normal force should equal the component of the skier’s weight perpendicular to the slope. (See the skier and free-body diagram in **Figure 2.15**.)

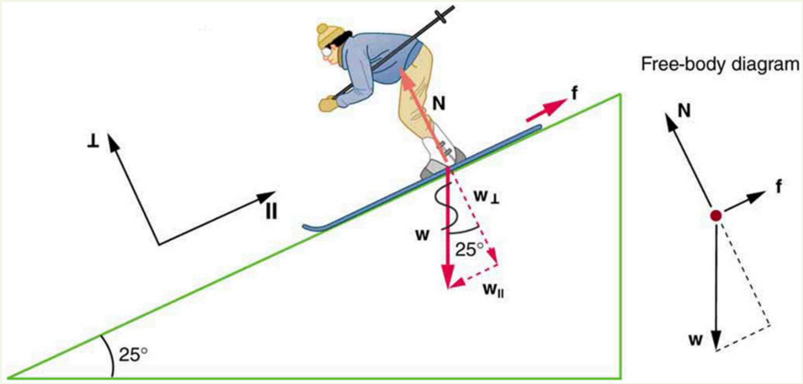


Figure 2.15 The motion of the skier and friction are parallel to the slope and so it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). $\mathbf{N}$ (the normal force) is perpendicular to the slope, and $\mathbf{f}$ (the friction) is parallel to the slope, but $\mathbf{w}$ (the skier's weight) has components along both axes, namely $\mathbf{w}_{\perp}$ and $\mathbf{W}_{//}$. $\mathbf{N}$ is equal in magnitude to $\mathbf{w}_{\perp}$, so there is no motion perpendicular to the slope. However, $\mathbf{f}$ is less than $\mathbf{W}_{//}$ in magnitude, so there is acceleration down the slope (along the *x*-axis).

That is,

$$N = w_{\perp} = w\cos 25^{\circ} = mg\cos 25^{\circ}. \tag{2.22}$$

Substituting this into our expression for kinetic friction, we get

$$f_{\mathrm{k}} = \mu_{\mathrm{k}} mg\cos 25^{\circ}, \tag{2.23}$$

which can now be solved for the coefficient of kinetic friction μ_{k}.

Solution

Solving for μ_{k} gives

$$\mu_{\mathrm{k}} = \frac{f_{\mathrm{k}}}{N} = \frac{f_{\mathrm{k}}}{w\cos 25^{\circ}} = \frac{f_{\mathrm{k}}}{mg\cos 25^{\circ}}. \tag{2.24}$$

Substituting known values on the right-hand side of the equation,

$$\mu_{\mathrm{k}} = \frac{45.0\ \mathrm{N}}{(62\ \mathrm{kg})(9.80\ \mathrm{m/s^2})(0.906)} = 0.082. \tag{2.25}$$

Discussion

This result is a little smaller than the coefficient listed in **Table 2.2** for waxed wood on snow, but it is still reasonable since values of the coefficients of friction can vary greatly. In situations like this, where an object of mass m slides down a slope that makes an angle θ with the horizontal, friction is given by $f_{\mathrm{k}} = \mu_{\mathrm{k}} mg\cos\theta$. All objects will slide down a slope with constant acceleration under these circumstances. Proof of this is left for this chapter's Problems and Exercises.

Take-Home Experiment

An object will slide down an inclined plane at a constant velocity if the net force on the object is zero. We can use this fact to measure the coefficient of kinetic friction between two objects. As shown in **Example 2.2**, the kinetic friction on a slope $f_{\mathrm{k}} = \mu_{\mathrm{k}} mg\cos\theta$. The component of the weight down the slope is equal to $mg\sin\theta$ (see the free-body diagram in **Figure 2.15**). These forces act in opposite directions, so when they have equal magnitude, the acceleration is zero. Writing these out:

$$f_{\mathrm{k}} = Fg_x \tag{2.26}$$

$$\mu_{\mathrm{k}} mg\cos\theta = mg\sin\theta. \tag{2.27}$$

Solving for μ_{k}, we find that

$$\mu_{\mathrm{k}} = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta. \tag{2.28}$$

Put a coin on a book and tilt it until the coin slides at a constant velocity down the book. You might need to tap the book lightly to get the coin to move. Measure the angle of tilt relative to the horizontal and find μ_k. Note that the coin will not start to slide at all until an angle greater than θ is attained, since the coefficient of static friction is larger than the coefficient of kinetic friction. Discuss how this may affect the value for μ_k and its uncertainty.

We have discussed that when an object rests on a horizontal surface, there is a normal force supporting it equal in magnitude to its weight. Furthermore, simple friction is always proportional to the normal force.

Making Connections: Submicroscopic Explanations of Friction

The simpler aspects of friction dealt with so far are its macroscopic (large-scale) characteristics. Great strides have been made in the atomic-scale explanation of friction during the past several decades. Researchers are finding that the atomic nature of friction seems to have several fundamental characteristics. These characteristics not only explain some of the simpler aspects of friction—they also hold the potential for the development of nearly friction-free environments that could save hundreds of billions of dollars in energy which is currently being converted (unnecessarily) to heat.

Figure 2.16 illustrates one macroscopic characteristic of friction that is explained by microscopic (small-scale) research. We have noted that friction is proportional to the normal force, but not to the area in contact, a somewhat counterintuitive notion. When two rough surfaces are in contact, the actual contact area is a tiny fraction of the total area since only high spots touch. When a greater normal force is exerted, the actual contact area increases, and it is found that the friction is proportional to this area.

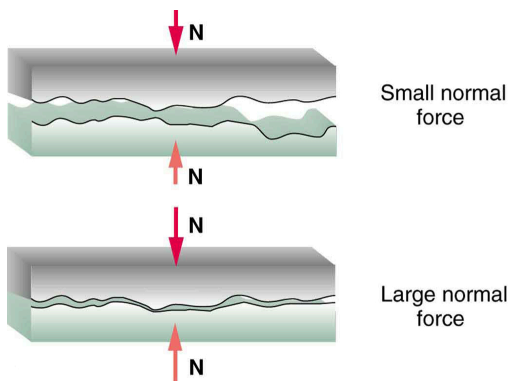


Figure 2.16 Two rough surfaces in contact have a much smaller area of actual contact than their total area. When there is a greater normal force as a result of a greater applied force, the area of actual contact increases as does friction.

But the atomic-scale view promises to explain far more than the simpler features of friction. The mechanism for how heat is generated is now being determined. In other words, why do surfaces get warmer when rubbed? Essentially, atoms are linked with one another to form lattices. When surfaces rub, the surface atoms adhere and cause atomic lattices to vibrate—essentially creating sound waves that penetrate the material. The sound waves diminish with distance and their energy is converted into heat. Chemical reactions that are related to frictional wear can also occur between atoms and molecules on the surfaces. **Figure 2.17** shows how the tip of a probe drawn across another material is deformed by atomic-scale friction. The force needed to drag the tip can be measured and is found to be related to shear stress, which will be discussed later in this chapter. The variation in shear stress is remarkable (more than a factor of 10^{12}) and difficult to predict theoretically, but shear stress is yielding a fundamental understanding of a large-scale phenomenon known since ancient times—friction.

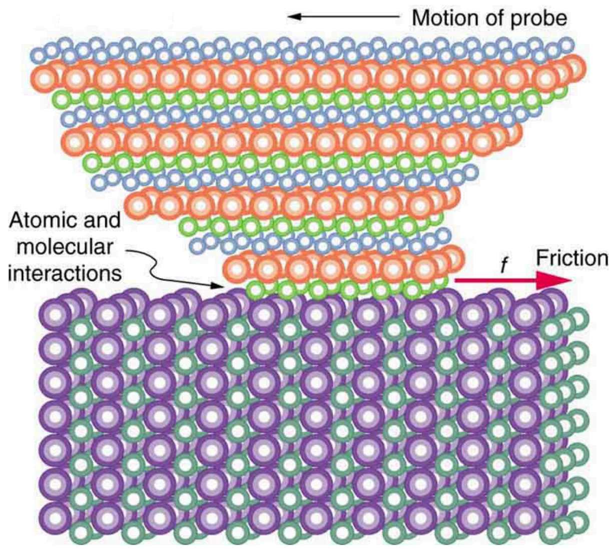


Figure 2.17 The tip of a probe is deformed sideways by frictional force as the probe is dragged across a surface. Measurements of how the force varies for different materials are yielding fundamental insights into the atomic nature of friction.

PhET Explorations: Forces and Motion

Explore the forces at work when you try to push a filing cabinet. Create an applied force and see the resulting friction force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. Draw a free-body diagram of all the forces (including gravitational and normal forces).

PhET Interactive Simulation

Figure 2.18 Forces and Motion (http://legacy.cnx.org/content/m64350/1.2/forces-and-motion_en.jar)

2.7 Elasticity: Stress and Strain

UMASS AMHERST Instructor's Notes

This section will be discussed in class, so reading this section is not required.

We now move from consideration of forces that affect the motion of an object (such as friction and drag) to those that affect an object's shape. If a bulldozer pushes a car into a wall, the car will not move but it will noticeably change shape. A change in shape due to the application of a force is a **deformation**. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed—that is, the deformation is elastic for small deformations. Second, the size of the deformation is proportional to the force—that is, for small deformations, Hooke's law is obeyed. In equation form, **Hooke's law** is given by

$$F = k\Delta L, \tag{2.29}$$

where ΔL is the amount of deformation (the change in length, for example) produced by the force F, and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force. Note that this force is a function of the deformation ΔL —it is not constant as a kinetic friction force is. Rearranging this to

$$\Delta L = \frac{F}{k} \tag{2.30}$$

makes it clear that the deformation is proportional to the applied force. **Figure 2.19** shows the Hooke's law relationship between the extension ΔL of a spring or of a human bone. For metals or springs, the straight line region in which Hooke's law pertains is much larger. Bones are brittle and the elastic region is small and the fracture abrupt. Eventually a large enough stress to the material will cause it to break or fracture. **Tensile strength** is the breaking stress that will cause permanent deformation or fracture of a material.

Hooke's Law

$$F = k\Delta L, \tag{2.31}$$

where ΔL is the amount of deformation (the change in length, for example) produced by the force F, and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force.

$$\Delta L = \frac{F}{k} \tag{2.32}$$

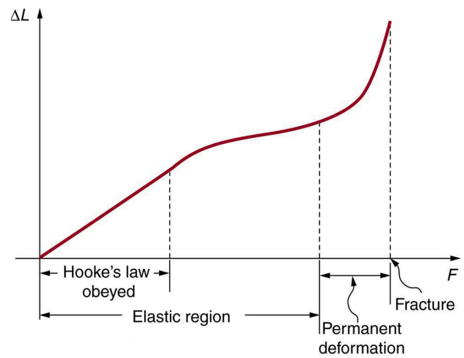


Figure 2.19 A graph of deformation ΔL versus applied force F. The straight segment is the linear region where Hooke's law is obeyed. The slope of the straight region is $\frac{1}{k}$. For larger forces, the graph is curved but the deformation is still elastic— ΔL will return to zero if the force is removed. Still greater forces permanently deform the object until it finally fractures. The shape of the curve near fracture depends on several factors, including how the force F is applied. Note that in this graph the slope increases just before fracture, indicating that a small increase in F is producing a large increase in L near the fracture.

The proportionality constant k depends upon a number of factors for the material. For example, a guitar string made of nylon stretches when it is tightened, and the elongation ΔL is proportional to the force applied (at least for small deformations). Thicker nylon strings and ones made of steel stretch less for the same applied force, implying they have a larger k (see **Figure 2.20**). Finally, all three strings return to their normal lengths when the force is removed, provided the deformation is small. Most materials will behave in this manner if the deformation is less than about 0.1% or about 1 part in 10^3.

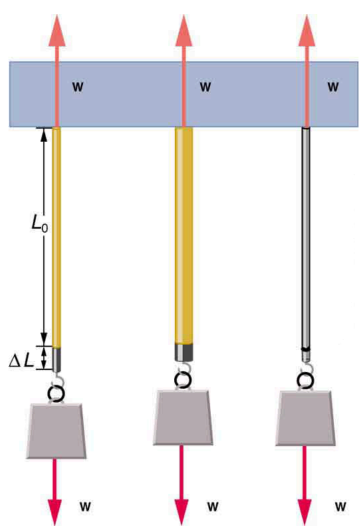


Figure 2.20 The same force, in this case a weight (w), applied to three different guitar strings of identical length produces the three different deformations shown as shaded segments. The string on the left is thin nylon, the one in the middle is thicker nylon, and the one on the right is steel.

Stretch Yourself a Little

How would you go about measuring the proportionality constant k of a rubber band? If a rubber band stretched 3 cm when a 100-g mass was attached to it, then how much would it stretch if two similar rubber bands were attached to the same mass—even if put together in parallel or alternatively if tied together in series?

We now consider three specific types of deformations: changes in length (tension and compression), sideways shear (stress), and changes in volume. All deformations are assumed to be small unless otherwise stated.

Changes in Length—Tension and Compression: Elastic Modulus

A change in length ΔL is produced when a force is applied to a wire or rod parallel to its length L_0, either stretching it (a tension) or compressing it. (See **Figure 2.21**.)

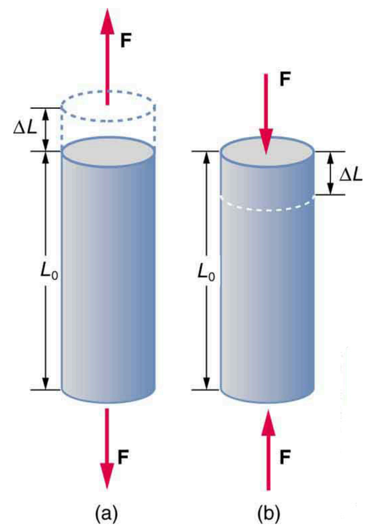


Figure 2.21 (a) Tension. The rod is stretched a length ΔL when a force is applied parallel to its length. (b) Compression. The same rod is compressed by forces with the same magnitude in the opposite direction. For very small deformations and uniform materials, ΔL is approximately the same for the same magnitude of tension or compression. For larger deformations, the cross-sectional area changes as the rod is compressed or stretched.

Experiments have shown that the change in length (ΔL) depends on only a few variables. As already noted, ΔL is proportional to the force F and depends on the substance from which the object is made. Additionally, the change in length is proportional to the original length L_0 and inversely proportional to the cross-sectional area of the wire or rod. For example, a long guitar string will stretch more than a short one, and a thick string will stretch less than a thin one. We can combine all these factors into one equation for ΔL :

$$\Delta L = \frac{1}{Y}\frac{F}{A}L_0, \tag{2.33}$$

where ΔL is the change in length, F the applied force, Y is a factor, called the elastic modulus or Young's modulus, that depends on the substance, A is the cross-sectional area, and L_0 is the original length. **Table 2.3** lists values of Y for several materials—those with a large Y are said to have a large tensile stifness because they deform less for a given tension or compression.

Table 2.3 Elastic Moduli[2]

Material	Young's modulus (tension–compression) *Y* $(10^9\ \text{N/m}^2)$	Shear modulus *S* $(10^9\ \text{N/m}^2)$	Bulk modulus *B* $(10^9\ \text{N/m}^2)$
Aluminum	70	25	75
Bone – tension	16	80	8
Bone – compression	9		
Brass	90	35	75
Brick	15		
Concrete	20		
Glass	70	20	30
Granite	45	20	45
Hair (human)	10		
Hardwood	15	10	
Iron, cast	100	40	90
Lead	16	5	50
Marble	60	20	70
Nylon	5		
Polystyrene	3		
Silk	6		
Spider thread	3		
Steel	210	80	130
Tendon	1		
Acetone			0.7
Ethanol			0.9
Glycerin			4.5
Mercury			25
Water			2.2

Young's moduli are not listed for liquids and gases in **Table 2.3** because they cannot be stretched or compressed in only one direction. Note that there is an assumption that the object does not accelerate, so that there are actually two applied forces of magnitude F acting in opposite directions. For example, the strings in **Figure 2.21** are being pulled down by a force of magnitude w and held up by the ceiling, which also exerts a force of magnitude w.

Example 2.3 The Stretch of a Long Cable

Suspension cables are used to carry gondolas at ski resorts. (See **Figure 2.22**) Consider a suspension cable that includes an unsupported span of 3020 m. Calculate the amount of stretch in the steel cable. Assume that the cable has a diameter of 5.6 cm and the maximum tension it can withstand is $3.0\times10^6\ \text{N}$.

2. Approximate and average values. Young's moduli Y for tension and compression sometimes differ but are averaged here. Bone has significantly different Young's moduli for tension and compression.

Figure 2.22 Gondolas travel along suspension cables at the Gala Yuzawa ski resort in Japan. (credit: Rudy Herman, Flickr)

Strategy

The force is equal to the maximum tension, or $F = 3.0\times10^{6}\ \text{N}$. The cross-sectional area is $\pi r^2 = 2.46\times10^{-3}\ \text{m}^2$. The equation $\Delta L = \frac{1}{Y}\frac{F}{A}L_0$ can be used to find the change in length.

Solution

All quantities are known. Thus,

$$\begin{aligned}\Delta L &= \left(\frac{1}{210\times10^{9}\ \text{N/m}^2}\right)\left(\frac{3.0\times10^{6}\ \text{N}}{2.46\times10^{-3}\ \text{m}^2}\right)(3020\ \text{m}) \\ &= 18\ \text{m}.\end{aligned} \tag{2.34}$$

Discussion

This is quite a stretch, but only about 0.6% of the unsupported length. Effects of temperature upon length might be important in these environments.

Bones, on the whole, do not fracture due to tension or compression. Rather they generally fracture due to sideways impact or bending, resulting in the bone shearing or snapping. The behavior of bones under tension and compression is important because it determines the load the bones can carry. Bones are classified as weight-bearing structures such as columns in buildings and trees. Weight-bearing structures have special features; columns in building have steel-reinforcing rods while trees and bones are fibrous. The bones in different parts of the body serve different structural functions and are prone to different stresses. Thus the bone in the top of the femur is arranged in thin sheets separated by marrow while in other places the bones can be cylindrical and filled with marrow or just solid. Overweight people have a tendency toward bone damage due to sustained compressions in bone joints and tendons.

Another biological example of Hooke's law occurs in tendons. Functionally, the tendon (the tissue connecting muscle to bone) must stretch easily at first when a force is applied, but offer a much greater restoring force for a greater strain. **Figure 2.23** shows a stress-strain relationship for a human tendon. Some tendons have a high collagen content so there is relatively little strain, or length change; others, like support tendons (as in the leg) can change length up to 10%. Note that this stress-strain curve is nonlinear, since the slope of the line changes in different regions. In the first part of the stretch called the toe region, the fibers in the tendon begin to align in the direction of the stress—this is called *uncrimping*. In the linear region, the fibrils will be stretched, and in the failure region individual fibers begin to break. A simple model of this relationship can be illustrated by springs in parallel: different springs are activated at different lengths of stretch. Examples of this are given in the problems at end of this chapter. Ligaments (tissue connecting bone to bone) behave in a similar way.

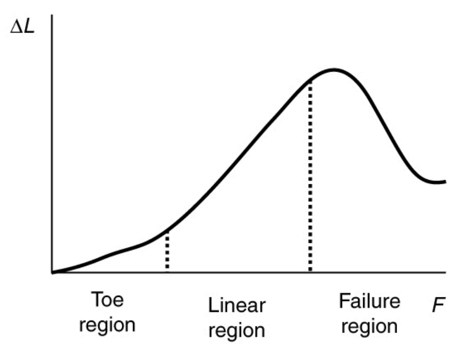


Figure 2.23 Typical stress-strain curve for mammalian tendon. Three regions are shown: (1) toe region (2) linear region, and (3) failure region.

Unlike bones and tendons, which need to be strong as well as elastic, the arteries and lungs need to be very stretchable. The elastic properties of the arteries are essential for blood flow. The pressure in the arteries increases and arterial walls stretch when the blood is pumped out of the heart. When the aortic valve shuts, the pressure in the arteries drops and the arterial walls relax to maintain the blood flow. When you feel your pulse, you are feeling exactly this—the elastic behavior of the arteries as the blood gushes through with each pump of the heart. If the arteries were rigid, you would not feel a pulse. The heart is also an organ with special elastic properties. The lungs expand with muscular effort when we breathe in but relax freely and elastically when we breathe out. Our skins are particularly elastic, especially for the young. A young person can go from 100 kg to 60 kg with no visible sag in their skins. The elasticity of all organs reduces with age. Gradual physiological aging through reduction in elasticity starts in the early 20s.

Example 2.4 Calculating Deformation: How Much Does Your Leg Shorten When You Stand on It?

Calculate the change in length of the upper leg bone (the femur) when a 70.0 kg man supports 62.0 kg of his mass on it, assuming the bone to be equivalent to a uniform rod that is 40.0 cm long and 2.00 cm in radius.

Strategy

The force is equal to the weight supported, or

$$F = mg = (62.0\text{ kg})\left(9.80\text{ m/s}^2\right) = 607.6\text{ N}, \tag{2.35}$$

and the cross-sectional area is $\pi r^2 = 1.257\times 10^{-3}\text{ m}^2$. The equation $\Delta L = \frac{1}{Y}\frac{F}{A}L_0$ can be used to find the change in length.

Solution

All quantities except ΔL are known. Note that the compression value for Young's modulus for bone must be used here. Thus,

$$\begin{aligned} \Delta L &= \left(\frac{1}{9\times 10^9\text{ N/m}^2}\right)\left(\frac{607.6\text{ N}}{1.257\times 10^{-3}\text{ m}^2}\right)(0.400\text{ m}) \\ &= 2\times 10^{-5}\text{ m}. \end{aligned} \tag{2.36}$$

Discussion

This small change in length seems reasonable, consistent with our experience that bones are rigid. In fact, even the rather large forces encountered during strenuous physical activity do not compress or bend bones by large amounts. Although bone is rigid compared with fat or muscle, several of the substances listed in Table 2.3 have larger values of Young's modulus Y. In other words, they are more rigid.

The equation for change in length is traditionally rearranged and written in the following form:

$$\frac{F}{A} = Y\frac{\Delta L}{L_0}. \tag{2.37}$$

The ratio of force to area, $\frac{F}{A}$, is defined as **stress** (measured in N/m^2), and the ratio of the change in length to length, $\frac{\Delta L}{L_0}$, is defined as **strain** (a unitless quantity). In other words,

$$\text{stress} = Y\times\text{strain}. \tag{2.38}$$

In this form, the equation is analogous to Hooke's law, with stress analogous to force and strain analogous to deformation. If we again rearrange this equation to the form

$$F = YA\frac{\Delta L}{L_0}, \tag{2.39}$$

we see that it is the same as Hooke's law with a proportionality constant

$$k = \frac{YA}{L_0}. \tag{2.40}$$

This general idea—that force and the deformation it causes are proportional for small deformations—applies to changes in length, sideways bending, and changes in volume.

Stress

The ratio of force to area, $\frac{F}{A}$, is defined as stress measured in N/m^2.

Strain

The ratio of the change in length to length, $\frac{\Delta L}{L_0}$, is defined as strain (a unitless quantity). In other words,

$$\text{stress} = Y \times \text{strain}. \tag{2.41}$$

Sideways Stress: Shear Modulus

Figure 2.24 illustrates what is meant by a sideways stress or a *shearing force*. Here the deformation is called Δx and it is perpendicular to L_0, rather than parallel as with tension and compression. Shear deformation behaves similarly to tension and compression and can be described with similar equations. The expression for **shear deformation** is

$$\Delta x = \frac{1}{S}\frac{F}{A}L_0, \tag{2.42}$$

where S is the shear modulus (see **Table 2.3**) and F is the force applied perpendicular to L_0 and parallel to the cross-sectional area A. Again, to keep the object from accelerating, there are actually two equal and opposite forces F applied across opposite faces, as illustrated in **Figure 2.24**. The equation is logical—for example, it is easier to bend a long thin pencil (small A) than a short thick one, and both are more easily bent than similar steel rods (large S).

Shear Deformation

$$\Delta x = \frac{1}{S}\frac{F}{A}L_0, \tag{2.43}$$

where S is the shear modulus and F is the force applied perpendicular to L_0 and parallel to the cross-sectional area A.

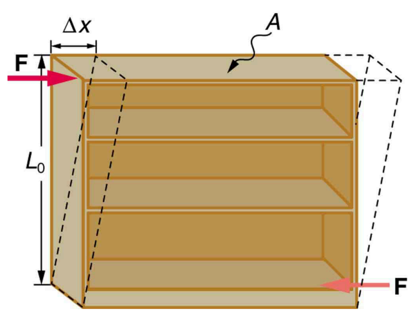


Figure 2.24 Shearing forces are applied perpendicular to the length L_0 and parallel to the area A, producing a deformation Δx. Vertical forces are not shown, but it should be kept in mind that in addition to the two shearing forces, $\mathbf{F}$, there must be supporting forces to keep the object from rotating. The distorting effects of these supporting forces are ignored in this treatment. The weight of the object also is not shown, since it is usually negligible compared with forces large enough to cause significant deformations.

Examination of the shear moduli in **Table 2.3** reveals some telling patterns. For example, shear moduli are less than Young's moduli for most materials. Bone is a remarkable exception. Its shear modulus is not only greater than its Young's modulus, but it is as large as that of steel. This is why bones are so rigid.

The spinal column (consisting of 26 vertebral segments separated by discs) provides the main support for the head and upper part of the body. The spinal column has normal curvature for stability, but this curvature can be increased, leading to increased shearing forces on the lower vertebrae. Discs are better at withstanding compressional forces than shear forces. Because the spine is not vertical, the weight of the upper body exerts some of both. Pregnant women and people that are overweight (with large abdomens) need to move their shoulders back to maintain balance, thereby increasing the curvature in their spine and so increasing the shear component of the stress. An increased angle due to more curvature increases the shear forces along the plane. These higher shear forces increase the risk of back injury through ruptured discs. The lumbosacral disc (the wedge shaped disc below the last vertebrae) is particularly at risk because of its location.

The shear moduli for concrete and brick are very small; they are too highly variable to be listed. Concrete used in buildings can withstand compression, as in pillars and arches, but is very poor against shear, as might be encountered in heavily loaded floors or during earthquakes. Modern structures were made possible by the use of steel and steel-reinforced concrete. Almost by definition, liquids and gases have shear moduli near zero, because they flow in response to shearing forces.

Example 2.5 Calculating Force Required to Deform: That Nail Does Not Bend Much Under a Load

Find the mass of the picture hanging from a steel nail as shown in **Figure 2.25**, given that the nail bends only $1.80\ \mu\text{m}$. (Assume the shear modulus is known to two significant figures.)

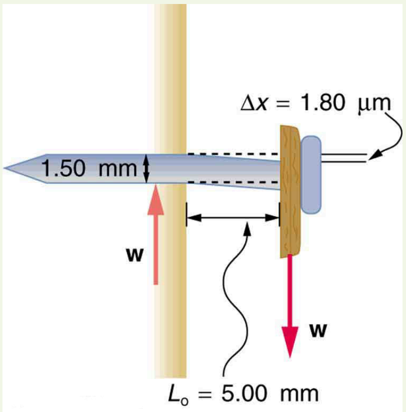


Figure 2.25 Side view of a nail with a picture hung from it. The nail flexes very slightly (shown much larger than actual) because of the shearing effect of the supported weight. Also shown is the upward force of the wall on the nail, illustrating that there are equal and opposite forces applied across opposite cross sections of the nail. See **Example 2.5** for a calculation of the mass of the picture.

Strategy

The force F on the nail (neglecting the nail's own weight) is the weight of the picture w . If we can find w , then the mass of the picture is just $\frac{w}{g}$. The equation $\Delta x = \frac{1}{S}\frac{F}{A}L_0$ can be solved for F .

Solution

Solving the equation $\Delta x = \frac{1}{S}\frac{F}{A}L_0$ for F , we see that all other quantities can be found:

$$F = \frac{SA}{L_0}\Delta x. \tag{2.44}$$

S is found in **Table 2.3** and is $S = 80\times10^9\ \text{N/m}^2$. The radius r is 0.750 mm (as seen in the figure), so the cross-sectional area is

$$A = \pi r^2 = 1.77\times10^{-6}\ \text{m}^2. \tag{2.45}$$

The value for L_0 is also shown in the figure. Thus,

$$F = \frac{(80\times10^9\ \text{N/m}^2)(1.77\times10^{-6}\ \text{m}^2)}{(5.00\times10^{-3}\ \text{m})}(1.80\times10^{-6}\ \text{m}) = 51\ \text{N}. \tag{2.46}$$

This 51 N force is the weight w of the picture, so the picture's mass is

$$m = \frac{w}{g} = \frac{F}{g} = 5.2\ \text{kg}. \tag{2.47}$$

Discussion

This is a fairly massive picture, and it is impressive that the nail flexes only $1.80\ \mu\text{m}$ —an amount undetectable to the unaided eye.

Changes in Volume: Bulk Modulus

An object will be compressed in all directions if inward forces are applied evenly on all its surfaces as in **Figure 2.26**. It is relatively easy to compress gases and extremely difficult to compress liquids and solids. For example, air in a wine bottle is compressed when it is corked. But if you try corking a brim-full bottle, you cannot compress the wine—some must be removed if the cork is to be inserted. The reason for these different compressibilities is that atoms and molecules are separated by large empty spaces in gases but packed close together in liquids and solids. To compress a gas, you must force its atoms and molecules closer together. To compress liquids and solids, you must actually compress their atoms and molecules, and very strong electromagnetic forces in them oppose this compression.

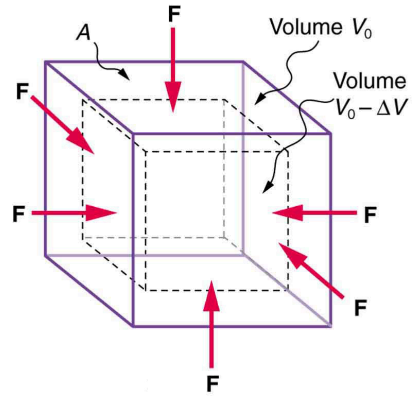


Figure 2.26 An inward force on all surfaces compresses this cube. Its change in volume is proportional to the force per unit area and its original volume, and is related to the compressibility of the substance.

We can describe the compression or volume deformation of an object with an equation. First, we note that a force "applied evenly" is defined to have the same stress, or ratio of force to area $\frac{F}{A}$ on all surfaces. The deformation produced is a change in volume ΔV, which is found to behave very similarly to the shear, tension, and compression previously discussed. (This is not surprising, since a compression of the entire object is equivalent to compressing each of its three dimensions.) The relationship of the change in volume to other physical quantities is given by

$$\Delta V = \frac{1}{B}\frac{F}{A}V_0, \tag{2.48}$$

where B is the bulk modulus (see **Table 2.3**), V_0 is the original volume, and $\frac{F}{A}$ is the force per unit area applied uniformly inward on all surfaces. Note that no bulk moduli are given for gases.

What are some examples of bulk compression of solids and liquids? One practical example is the manufacture of industrial-grade diamonds by compressing carbon with an extremely large force per unit area. The carbon atoms rearrange their crystalline structure into the more tightly packed pattern of diamonds. In nature, a similar process occurs deep underground, where extremely large forces result from the weight of overlying material. Another natural source of large compressive forces is the pressure created by the weight of water, especially in deep parts of the oceans. Water exerts an inward force on all surfaces of a submerged object, and even on the water itself. At great depths, water is measurably compressed, as the following example illustrates.

Example 2.6 Calculating Change in Volume with Deformation: How Much Is Water Compressed at Great Ocean Depths?

Calculate the fractional decrease in volume ($\frac{\Delta V}{V_0}$) for seawater at 5.00 km depth, where the force per unit area is $5.00\times10^7\ \mathrm{N/m^2}$.

Strategy

Equation $\Delta V = \frac{1}{B}\frac{F}{A}V_0$ is the correct physical relationship. All quantities in the equation except $\frac{\Delta V}{V_0}$ are known.

Solution

Solving for the unknown $\frac{\Delta V}{V_0}$ gives

$$\frac{\Delta V}{V_0} = \frac{1}{B}\frac{F}{A}. \tag{2.49}$$

Substituting known values with the value for the bulk modulus B from **Table 2.3**,

$$\begin{aligned}\frac{\Delta V}{V_0} &= \frac{5.00\times10^7\ \mathrm{N/m^2}}{2.2\times10^9\ \mathrm{N/m^2}}\\ &= 0.023 = 2.3\%.\end{aligned} \tag{2.50}$$

Discussion

Although measurable, this is not a significant decrease in volume considering that the force per unit area is about 500 atmospheres (1 million pounds per square foot). Liquids and solids are extraordinarily difficult to compress.

Conversely, very large forces are created by liquids and solids when they try to expand but are constrained from doing so—which is equivalent to compressing them to less than their normal volume. This often occurs when a contained material warms up, since most materials expand when their temperature increases. If the materials are tightly constrained, they deform or break their container. Another very common example occurs when water freezes. Water, unlike most materials, expands when it freezes, and it can easily fracture a boulder, rupture a biological cell, or crack an engine block that gets in its way.

Other types of deformations, such as torsion or twisting, behave analogously to the tension, shear, and bulk deformations considered here.

2.8 Drag Forces

UMASS AMHERST Instructor's Notes

This section has been included for completeness, and the content here may only be briefly touched upon in class.

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either a gas or a liquid). You feel the drag force when you move your hand through water. You might also feel it if you move your hand during a strong wind. The faster you move your hand, the harder it is to move. You feel a smaller drag force when you tilt your hand so only the side goes through the air—you have decreased the area of your hand that faces the direction of motion. Like friction, the **drag force** always opposes the motion of an object. Unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid. This functionality is complicated and depends upon the shape of the object, its size, its velocity, and the fluid it is in. For most large objects such as bicyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force F_D is found to be proportional to the square of the speed of the object. We can write this relationship mathematically as $F_D \propto v^2$. When taking into account other factors, this relationship becomes

$$F_\text{D} = \frac{1}{2}\text{C}\rho A v^2, \tag{2.51}$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid. (Recall that density is mass per unit volume.) This equation can also be written in a more generalized fashion as $F_\text{D} = bv^2$, where b is a constant equivalent to $0.5C\rho A$. We have set the exponent for these equations as 2 because, when an object is moving at high velocity through air, the magnitude of the drag force is proportional to the square of the speed. As we shall see in a few pages on fluid dynamics, for small particles moving at low speeds in a fluid, the exponent is equal to 1.

Drag Force

Drag force F_D is found to be proportional to the square of the speed of the object. Mathematically

$$F_\text{D} \propto v^2 \tag{2.52}$$

$$F_\text{D} = \frac{1}{2}C\rho A v^2, \tag{2.53}$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid.

Athletes as well as car designers seek to reduce the drag force to lower their race times. (See **Figure 2.27**). “Aerodynamic” shaping of an automobile can reduce the drag force and so increase a car’s gas mileage.

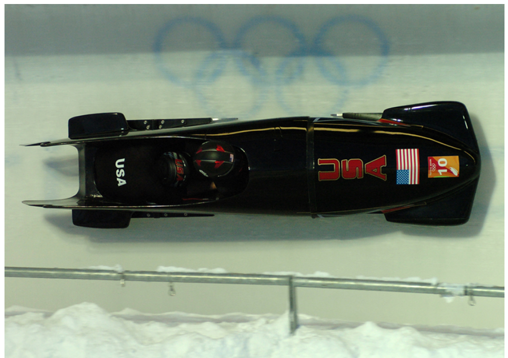


Figure 2.27 From racing cars to bobsled racers, aerodynamic shaping is crucial to achieving top speeds. Bobsleds are designed for speed. They are shaped like a bullet with tapered fins. (credit: U.S. Army, via Wikimedia Commons)

The value of the drag coefficient, C , is determined empirically, usually with the use of a wind tunnel. (See **Figure 2.28**).

Figure 2.28 NASA researchers test a model plane in a wind tunnel. (credit: NASA/Ames)

The drag coefficient can depend upon velocity, but we will assume that it is a constant here. **Table 2.4** lists some typical drag coefficients for a variety of objects. Notice that the drag coefficient is a dimensionless quantity. At highway speeds, over 50% of the power of a car is used to overcome air drag. The most fuel-efficient cruising speed is about 70–80 km/h (about 45–50 mi/h). For this reason, during the 1970s oil crisis in the United States, maximum speeds on highways were set at about 90 km/h (55 mi/h).

Table 2.4 Drag Coefficient Values **Typical values of drag coefficient C.**

Object	C
Airfoil	0.05
Toyota Camry	0.28
Ford Focus	0.32
Honda Civic	0.36
Ferrari Testarossa	0.37
Dodge Ram pickup	0.43
Sphere	0.45
Hummer H2 SUV	0.64
Skydiver (feet first)	0.70
Bicycle	0.90
Skydiver (horizontal)	1.0
Circular flat plate	1.12

Substantial research is under way in the sporting world to minimize drag. The dimples on golf balls are being redesigned as are the clothes that athletes wear. Bicycle racers and some swimmers and runners wear full bodysuits. Australian Cathy Freeman wore a full body suit in the 2000 Sydney Olympics, and won the gold medal for the 400 m race. Many swimmers in the 2008 Beijing Olympics wore (Speedo) body suits; it might have made a difference in breaking many world records (See **Figure 2.29**). Most elite swimmers (and cyclists) shave their body hair. Such innovations can have the effect of slicing away milliseconds in a race, sometimes making the difference between a gold and a silver medal. One consequence is that careful and precise guidelines must be continuously developed to maintain the integrity of the sport.

Figure 2.29 Body suits, such as this LZR Racer Suit, have been credited with many world records after their release in 2008. Smoother "skin" and more compression forces on a swimmer's body provide at least 10% less drag. (credit: NASA/Kathy Barnstorff)

Some interesting situations connected to Newton's second law occur when considering the effects of drag forces upon a moving object. For instance, consider a skydiver falling through air under the influence of gravity. The two forces acting on him are the force of gravity and the drag force (ignoring the buoyant force). The downward force of gravity remains constant regardless of the velocity at which the person is moving. However, as the person's velocity increases, the magnitude of the drag force increases until the magnitude of the drag force is equal to the gravitational force, thus producing a net force of zero. A zero net force means that there is no acceleration, as given by Newton's second law. At this point, the person's velocity remains constant and we say that the person has reached his *terminal velocity* (v_t). Since F_D is proportional to the speed, a heavier skydiver must go faster for F_D to equal his weight. Let's see how this works out more quantitatively.

At the terminal velocity,

$$F_{net} = mg - F_D = ma = 0. \tag{2.54}$$

Thus,

$$mg = F_D. \tag{2.55}$$

Using the equation for drag force, we have

$$mg = \frac{1}{2}\rho C A v^2. \tag{2.56}$$

Solving for the velocity, we obtain

$$v = \sqrt{\frac{2mg}{\rho CA}}. \tag{2.57}$$

Assume the density of air is $\rho = 1.21\ \text{kg/m}^3$. A 75-kg skydiver descending head first will have an area approximately $A = 0.18\ \text{m}^2$ and a drag coefficient of approximately $C = 0.70$. We find that

$$\begin{aligned} v &= \sqrt{\frac{2(75\ \text{kg})(9.80\ \text{m/s}^2)}{(1.21\ \text{kg/m}^3)(0.70)(0.18\ \text{m}^2)}} \\ &= 98\ \text{m/s} \\ &= 350\ \text{km/h}. \end{aligned} \tag{2.58}$$

This means a skydiver with a mass of 75 kg achieves a maximum terminal velocity of about 350 km/h while traveling in a pike (head first) position, minimizing the area and his drag. In a spread-eagle position, that terminal velocity may decrease to about 200 km/h as the area increases. This terminal velocity becomes much smaller after the parachute opens.

Take-Home Experiment

This interesting activity examines the effect of weight upon terminal velocity. Gather together some nested coffee filters. Leaving them in their original shape, measure the time it takes for one, two, three, four, and five nested filters to fall to the floor from the same height (roughly 2 m). (Note that, due to the way the filters are nested, drag is constant and only mass varies.) They obtain terminal velocity quite quickly, so find this velocity as a function of mass. Plot the terminal velocity v versus mass. Also plot v^2 versus mass. Which of these relationships is more linear? What can you conclude from these graphs?

Example 2.7 A Terminal Velocity

Find the terminal velocity of an 85-kg skydiver falling in a spread-eagle position.

Strategy

At terminal velocity, $F_{\text{net}} = 0$. Thus the drag force on the skydiver must equal the force of gravity (the person's weight). Using the equation of drag force, we find $mg = \frac{1}{2}\rho CAv^2$.

Thus the terminal velocity v_{t} can be written as

$$v_{\text{t}} = \sqrt{\frac{2mg}{\rho CA}}. \tag{2.59}$$

Solution

All quantities are known except the person's projected area. This is an adult (82 kg) falling spread eagle. We can estimate the frontal area as

$$A = (2\text{ m})(0.35\text{ m}) = 0.70\text{ m}^2. \tag{2.60}$$

Using our equation for v_{t}, we find that

$$\begin{aligned} v_{\text{t}} &= \sqrt{\frac{2(85\text{ kg})(9.80\text{ m/s}^2)}{(1.21\text{ kg/m}^3)(1.0)(0.70\text{ m}^2)}} \\ &= 44\text{ m/s}. \end{aligned} \tag{2.61}$$

Discussion

This result is consistent with the value for v_{t} mentioned earlier. The 75-kg skydiver going feet first had a $v = 98\text{ m/s}$. He weighed less but had a smaller frontal area and so a smaller drag due to the air.

The size of the object that is falling through air presents another interesting application of air drag. If you fall from a 5-m high branch of a tree, you will likely get hurt—possibly fracturing a bone. However, a small squirrel does this all the time, without getting hurt. You don't reach a terminal velocity in such a short distance, but the squirrel does.

The following interesting quote on animal size and terminal velocity is from a 1928 essay by a British biologist, J.B.S. Haldane, titled "On Being the Right Size."

To the mouse and any smaller animal, [gravity] presents practically no dangers. You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, and a horse splashes. For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal's length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only to a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force.

The above quadratic dependence of air drag upon velocity does not hold if the object is very small, is going very slow, or is in a denser medium than air. Then we find that the drag force is proportional just to the velocity. This relationship is given by **Stokes' law**, which states that

$$F_{\text{s}} = 6\pi r\eta v, \tag{2.62}$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity.

Stokes' Law

$$F_{\text{s}} = 6\pi r\eta v, \tag{2.63}$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity.

Good examples of this law are provided by microorganisms, pollen, and dust particles. Because each of these objects is so small, we find that many of these objects travel unaided only at a constant (terminal) velocity. Terminal velocities for bacteria (size about $1\ \mu\text{m}$) can be about $2\ \mu\text{m/s}$. To move at a greater speed, many bacteria swim using flagella (organelles shaped like little tails) that are powered by little motors embedded in the cell. Sediment in a lake can move at a greater terminal velocity (about $5\ \mu\text{m/s}$), so it can take days to reach the bottom of the lake after being deposited on the surface.

If we compare animals living on land with those in water, you can see how drag has influenced evolution. Fishes, dolphins, and even massive whales are streamlined in shape to reduce drag forces. Birds are streamlined and migratory species that fly large distances often have particular features such as long necks. Flocks of birds fly in the shape of a spear head as the flock forms a streamlined pattern (see **Figure 2.30**). In humans, one important example of streamlining is the shape of sperm, which need to be efficient in their use of energy.

Figure 2.30 Geese fly in a V formation during their long migratory travels. This shape reduces drag and energy consumption for individual birds, and also allows them a better way to communicate. (credit: Julo, Wikimedia Commons)

Galileo's Experiment

Galileo is said to have dropped two objects of different masses from the Tower of Pisa. He measured how long it took each to reach the ground. Since stopwatches weren't readily available, how do you think he measured their fall time? If the objects were the same size, but with different masses, what do you think he should have observed? Would this result be different if done on the Moon?

PhET Explorations: Masses & Springs

A realistic mass and spring laboratory. Hang masses from springs and adjust the spring stiffness and damping. You can even slow time. Transport the lab to different planets. A chart shows the kinetic, potential, and thermal energy for each spring.

PhET Interactive Simulation

Figure 2.31 Masses & Springs (http://legacy.cnx.org/content/m64348/1.2/mass-spring-lab_en.jar)

Glossary

carrier particle: a fundamental particle of nature that is surrounded by a characteristic force field; photons are carrier particles of the electromagnetic force

deformation: change in shape due to the application of force

drag force: F_D, found to be proportional to the square of the speed of the object; mathematically

$$F_\text{D} \propto v^2$$

$$F_D = \frac{1}{2}C\rho A v^2,$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid

force field: a region in which a test particle will experience a force

friction: a force that opposes relative motion or attempts at motion between systems in contact

Hooke's law: proportional relationship between the force F on a material and the deformation ΔL it causes, $F = k\Delta L$

inertial frame of reference: a coordinate system that is not accelerating; all forces acting in an inertial frame of reference are real forces, as opposed to fictitious forces that are observed due to an accelerating frame of reference

kinetic friction: a force that opposes the motion of two systems that are in contact and moving relative to one another

magnitude of kinetic friction: $f_k = \mu_k N$, where μ_k is the coefficient of kinetic friction

magnitude of static friction: $f_s \le \mu_s N$, where μ_s is the coefficient of static friction and N is the magnitude of the normal force

normal force: the force that a surface applies to an object to support the weight of the object; acts perpendicular to the surface on which the object rests

shear deformation: deformation perpendicular to the original length of an object

static friction: a force that opposes the motion of two systems that are in contact and are not moving relative to one another

Stokes' law: $F_s = 6\pi r\eta v$, where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity

strain: ratio of change in length to original length

stress: ratio of force to area

tensile strength: the breaking stress that will cause permanent deformation or fraction of a material

tension: the pulling force that acts along a medium, especially a stretched flexible connector, such as a rope or cable; when a rope supports the weight of an object, the force on the object due to the rope is called a tension force

Section Summary

2.1 The Fundamental Forces

- The various types of forces that are categorized for use in many applications are all manifestations of the *four basic forces* in nature.
- The properties of these forces are summarized in **Table 2.1**.
- Everything we experience directly without sensitive instruments is due to either electromagnetic forces or gravitational forces. The nuclear forces are responsible for the submicroscopic structure of matter, but they are not directly sensed because of their short ranges. Attempts are being made to show all four forces are different manifestations of a single unified force.
- A force field surrounds an object creating a force and is the carrier of that force.

2.5 Friction

- Friction is a contact force between systems that opposes the motion or attempted motion between them. Simple friction is proportional to the normal force N pushing the systems together. (A normal force is always perpendicular to the contact surface between systems.) Friction depends on both of the materials involved. The magnitude of static friction f_s between systems stationary relative to one another is given by

  $$f_s \le \mu_s N,$$

  where μ_s is the coefficient of static friction, which depends on both of the materials.
- The kinetic friction force f_k between systems moving relative to one another is given by

  $$f_k = \mu_k N,$$

  where μ_k is the coefficient of kinetic friction, which also depends on both materials.

2.6 Elasticity: Stress and Strain

- Hooke's law is given by

$$F = k\Delta L,$$

where ΔL is the amount of deformation (the change in length), F is the applied force, and k is a proportionality constant that depends on the shape and composition of the object and the direction of the force. The relationship between the deformation and the applied force can also be written as

$$\Delta L = \frac{1}{Y}\frac{F}{A}L_0,$$

where Y is *Young's modulus*, which depends on the substance, A is the cross-sectional area, and L_0 is the original length.

- The ratio of force to area, $\frac{F}{A}$, is defined as *stress*, measured in N/m^2.
- The ratio of the change in length to length, $\frac{\Delta L}{L_0}$, is defined as *strain* (a unitless quantity). In other words,

$$\text{stress} = Y \times \text{strain}.$$

- The expression for shear deformation is

$$\Delta x = \frac{1}{S}\frac{F}{A}L_0,$$

where S is the shear modulus and F is the force applied perpendicular to L_0 and parallel to the cross-sectional area A.

- The relationship of the change in volume to other physical quantities is given by

$$\Delta V = \frac{1}{B}\frac{F}{A}V_0,$$

where B is the bulk modulus, V_0 is the original volume, and $\frac{F}{A}$ is the force per unit area applied uniformly inward on all surfaces.

2.7 Drag Forces

- Drag forces acting on an object moving in a fluid oppose the motion. For larger objects (such as a baseball) moving at a velocity v in air, the drag force is given by

$$F_{\text{D}} = \frac{1}{2}C\rho A v^2,$$

where C is the drag coefficient (typical values are given in **Table 2.4**), A is the area of the object facing the fluid, and ρ is the fluid density.

- For small objects (such as a bacterium) moving in a denser medium (such as water), the drag force is given by Stokes' law,

$$F_{\text{s}} = 6\pi\eta r v,$$

where r is the radius of the object, η is the fluid viscosity, and v is the object's velocity.

Conceptual Questions

2.1 The Fundamental Forces

1. Explain, in terms of the properties of the four basic forces, why people notice the gravitational force acting on their bodies if it is such a comparatively weak force.

2. What is the dominant force between astronomical objects? Why are the other three basic forces less significant over these very large distances?

3. Give a detailed example of how the exchange of a particle can result in an *attractive* force. (For example, consider one child pulling a toy out of the hands of another.)

2.5 Friction

4. Define normal force. What is its relationship to friction when friction behaves simply?

5. The glue on a piece of tape can exert forces. Can these forces be a type of simple friction? Explain, considering especially that tape can stick to vertical walls and even to ceilings.

6. When you learn to drive, you discover that you need to let up slightly on the brake pedal as you come to a stop or the car will stop with a jerk. Explain this in terms of the relationship between static and kinetic friction.

7. When you push a piece of chalk across a chalkboard, it sometimes screeches because it rapidly alternates between slipping and sticking to the board. Describe this process in more detail, in particular explaining how it is related to the fact that kinetic friction is less than static friction. (The same slip-grab process occurs when tires screech on pavement.)

2.6 Elasticity: Stress and Strain

8. The elastic properties of the arteries are essential for blood flow. Explain the importance of this in terms of the characteristics of the flow of blood (pulsating or continuous).

9. What are you feeling when you feel your pulse? Measure your pulse rate for 10 s and for 1 min. Is there a factor of 6 difference?

10. Examine different types of shoes, including sports shoes and thongs. In terms of physics, why are the bottom surfaces designed as they are? What differences will dry and wet conditions make for these surfaces?

11. Would you expect your height to be different depending upon the time of day? Why or why not?

12. Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?

13. Explain why pregnant women often suffer from back strain late in their pregnancy.

14. An old carpenter’s trick to keep nails from bending when they are pounded into hard materials is to grip the center of the nail firmly with pliers. Why does this help?

15. When a glass bottle full of vinegar warms up, both the vinegar and the glass expand, but vinegar expands significantly more with temperature than glass. The bottle will break if it was filled to its tightly capped lid. Explain why, and also explain how a pocket of air above the vinegar would prevent the break. (This is the function of the air above liquids in glass containers.)

2.7 Drag Forces

16. Athletes such as swimmers and bicyclists wear body suits in competition. Formulate a list of pros and cons of such suits.

17. Two expressions were used for the drag force experienced by a moving object in a liquid. One depended upon the speed, while the other was proportional to the square of the speed. In which types of motion would each of these expressions be more applicable than the other one?

18. As cars travel, oil and gasoline leaks onto the road surface. If a light rain falls, what does this do to the control of the car? Does a heavy rain make any difference?

19. Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?

Problems & Exercises

2.1 The Fundamental Forces

1. (a) What is the strength of the weak nuclear force relative to the strong nuclear force? (b) What is the strength of the weak nuclear force relative to the electromagnetic force? Since the weak nuclear force acts at only very short distances, such as inside nuclei, where the strong and electromagnetic forces also act, it might seem surprising that we have any knowledge of it at all. We have such knowledge because the weak nuclear force is responsible for beta decay, a type of nuclear decay not explained by other forces.

2. (a) What is the ratio of the strength of the gravitational force to that of the strong nuclear force? (b) What is the ratio of the strength of the gravitational force to that of the weak nuclear force? (c) What is the ratio of the strength of the gravitational force to that of the electromagnetic force? What do your answers imply about the influence of the gravitational force on atomic nuclei?

3. What is the ratio of the strength of the strong nuclear force to that of the electromagnetic force? Based on this ratio, you might expect that the strong force dominates the nucleus, which is true for small nuclei. Large nuclei, however, have sizes greater than the range of the strong nuclear force. At these sizes, the electromagnetic force begins to affect nuclear stability. These facts will be used to explain nuclear fusion and fission later in this text.

2.5 Friction

4. A physics major is cooking breakfast when he notices that the frictional force between his steel spatula and his Teflon frying pan is only 0.200 N. Knowing the coefficient of kinetic friction between the two materials, he quickly calculates the normal force. What is it?

5. (a) When rebuilding her car's engine, a physics major must exert 300 N of force to insert a dry steel piston into a steel cylinder. What is the magnitude of the normal force between the piston and cylinder? (b) What is the magnitude of the force would she have to exert if the steel parts were oiled?

6. (a) What is the maximum frictional force in the knee joint of a person who supports 66.0 kg of her mass on that knee? (b) During strenuous exercise it is possible to exert forces to the joints that are easily ten times greater than the weight being supported. What is the maximum force of friction under such conditions? The frictional forces in joints are relatively small in all circumstances except when the joints deteriorate, such as from injury or arthritis. Increased frictional forces can cause further damage and pain.

7. Suppose you have a 120-kg wooden crate resting on a wood floor. (a) What maximum force can you exert horizontally on the crate without moving it? (b) If you continue to exert this force once the crate starts to slip, what will the magnitude of its acceleration then be?

8. (a) If half of the weight of a small $1.00\times10^3\text{ kg}$ utility truck is supported by its two drive wheels, what is the magnitude of the maximum acceleration it can achieve on dry concrete? (b) Will a metal cabinet lying on the wooden bed of the truck slip if it accelerates at this rate? (c) Solve both problems assuming the truck has four-wheel drive.

9. A team of eight dogs pulls a sled with waxed wood runners on wet snow (mush!). The dogs have average masses of 19.0 kg, and the loaded sled with its rider has a mass of 210 kg. (a) Calculate the magnitude of the acceleration starting from rest if each dog exerts an average force of 185 N backward on the snow. (b) What is the magnitude of the acceleration once the sled starts to move? (c) For both situations, calculate the magnitude of the force in the coupling between the dogs and the sled.

10. Consider the 65.0-kg ice skater being pushed by two others shown in Figure 2.32. (a) Find the direction and magnitude of $\mathbf{F}_{\text{tot}}$, the total force exerted on her by the others, given that the magnitudes F_1 and F_2 are 26.4 N and 18.6 N, respectively. (b) What is her initial acceleration if she is initially stationary and wearing steel-bladed skates that point in the direction of $\mathbf{F}_{\text{tot}}$? (c) What is her acceleration assuming she is already moving in the direction of $\mathbf{F}_{\text{tot}}$? (Remember that friction always acts in the direction opposite that of motion or attempted motion between surfaces in contact.)

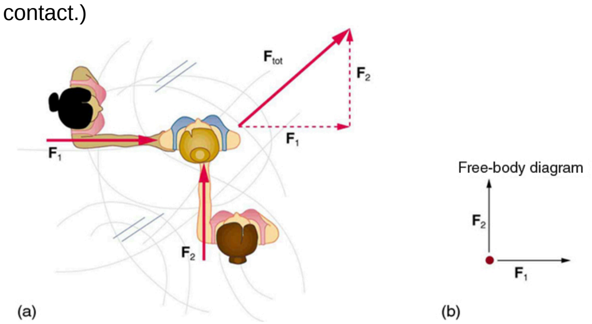


Figure 2.32

11. Show that the acceleration of any object down a frictionless incline that makes an angle θ with the horizontal is $a = g\sin\theta$. (Note that this acceleration is independent of mass.)

12. Show that the acceleration of any object down an incline where friction behaves simply (that is, where $f_{\text{k}} = \mu_{\text{k}}N$) is $a = g(\sin\theta - \mu_{\text{k}}\cos\theta)$. Note that the acceleration is independent of mass and reduces to the expression found in the previous problem when friction becomes negligibly small $(\mu_{\text{k}} = 0)$.

13. Calculate the deceleration of a snow boarder going up a 5.0^{o}, slope assuming the coefficient of friction for waxed wood on wet snow. The result of Exercise 2.12 may be useful, but be careful to consider the fact that the snow boarder is going uphill. Explicitly show how you follow the steps in **Problem-Solving Strategies (https://legacy.cnx.org/content/m42076/latest/)** .

14. (a) Calculate the acceleration of a skier heading down a 10.0^{o} slope, assuming the coefficient of friction for waxed wood on wet snow. (b) Find the angle of the slope down which this skier could coast at a constant velocity. You can neglect air resistance in both parts, and you will find the result of Exercise 2.12 to be useful. Explicitly show how you follow the steps in the **Problem-Solving Strategies (https://legacy.cnx.org/content/m42076/latest/)** .

15. If an object is to rest on an incline without slipping, then friction must equal the component of the weight of the object parallel to the incline. This requires greater and greater friction for steeper slopes. Show that the maximum angle of an incline above the horizontal for which an object will not slide down is $\theta = \tan^{-1} \mu_s$. You may use the result of the previous problem. Assume that $a = 0$ and that static friction has reached its maximum value.

16. Calculate the maximum deceleration of a car that is heading down a 6° slope (one that makes an angle of 6° with the horizontal) under the following road conditions. You may assume that the weight of the car is evenly distributed on all four tires and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the deceleration. (Ignore rolling.) Calculate for a car: (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that $\mu_s = 0.100$, the same as for shoes on ice.

17. Calculate the maximum acceleration of a car that is heading up a 4° slope (one that makes an angle of 4° with the horizontal) under the following road conditions. Assume that only half the weight of the car is supported by the two drive wheels and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the acceleration. (Ignore rolling.) (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that $\mu_s = 0.100$, the same as for shoes on ice.

18. Repeat **Exercise 2.17** for a car with four-wheel drive.

19. A freight train consists of two $8.00\times10^5\text{-kg}$ engines and 45 cars with average masses of $5.50\times10^5 \text{ kg}$. (a) What force must each engine exert backward on the track to accelerate the train at a rate of $5.00\times10^{-2} \text{ m/s}^2$ if the force of friction is $7.50\times10^5 \text{ N}$, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently trains are very energy-efficient transportation systems. (b) What is the magnitude of the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?

20. Consider the 52.0-kg mountain climber in **Figure 2.33**. (a) Find the tension in the rope and the force that the mountain climber must exert with her feet on the vertical rock face to remain stationary. Assume that the force is exerted parallel to her legs. Also, assume negligible force exerted by her arms. (b) What is the minimum coefficient of friction between her shoes and the cliff?

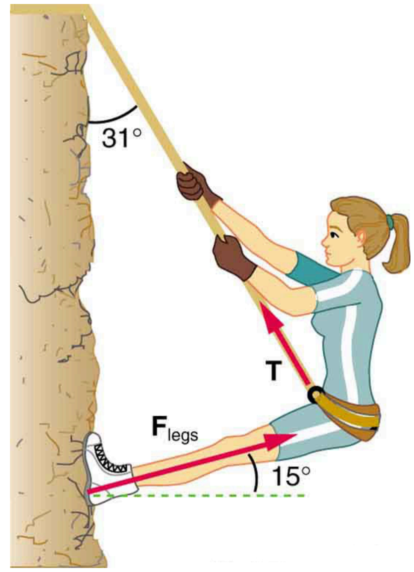


Figure 2.33 Part of the climber's weight is supported by her rope and part by friction between her feet and the rock face.

21. A contestant in a winter sporting event pushes a 45.0-kg block of ice across a frozen lake as shown in **Figure 2.34**(a). (a) Calculate the minimum force F he must exert to get the block moving. (b) What is the magnitude of its acceleration once it starts to move, if that force is maintained?

22. Repeat **Exercise 2.21** with the contestant pulling the block of ice with a rope over his shoulder at the same angle above the horizontal as shown in **Figure 2.34**(b).

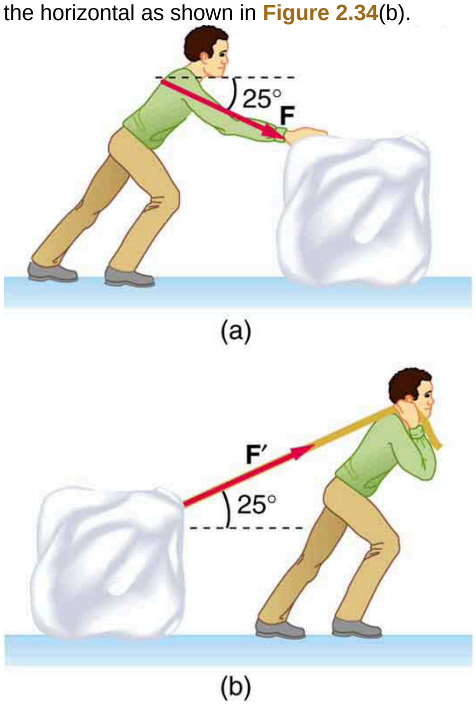


Figure 2.34 Which method of sliding a block of ice requires less force—(a) pushing or (b) pulling at the same angle above the horizontal?

2.6 Elasticity: Stress and Strain

23. During a circus act, one performer swings upside down hanging from a trapeze holding another, also upside-down, performer by the legs. If the upward force on the lower performer is three times her weight, how much do the bones (the femurs) in her upper legs stretch? You may assume each is equivalent to a uniform rod 35.0 cm long and 1.80 cm in radius. Her mass is 60.0 kg.

24. During a wrestling match, a 150 kg wrestler briefly stands on one hand during a maneuver designed to perplex his already moribund adversary. By how much does the upper arm bone shorten in length? The bone can be represented by a uniform rod 38.0 cm in length and 2.10 cm in radius.

25. (a) The “lead” in pencils is a graphite composition with a Young’s modulus of about $1\times10^9\ \mathrm{N/m^2}$. Calculate the change in length of the lead in an automatic pencil if you tap it straight into the pencil with a force of 4.0 N. The lead is 0.50 mm in diameter and 60 mm long. (b) Is the answer reasonable? That is, does it seem to be consistent with what you have observed when using pencils?

26. TV broadcast antennas are the tallest artificial structures on Earth. In 1987, a 72.0-kg physicist placed himself and 400 kg of equipment at the top of one 610-m high antenna to perform gravity experiments. By how much was the antenna compressed, if we consider it to be equivalent to a steel cylinder 0.150 m in radius?

27. (a) By how much does a 65.0-kg mountain climber stretch her 0.800-cm diameter nylon rope when she hangs 35.0 m below a rock outcropping? (b) Does the answer seem to be consistent with what you have observed for nylon ropes? Would it make sense if the rope were actually a bungee cord?

28. A 20.0-m tall hollow aluminum flagpole is equivalent in stiffness to a solid cylinder 4.00 cm in diameter. A strong wind bends the pole much as a horizontal force of 900 N exerted at the top would. How far to the side does the top of the pole flex?

29. As an oil well is drilled, each new section of drill pipe supports its own weight and that of the pipe and drill bit beneath it. Calculate the stretch in a new 6.00 m length of steel pipe that supports 3.00 km of pipe having a mass of 20.0 kg/m and a 100-kg drill bit. The pipe is equivalent in stiffness to a solid cylinder 5.00 cm in diameter.

30. Calculate the force a piano tuner applies to stretch a steel piano wire 8.00 mm, if the wire is originally 0.850 mm in diameter and 1.35 m long.

31. A vertebra is subjected to a shearing force of 500 N. Find the shear deformation, taking the vertebra to be a cylinder 3.00 cm high and 4.00 cm in diameter.

32. A disk between vertebrae in the spine is subjected to a shearing force of 600 N. Find its shear deformation, taking it to have the shear modulus of $1\times10^9\ \mathrm{N/m^2}$. The disk is equivalent to a solid cylinder 0.700 cm high and 4.00 cm in diameter.

33. When using a pencil eraser, you exert a vertical force of 6.00 N at a distance of 2.00 cm from the hardwood-eraser joint. The pencil is 6.00 mm in diameter and is held at an angle of 20.0° to the horizontal. (a) By how much does the wood flex perpendicular to its length? (b) How much is it compressed lengthwise?

34. To consider the effect of wires hung on poles, we take data from **Example 3.6**, in which tensions in wires supporting a traffic light were calculated. The left wire made an angle 30.0° below the horizontal with the top of its pole and carried a tension of 108 N. The 12.0 m tall hollow aluminum pole is equivalent in stiffness to a 4.50 cm diameter solid cylinder. (a) How far is it bent to the side? (b) By how much is it compressed?

35. A farmer making grape juice fills a glass bottle to the brim and caps it tightly. The juice expands more than the glass when it warms up, in such a way that the volume increases by 0.2% (that is, $\Delta V/V_0 = 2\times10^{-3}$) relative to the space available. Calculate the magnitude of the normal force exerted by the juice per square centimeter if its bulk modulus is $1.8\times10^9\ \mathrm{N/m^2}$, assuming the bottle does not break. In view of your answer, do you think the bottle will survive?

36. (a) When water freezes, its volume increases by 9.05% (that is, $\Delta V/V_0 = 9.05\times10^{-2}$). What force per unit area is water capable of exerting on a container when it freezes? (It is acceptable to use the bulk modulus of water in this problem.) (b) Is it surprising that such forces can fracture engine blocks, boulders, and the like?

37. This problem returns to the tightrope walker studied in **m42075 (https://legacy.cnx.org/content/m42075/latest/#fs-id986136)** , who created a tension of $3.94\times10^3\ \mathrm{N}$ in a wire making an angle 5.0° below the horizontal with each supporting pole. Calculate how much this tension stretches the steel wire if it was originally 15 m long and 0.50 cm in diameter.

38. The pole in **Figure 2.35** is at a 90.0° bend in a power line and is therefore subjected to more shear force than poles in straight parts of the line. The tension in each line is $4.00\times10^{4}\ \text{N}$, at the angles shown. The pole is 15.0 m tall, has an 18.0 cm diameter, and can be considered to have half the stiffness of hardwood. (a) Calculate the compression of the pole. (b) Find how much it bends and in what direction. (c) Find the tension in a guy wire used to keep the pole straight if it is attached to the top of the pole at an angle of 30.0° with the vertical. (Clearly, the guy wire must be in the opposite direction of the bend.)

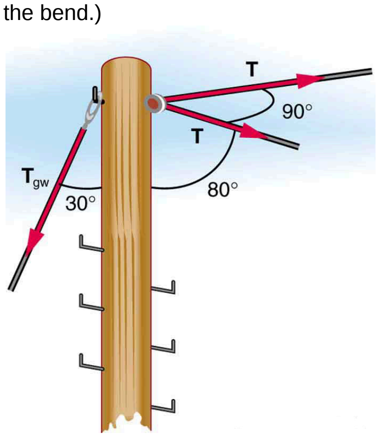


Figure 2.35 This telephone pole is at a 90° bend in a power line. A guy wire is attached to the top of the pole at an angle of 30° with the vertical.

2.7 Drag Forces

39. The terminal velocity of a person falling in air depends upon the weight and the area of the person facing the fluid. Find the terminal velocity (in meters per second and kilometers per hour) of an 80.0-kg skydiver falling in a pike (headfirst) position with a surface area of $0.140\ \text{m}^2$.

40. A 60-kg and a 90-kg skydiver jump from an airplane at an altitude of 6000 m, both falling in the pike position. Make some assumption on their frontal areas and calculate their terminal velocities. How long will it take for each skydiver to reach the ground (assuming the time to reach terminal velocity is small)? Assume all values are accurate to three significant digits.

41. A 560-g squirrel with a surface area of $930\ \text{cm}^2$ falls from a 5.0-m tree to the ground. Estimate its terminal velocity. (Use a drag coefficient for a horizontal skydiver.) What will be the velocity of a 56-kg person hitting the ground, assuming no drag contribution in such a short distance?

42. To maintain a constant speed, the force provided by a car's engine must equal the drag force plus the force of friction of the road (the rolling resistance). (a) What are the magnitudes of drag forces at 70 km/h and 100 km/h for a Toyota Camry? (Drag area is $0.70\ \text{m}^2$) (b) What is the magnitude of drag force at 70 km/h and 100 km/h for a Hummer H2? (Drag area is $2.44\ \text{m}^2$) Assume all values are accurate to three significant digits.

43. By what factor does the drag force on a car increase as it goes from 65 to 110 km/h?

44. Calculate the speed a spherical rain drop would achieve falling from 5.00 km (a) in the absence of air drag (b) with air drag. Take the size across of the drop to be 4 mm, the density to be $1.00\times10^{3}\ \text{kg/m}^3$, and the surface area to be πr^2.

45. Using Stokes' law, verify that the units for viscosity are kilograms per meter per second.

46. Find the terminal velocity of a spherical bacterium (diameter $2.00\ \mu\text{m}$) falling in water. You will first need to note that the drag force is equal to the weight at terminal velocity. Take the density of the bacterium to be $1.10\times10^{3}\ \text{kg/m}^3$.

47. Stokes' law describes sedimentation of particles in liquids and can be used to measure viscosity. Particles in liquids achieve terminal velocity quickly. One can measure the time it takes for a particle to fall a certain distance and then use Stokes' law to calculate the viscosity of the liquid. Suppose a steel ball bearing (density $7.8\times10^{3}\ \text{kg/m}^3$, diameter $3.0\ \text{mm}$) is dropped in a container of motor oil. It takes 12 s to fall a distance of 0.60 m. Calculate the viscosity of the oil.

3 EXAMPLES OF APPLICATIONS OF NEWTON'S LAWS

3.1 Introduction

This chapter covers some problem-solving strategies and a few examples of problems when working with forces. We will be going over these in class, so this section is entirely for your reference. If you do feel like you need more practice with force problems and free body diagrams, or if you are looking for a way to study for the exam, these sections are a good place to start. However, if you do so, it is highly recommended that you work through the problems yourselves as well. While reading about it alone can be somewhat helpful, you will get a lot more out of it if you work through the physics alongside.

3.2 Problem Solving Strategy

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills.

Example 3.1 Drag Force on a Barge

Suppose two tugboats push on a barge at different angles, as shown in Figure 3.1. The first tugboat exerts a force of $2.7\times10^5\ \mathrm{N}$ in the *x*-direction, and the second tugboat exerts a force of $3.6\times10^5\ \mathrm{N}$ in the *y*-direction.

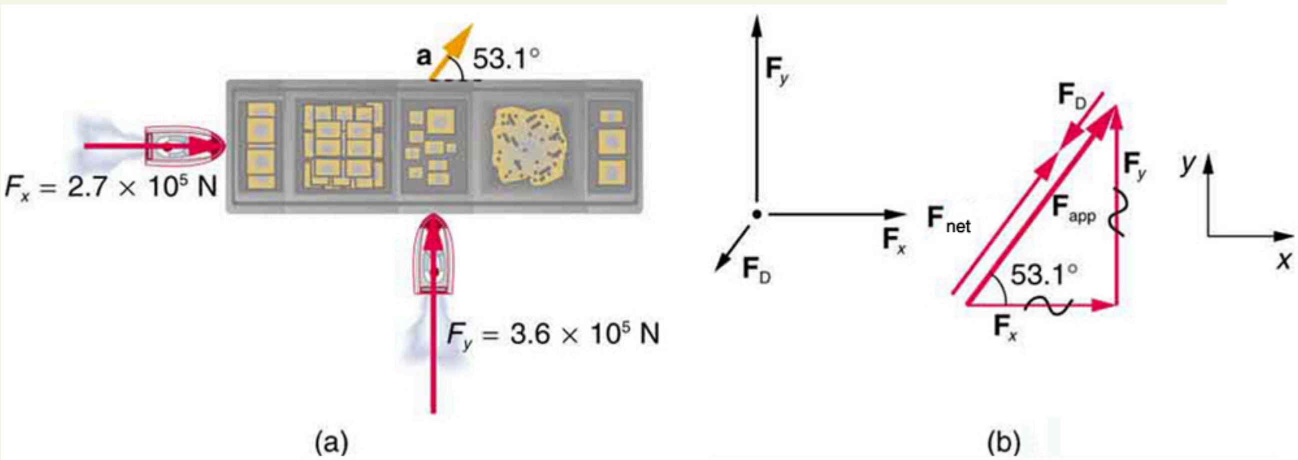


Figure 3.1 (a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Since the applied forces are perpendicular, the *x*- and *y*-axes are in the same direction as $\mathbf{F}_x$ and $\mathbf{F}_y$. The problem quickly becomes a one-dimensional problem along the direction of $\mathbf{F}_{\text{app}}$, since friction is in the direction opposite to $\mathbf{F}_{\text{app}}$.

If the mass of the barge is $5.0\times10^6\ \mathrm{kg}$ and its acceleration is observed to be $7.5\times10^{-2}\ \mathrm{m/s^2}$ in the direction shown, what is the drag force of the water on the barge resisting the motion? (Note: drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object.)

Strategy

The directions and magnitudes of acceleration and the applied forces are given in Figure 3.1**(a)**. We will define the total force of the tugboats on the barge as $\mathbf{F}_{\text{app}}$ so that:

$$\mathbf{F}_{\text{app}}=\mathbf{F}_x+\mathbf{F}_y \tag{3.1}$$

Since the barge is flat bottomed, the drag of the water $\mathbf{F}_{\text{D}}$ will be in the direction opposite to $\mathbf{F}_{\text{app}}$, as shown in the free-body diagram in Figure 3.1(b). The system of interest here is the barge, since the forces on *it* are given as well as its acceleration. Our strategy is to find the magnitude and direction of the net applied force $\mathbf{F}_{\text{app}}$, and then apply Newton's second law to solve for the drag force $\mathbf{F}_{\text{D}}$.

Solution

Since $\mathbf{F}_x$ and $\mathbf{F}_y$ are perpendicular, the magnitude and direction of $\mathbf{F}_{\text{app}}$ are easily found. First, the resultant magnitude is given by the Pythagorean theorem:

$$\begin{aligned} F_{\text{app}} &= \sqrt{\mathrm{F}_x^2 + \mathrm{F}_y^2} \\ F_{\text{app}} &= \sqrt{(2.7\times10^5\ \mathrm{N})^2 + (3.6\times10^5\ \mathrm{N})^2} = 4.5\times10^5\ \mathrm{N}. \end{aligned} \tag{3.2}$$

The angle is given by

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{F_y}{F_x}\right) \\ \theta &= \tan^{-1}\left(\frac{3.6\times10^5\ \mathrm{N}}{2.7\times10^5\ \mathrm{N}}\right) = 53^\circ, \end{aligned} \tag{3.3}$$

which we know, because of Newton's first law, is the same direction as the acceleration. $\mathbf{F}_{\text{D}}$ is in the opposite direction of $\mathbf{F}_{\text{app}}$, since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as $\mathbf{F}_{\text{app}}$, but its magnitude is slightly less than $\mathbf{F}_{\text{app}}$. The problem is now one-dimensional. From **Figure 3.1(b)**, we can see that

$$F_{\text{net}} = F_{\text{app}} - F_{\text{D}}. \tag{3.4}$$

But Newton's second law states that

$$F_{\text{net}} = ma. \tag{3.5}$$

Thus,

$$F_{\text{app}} - F_{\text{D}} = ma. \tag{3.6}$$

This can be solved for the magnitude of the drag force of the water F_{D} in terms of known quantities:

$$F_{\text{D}} = F_{\text{app}} - ma. \tag{3.7}$$

Substituting known values gives

$$\mathrm{F}_{\text{D}} = (4.5\times10^5\ \mathrm{N}) - (5.0\times10^6\ \mathrm{kg})(7.5\times10^{-2}\ \mathrm{m/s}^2) = 7.5\times10^4\ \mathrm{N}. \tag{3.8}$$

The direction of $\mathbf{F}_{\text{D}}$ has already been determined to be in the direction opposite to $\mathbf{F}_{\text{app}}$, or at an angle of 53° south of west.

Discussion

The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where F_{D} is less than 1/600th of the weight of the ship.

In the earlier example of a tightrope walker we noted that the tensions in wires supporting a mass were equal only because the angles on either side were equal. Consider the following example, where the angles are not equal; slightly more trigonometry is involved.

Example 3.2 Different Tensions at Different Angles

Consider the traffic light (mass 15.0 kg) suspended from two wires as shown in **Figure 3.2**. Find the tension in each wire, neglecting the masses of the wires.

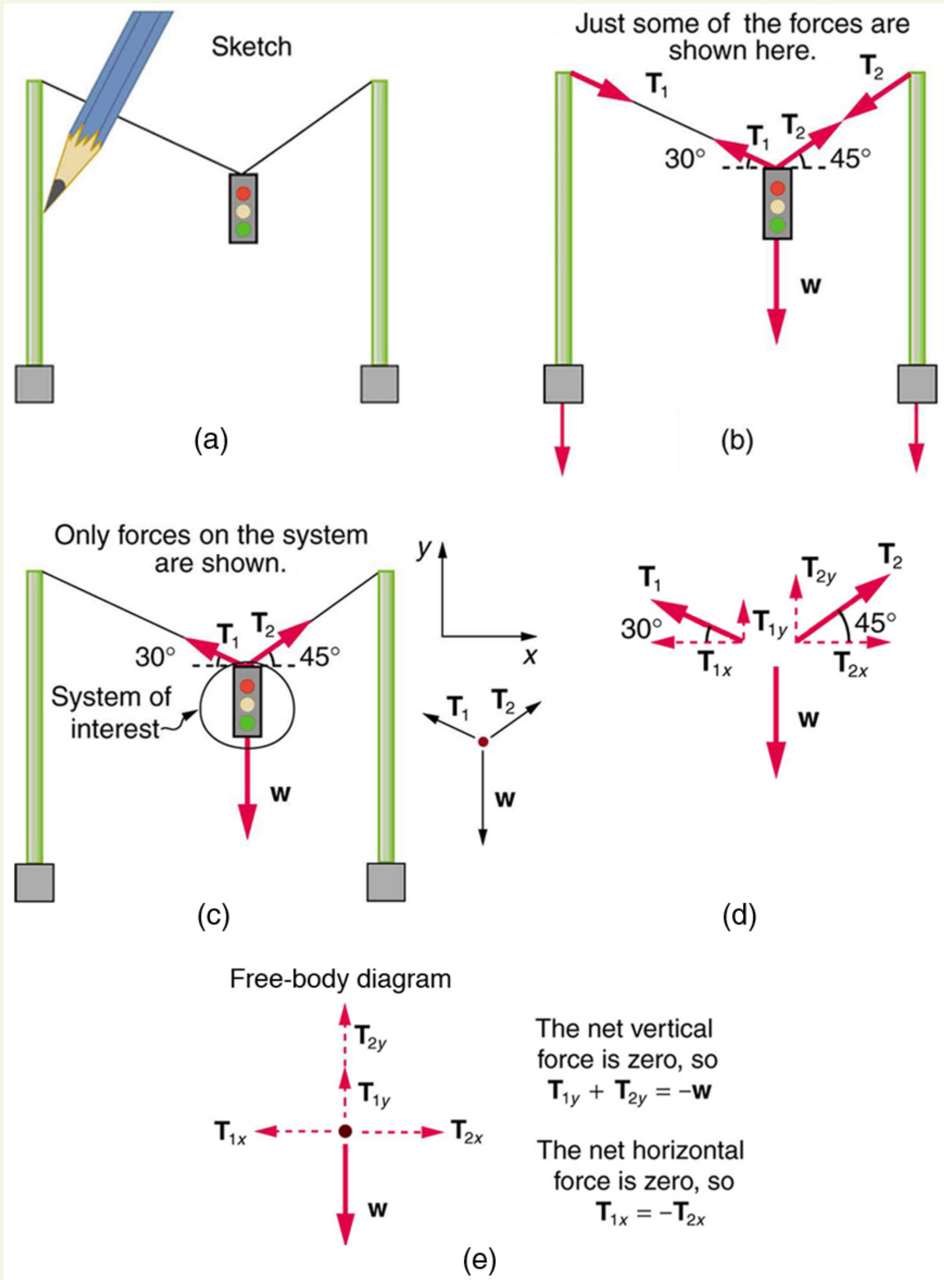


Figure 3.2 A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical (*y*) and horizontal (*x*) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.

Strategy

The system of interest is the traffic light, and its free-body diagram is shown in **Figure 3.2**(c). The three forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in part (d) of the figure. There are two unknowns in this problem (T_1 and T_2), so two equations are needed to find them. These two equations come from applying Newton's second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

Solution

First consider the horizontal or *x*-axis:

$$F_{\text{net}x} = T_{2x} - T_{1x} = 0. \tag{3.9}$$

Thus, as you might expect,

$$T_{1x} = T_{2x}. \tag{3.10}$$

This gives us the following relationship between T_1 and T_2 :

$$T_1 \cos(30°) = T_2 \cos(45°). \tag{3.11}$$

Thus,

$$T_2 = (1.225)T_1. \tag{3.12}$$

Note that T_1 and T_2 are not equal in this case, because the angles on either side are not equal. It is reasonable that T_2 ends up being greater than T_1 , because it is exerted more vertically than T_1 .

Now consider the force components along the vertical or *y*-axis:

$$F_{\text{net } y} = T_{1y} + T_{2y} - w = 0. \tag{3.13}$$

This implies

$$T_{1y} + T_{2y} = w. \tag{3.14}$$

Substituting the expressions for the vertical components gives

$$T_1 \sin(30°) + T_2 \sin(45°) = w. \tag{3.15}$$

There are two unknowns in this equation, but substituting the expression for T_2 in terms of T_1 reduces this to one equation with one unknown:

$$T_1(0.500) + (1.225T_1)(0.707) = w = mg, \tag{3.16}$$

which yields

$$(1.366)T_1 = (15.0\text{ kg})(9.80\text{ m/s}^2). \tag{3.17}$$

Solving this last equation gives the magnitude of T_1 to be

$$T_1 = 108\text{ N}. \tag{3.18}$$

Finally, the magnitude of T_2 is determined using the relationship between them, T_2 = 1.225 T_1 , found above. Thus we obtain

$$T_2 = 132\text{ N}. \tag{3.19}$$

Discussion

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker).

The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride? Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed: will the scale still read more than your weight at rest? Consider the following example.

Example 3.3 What Does the Bathroom Scale Read in an Elevator?

Figure 3.3 shows a 75.0-kg man (weight of about 165 lb) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate of 1.20 m/s^2 , and (b) if the elevator moves upward at a constant speed of 1 m/s.

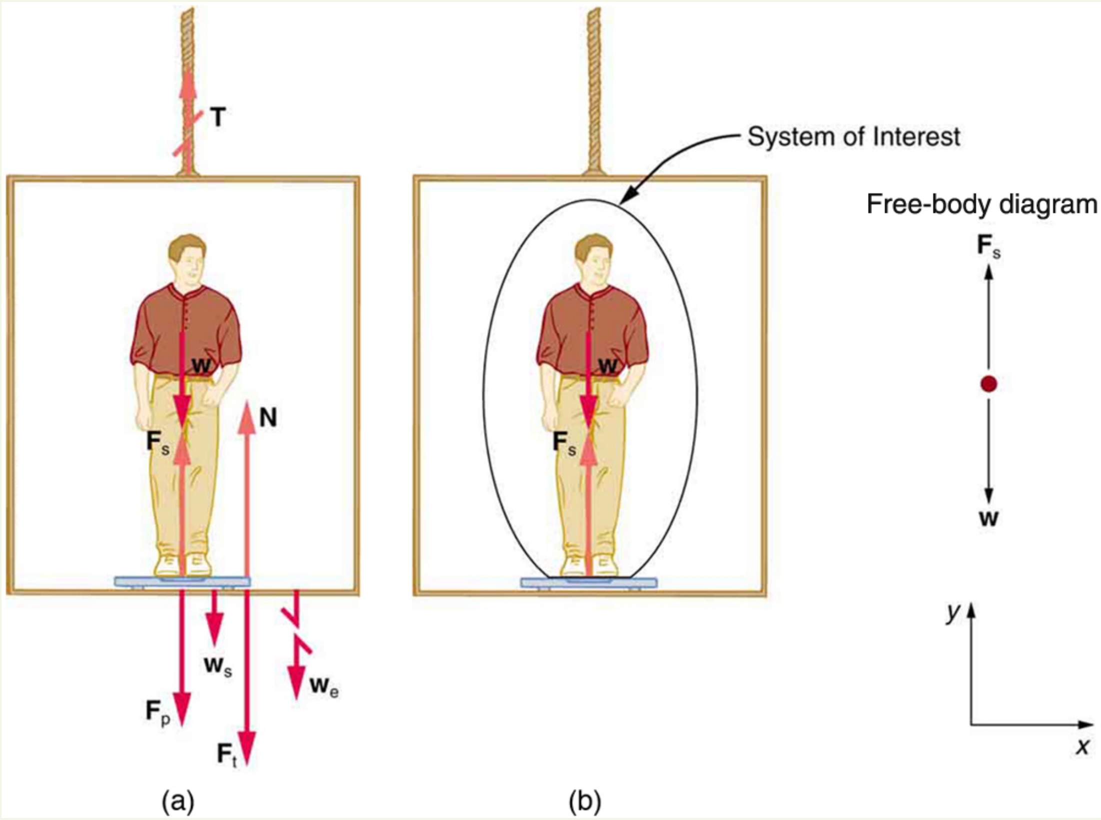


Figure 3.3 (a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. $\mathbf{T}$ is the tension in the supporting cable, $\mathbf{w}$ is the weight of the person, $\mathbf{w}_\text{s}$ is the weight of the scale, $\mathbf{w}_\text{e}$ is the weight of the elevator, $\mathbf{F}_\text{s}$ is the force of the scale on the person, $\mathbf{F}_\text{p}$ is the force of the person on the scale, $\mathbf{F}_\text{t}$ is the force of the scale on the floor of the elevator, and $\mathbf{N}$ is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person.

Strategy

If the scale is accurate, its reading will equal F_p, the magnitude of the force the person exerts downward on it. **Figure 3.3**(a) shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn as in **Figure 3.3**(b). Analysis of the free-body diagram using Newton's laws can produce answers to both parts (a) and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight $\mathbf{w}$ and the upward force of the scale $\mathbf{F}_\text{s}$. According to Newton's third law $\mathbf{F}_\text{p}$ and $\mathbf{F}_\text{s}$ are equal in magnitude and opposite in direction, so that we need to find F_s in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,

$$F_\text{net} = ma. \tag{3.20}$$

From the free-body diagram we see that $F_\text{net} = F_\text{s} - w$, so that

$$F_\text{s} - w = ma. \tag{3.21}$$

Solving for F_s gives an equation with only one unknown:

$$F_\text{s} = ma + w, \tag{3.22}$$

or, because $w = mg$, simply

$$F_\text{s} = ma + mg. \tag{3.23}$$

No assumptions were made about the acceleration, and so this solution should be valid for a variety of accelerations in addition to the ones in this exercise.

Solution for (a)

In this part of the problem, $a = 1.20\ \text{m/s}^2$, so that

$$F_\text{s} = (75.0\ \text{kg})(1.20\ \text{m/s}^2) + (75.0\ \text{kg})(9.80\ \text{m/s}^2), \tag{3.24}$$

yielding

$$F_\text{s} = 825\ \text{N}. \tag{3.25}$$

Discussion for (a)

This is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

$$\begin{aligned} F_\text{net} &= ma = 0 = F_\text{s} - w \\ F_\text{s} &= w = mg \\ F_\text{s} &= (75.0\ \text{kg})(9.80\ \text{m/s}^2) \\ F_\text{s} &= 735\ \text{N}. \end{aligned} \tag{3.26}$$

So, the scale reading in the elevator is greater than his 735-N (165 lb) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward. Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators.

Solution for (b)

Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight? For any constant velocity—up, down, or stationary—acceleration is zero because $a = \frac{\Delta v}{\Delta t}$, and $\Delta v = 0$.

Thus,

$$F_\text{s} = ma + mg = 0 + mg. \tag{3.27}$$

Now

$$F_\text{s} = (75.0\ \text{kg})(9.80\ \text{m/s}^2), \tag{3.28}$$

which gives

$$F_\text{s} = 735\ \text{N}. \tag{3.29}$$

Discussion for (b)

The scale reading is 735 N, which equals the person’s weight. This will be the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward, a is negative, and the scale reading is *less* than the weight of the person, until a constant downward velocity is reached, at which time the scale reading again becomes equal to the person’s weight. If the elevator is in free-fall and accelerating downward at g , then the scale reading will be zero and the person will *appear* to be weightless.

Integrating Concepts: Newton’s Laws of Motion and Kinematics

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton’s laws of motion can also be integrated with other concepts that have been discussed previously in this text to solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters. When approaching problems that involve various types of forces, acceleration, velocity, and/or position, use the following steps to approach the problem:

Problem-Solving Strategy

Step 1. *Identify which physical principles are involved*. Listing the givens and the quantities to be calculated will allow you to identify the principles involved.
Step 2. *Solve the problem using strategies outlined in the text*. If these are available for the specific topic, you should refer to them. You should also refer to the sections of the text that deal with a particular topic. The following worked example illustrates how these strategies are applied to an integrated concept problem.

Example 3.4 What Force Must a Soccer Player Exert to Reach Top Speed?

A soccer player starts from rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What was his average acceleration? (b) What average force did he exert backward on the ground to achieve this acceleration? The player’s mass is 70.0 kg, and air resistance is negligible.

Strategy

1. To solve an *integrated concept problem*, we must first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example considers *acceleration* along a straight line. This is a topic of *kinematics*. Part (b) deals with *force*, a topic of *dynamics* found in this chapter.
2. The following solutions to each part of the example illustrate how the specific problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so forth.

Solution for (a)

We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is $\Delta v = 8.00 \text{ m/s}$. We are given the elapsed time, and so $\Delta t = 2.50 \text{ s}$. The unknown is acceleration, which can be found from its definition:

$$a = \frac{\Delta v}{\Delta t}. \tag{3.30}$$

Substituting the known values yields

$$\begin{aligned} a &= \frac{8.00 \text{ m/s}}{2.50 \text{ s}} \\ &= 3.20 \text{ m/s}^2. \end{aligned} \tag{3.31}$$

Discussion for (a)

This is an attainable acceleration for an athlete in good condition.

Solution for (b)

Here we are asked to find the average force the player exerts backward to achieve this forward acceleration. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes his acceleration. Since we now know the player's acceleration and are given his mass, we can use Newton's second law to find the force exerted. That is,

$$F_{\text{net}} = ma. \tag{3.32}$$

Substituting the known values of m and a gives

$$\begin{aligned} F_{\text{net}} &= (70.0 \text{ kg})(3.20 \text{ m/s}^2) \\ &= 224 \text{ N}. \end{aligned} \tag{3.33}$$

Discussion for (b)

This is about 50 pounds, a reasonable average force.

This worked example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These strategies are found throughout the text, and many worked examples show how to use them for single topics. You will find these techniques for integrated concept problems useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

3.3 Further Applications of Newton's Laws of Motion

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills.

Example 3.5 Drag Force on a Barge

Suppose two tugboats push on a barge at different angles, as shown in **Figure 3.4**. The first tugboat exerts a force of $2.7\times10^5 \text{ N}$ in the *x*-direction, and the second tugboat exerts a force of $3.6\times10^5 \text{ N}$ in the *y*-direction.

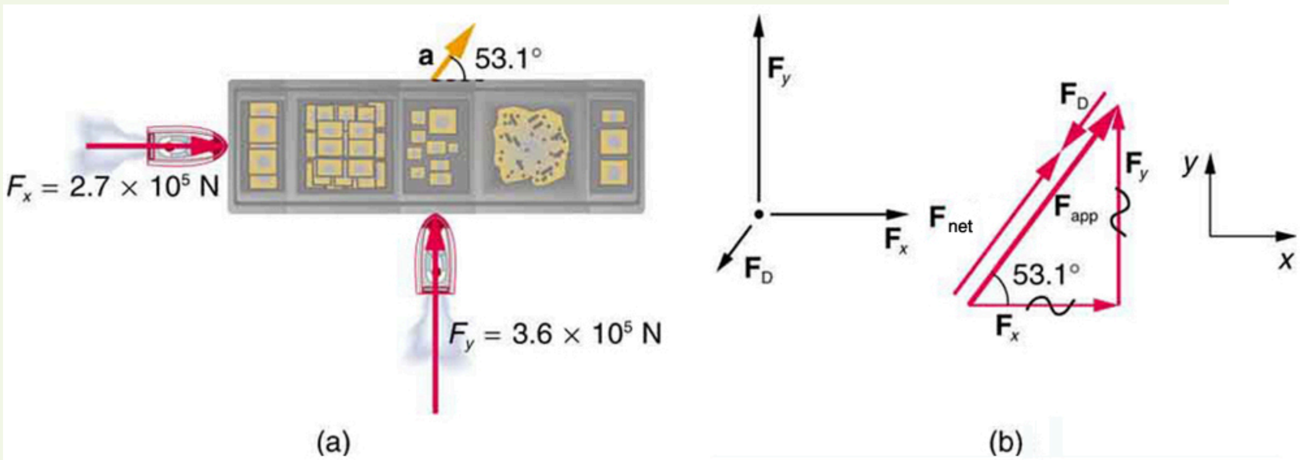


Figure 3.4 (a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Since the applied forces are perpendicular, the *x*- and *y*-axes are in the same direction as $\mathbf{F}_x$ and $\mathbf{F}_y$. The problem quickly becomes a one-dimensional problem along the direction of $\mathbf{F}_{\text{app}}$, since friction is in the direction opposite to $\mathbf{F}_{\text{app}}$.

If the mass of the barge is $5.0\times10^6\text{ kg}$ and its acceleration is observed to be $7.5\times10^{-2}\text{ m/s}^2$ in the direction shown, what is the drag force of the water on the barge resisting the motion? (Note: drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object.)

Strategy

The directions and magnitudes of acceleration and the applied forces are given in **Figure 3.4(a)**. We will define the total force of the tugboats on the barge as $\mathbf{F}_{\text{app}}$ so that:

$$\mathbf{F}_{\text{app}}=\mathbf{F}_x+\mathbf{F}_y \tag{3.34}$$

Since the barge is flat bottomed, the drag of the water $\mathbf{F}_{\text{D}}$ will be in the direction opposite to $\mathbf{F}_{\text{app}}$, as shown in the free-body diagram in **Figure 3.4**(b). The system of interest here is the barge, since the forces on *it* are given as well as its acceleration. Our strategy is to find the magnitude and direction of the net applied force $\mathbf{F}_{\text{app}}$, and then apply Newton's second law to solve for the drag force $\mathbf{F}_{\text{D}}$.

Solution

Since $\mathbf{F}_x$ and $\mathbf{F}_y$ are perpendicular, the magnitude and direction of $\mathbf{F}_{\text{app}}$ are easily found. First, the resultant magnitude is given by the Pythagorean theorem:

$$\begin{aligned} F_{\text{app}} &= \sqrt{\text{F}_x^2+\text{F}_y^2} \\ F_{\text{app}} &= \sqrt{(2.7\times10^5\text{ N})^2+(3.6\times10^5\text{ N})^2} \;=\; 4.5\times10^5\text{ N.} \end{aligned} \tag{3.35}$$

The angle is given by

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{F_y}{F_x}\right) \\ \theta &= \tan^{-1}\left(\frac{3.6\times10^5\text{ N}}{2.7\times10^5\text{ N}}\right) = 53^{\text{o}}, \end{aligned} \tag{3.36}$$

which we know, because of Newton's first law, is the same direction as the acceleration. $\mathbf{F}_{\text{D}}$ is in the opposite direction of $\mathbf{F}_{\text{app}}$, since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as $\mathbf{F}_{\text{app}}$, but its magnitude is slightly less than $\mathbf{F}_{\text{app}}$. The problem is now one-dimensional. From **Figure 3.4(b)**, we can see that

$$F_{\text{net}} = F_{\text{app}} - F_{\text{D}}. \tag{3.37}$$

But Newton's second law states that

$$F_{\text{net}} = ma. \tag{3.38}$$

Thus,

$$F_{\text{app}} - F_{\text{D}} = ma. \tag{3.39}$$

This can be solved for the magnitude of the drag force of the water F_{D} in terms of known quantities:

$$F_{\text{D}} = F_{\text{app}} - ma. \tag{3.40}$$

Substituting known values gives

$$\text{F}_{\text{D}} = (4.5\times10^{5}\text{ N}) - (5.0\times10^{6}\text{ kg})(7.5\times10^{-2}\text{ m/s}^{2}) = 7.5\times10^{4}\text{ N}. \tag{3.41}$$

The direction of $\mathbf{F}_{\text{D}}$ has already been determined to be in the direction opposite to $\mathbf{F}_{\text{app}}$, or at an angle of 53° south of west.

Discussion

The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where F_{D} is less than 1/600th of the weight of the ship.

In the earlier example of a tightrope walker we noted that the tensions in wires supporting a mass were equal only because the angles on either side were equal. Consider the following example, where the angles are not equal; slightly more trigonometry is involved.

Example 3.6 Different Tensions at Different Angles

Consider the traffic light (mass 15.0 kg) suspended from two wires as shown in **Figure 3.5**. Find the tension in each wire, neglecting the masses of the wires.

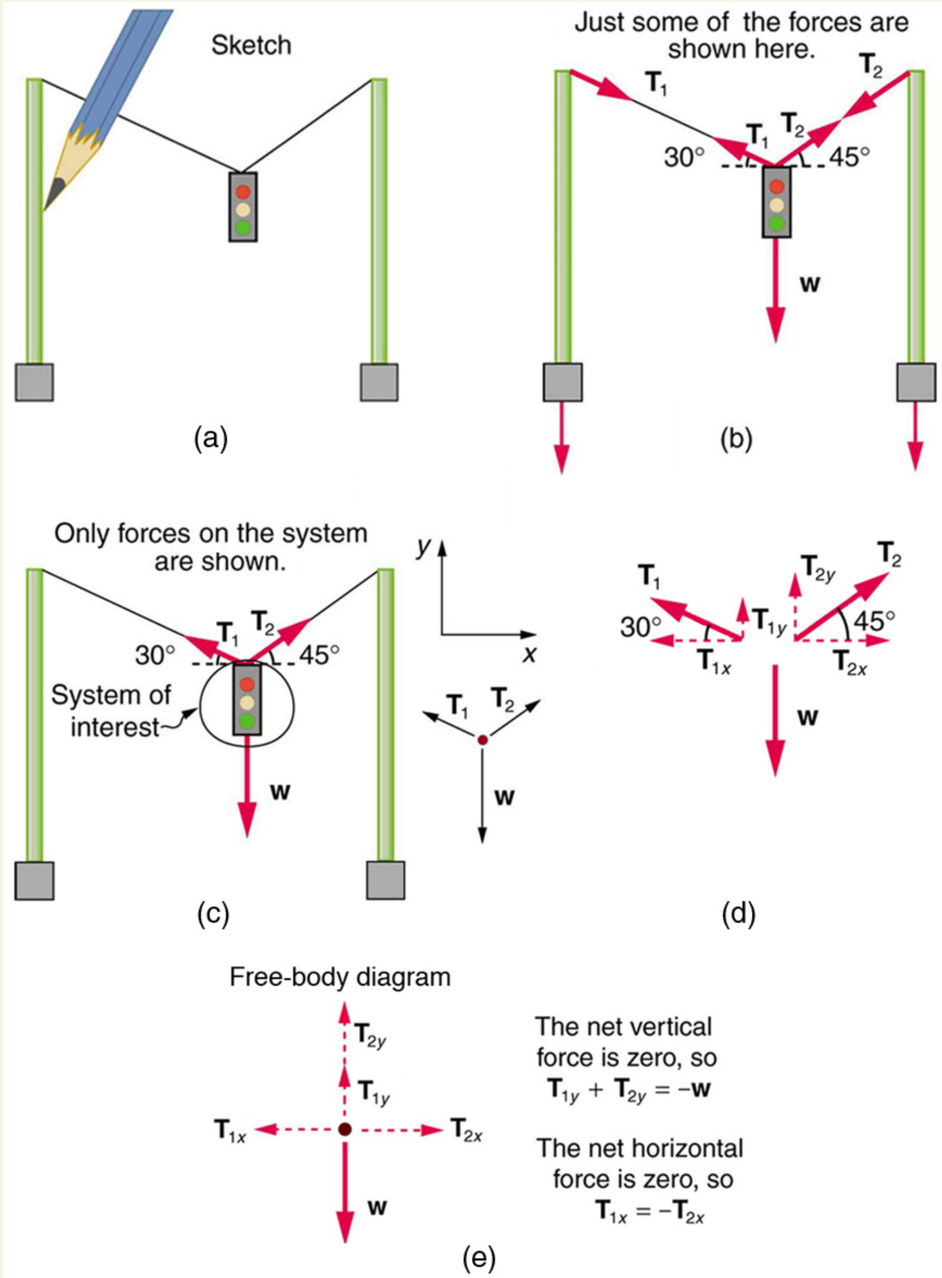


Figure 3.5 A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical (*y*) and horizontal (*x*) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.

Strategy

The system of interest is the traffic light, and its free-body diagram is shown in **Figure 3.5**(c). The three forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in part (d) of the figure. There are two unknowns in this problem (T_1 and T_2), so two equations are needed to find them. These two equations come from applying Newton's second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

Solution

First consider the horizontal or *x*-axis:

$$F_{\text{net}x} = T_{2x} - T_{1x} = 0. \tag{3.42}$$

Thus, as you might expect,

$$T_{1x} = T_{2x}. \tag{3.43}$$

This gives us the following relationship between T_1 and T_2 :

$$T_1 \cos(30^\circ) = T_2 \cos(45^\circ). \tag{3.44}$$

Thus,

$$T_2 = (1.225)T_1. \tag{3.45}$$

Note that T_1 and T_2 are not equal in this case, because the angles on either side are not equal. It is reasonable that T_2 ends up being greater than T_1 , because it is exerted more vertically than T_1 .

Now consider the force components along the vertical or *y*-axis:

$$F_{\text{net } y} = T_{1y} + T_{2y} - w = 0. \tag{3.46}$$

This implies

$$T_{1y} + T_{2y} = w. \tag{3.47}$$

Substituting the expressions for the vertical components gives

$$T_1 \sin(30^\circ) + T_2 \sin(45^\circ) = w. \tag{3.48}$$

There are two unknowns in this equation, but substituting the expression for T_2 in terms of T_1 reduces this to one equation with one unknown:

$$T_1(0.500) + (1.225T_1)(0.707) = w = mg, \tag{3.49}$$

which yields

$$(1.366)T_1 = (15.0 \text{ kg})(9.80 \text{ m/s}^2). \tag{3.50}$$

Solving this last equation gives the magnitude of T_1 to be

$$T_1 = 108 \text{ N}. \tag{3.51}$$

Finally, the magnitude of T_2 is determined using the relationship between them, T_2 = 1.225 T_1 , found above. Thus we obtain

$$T_2 = 132 \text{ N}. \tag{3.52}$$

Discussion

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker).

The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride? Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed: will the scale still read more than your weight at rest? Consider the following example.

Example 3.7 What Does the Bathroom Scale Read in an Elevator?

Figure 3.6 shows a 75.0-kg man (weight of about 165 lb) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate of 1.20 m/s^2 , and (b) if the elevator moves upward at a constant speed of 1 m/s.

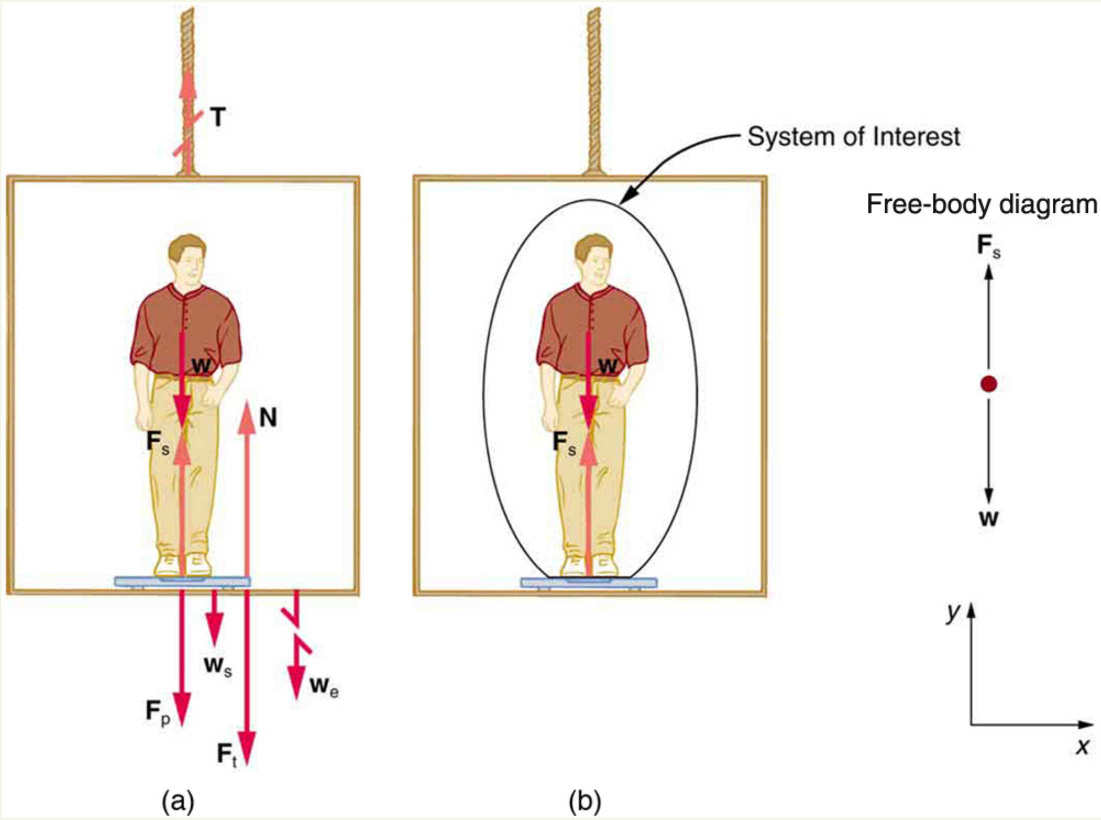


Figure 3.6 (a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. $\mathbf{T}$ is the tension in the supporting cable, $\mathbf{w}$ is the weight of the person, $\mathbf{w}_\text{s}$ is the weight of the scale, $\mathbf{w}_\text{e}$ is the weight of the elevator, $\mathbf{F}_\text{s}$ is the force of the scale on the person, $\mathbf{F}_\text{p}$ is the force of the person on the scale, $\mathbf{F}_\text{t}$ is the force of the scale on the floor of the elevator, and $\mathbf{N}$ is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person.

Strategy

If the scale is accurate, its reading will equal F_p , the magnitude of the force the person exerts downward on it. **Figure 3.6**(a) shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn as in **Figure 3.6**(b). Analysis of the free-body diagram using Newton's laws can produce answers to both parts (a) and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight $\mathbf{w}$ and the upward force of the scale $\mathbf{F}_\text{s}$. According to Newton's third law $\mathbf{F}_\text{p}$ and $\mathbf{F}_\text{s}$ are equal in magnitude and opposite in direction, so that we need to find F_s in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,

$$F_\text{net} = ma. \tag{3.53}$$

From the free-body diagram we see that $F_\text{net} = F_\text{s} - w$, so that

$$F_\text{s} - w = ma. \tag{3.54}$$

Solving for F_s gives an equation with only one unknown:

$$F_\text{s} = ma + w, \tag{3.55}$$

or, because $w = mg$, simply

$$F_\text{s} = ma + mg. \tag{3.56}$$

No assumptions were made about the acceleration, and so this solution should be valid for a variety of accelerations in addition to the ones in this exercise.

Solution for (a)

In this part of the problem, $a = 1.20\ \text{m/s}^2$, so that

$$F_\text{s} = (75.0\ \text{kg})(1.20\ \text{m/s}^2) + (75.0\ \text{kg})(9.80\ \text{m/s}^2), \tag{3.57}$$

yielding

$$F_\text{s} = 825\ \text{N}. \tag{3.58}$$

Discussion for (a)

This is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

$$\begin{aligned} F_\text{net} &= ma = 0 = F_\text{s} - w \\ F_\text{s} &= w = mg \\ F_\text{s} &= (75.0\ \text{kg})(9.80\ \text{m/s}^2) \\ F_\text{s} &= 735\ \text{N}. \end{aligned} \tag{3.59}$$

So, the scale reading in the elevator is greater than his 735-N (165 lb) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward. Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators.

Solution for (b)

Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight? For any constant velocity—up, down, or stationary—acceleration is zero because $a = \frac{\Delta v}{\Delta t}$, and $\Delta v = 0$.

Thus,

$$F_\text{s} = ma + mg = 0 + mg. \tag{3.60}$$

Now

$$F_\text{s} = (75.0\ \text{kg})(9.80\ \text{m/s}^2), \tag{3.61}$$

which gives

$$F_\text{s} = 735\ \text{N}. \tag{3.62}$$

Discussion for (b)

The scale reading is 735 N, which equals the person's weight. This will be the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward, a is negative, and the scale reading is *less* than the weight of the person, until a constant downward velocity is reached, at which time the scale reading again becomes equal to the person's weight. If the elevator is in free-fall and accelerating downward at g, then the scale reading will be zero and the person will *appear* to be weightless.

Integrating Concepts: Newton's Laws of Motion and Kinematics

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton's laws of motion can also be integrated with other concepts that have been discussed previously in this text to solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters. When approaching problems that involve various types of forces, acceleration, velocity, and/or position, use the following steps to approach the problem:

Problem-Solving Strategy

Step 1. *Identify which physical principles are involved*. Listing the givens and the quantities to be calculated will allow you to identify the principles involved.
Step 2. *Solve the problem using strategies outlined in the text*. If these are available for the specific topic, you should refer to them. You should also refer to the sections of the text that deal with a particular topic. The following worked example illustrates how these strategies are applied to an integrated concept problem.

Example 3.8 What Force Must a Soccer Player Exert to Reach Top Speed?

A soccer player starts from rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What was his average acceleration? (b) What average force did he exert backward on the ground to achieve this acceleration? The player's mass is 70.0 kg, and air resistance is negligible.

Strategy

1. To solve an *integrated concept problem*, we must first identify the physical principles involved and identify the chapters in which they are found. Part (a) of this example considers *acceleration* along a straight line. This is a topic of *kinematics*. Part (b) deals with *force*, a topic of *dynamics* found in this chapter.
2. The following solutions to each part of the example illustrate how the specific problem-solving strategies are applied. These involve identifying knowns and unknowns, checking to see if the answer is reasonable, and so forth.

Solution for (a)

We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is $\Delta v = 8.00\text{ m/s}$. We are given the elapsed time, and so $\Delta t = 2.50\text{ s}$. The unknown is acceleration, which can be found from its definition:

$$a = \frac{\Delta v}{\Delta t}. \tag{3.63}$$

Substituting the known values yields

$$\begin{aligned} a &= \frac{8.00\text{ m/s}}{2.50\text{ s}} \\ &= 3.20\text{ m/s}^2. \end{aligned} \tag{3.64}$$

Discussion for (a)

This is an attainable acceleration for an athlete in good condition.

Solution for (b)

Here we are asked to find the average force the player exerts backward to achieve this forward acceleration. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes his acceleration. Since we now know the player's acceleration and are given his mass, we can use Newton's second law to find the force exerted. That is,

$$F_{\text{net}} = ma. \tag{3.65}$$

Substituting the known values of m and a gives

$$\begin{aligned} F_{\text{net}} &= (70.0\text{ kg})(3.20\text{ m/s}^2) \\ &= 224\text{ N}. \end{aligned} \tag{3.66}$$

Discussion for (b)

This is about 50 pounds, a reasonable average force.

This worked example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles involved in the problem. The second step is to solve for the unknown using familiar problem-solving strategies. These strategies are found throughout the text, and many worked examples show how to use them for single topics. You will find these techniques for integrated concept problems useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life. The following problems will build your skills in the broad application of physical principles.

Section Summary

3.1 Problem Solving Strategy

- Newton's laws of motion can be applied in numerous situations to solve problems of motion.
- Some problems will contain multiple force vectors acting in different directions on an object. Be sure to draw diagrams, resolve all force vectors into horizontal and vertical components, and draw a free-body diagram. Always analyze the direction in which an object accelerates so that you can determine whether $F_{\text{net}} = ma$ or $F_{\text{net}} = 0$.
- The normal force on an object is not always equal in magnitude to the weight of the object. If an object is accelerating, the normal force will be less than or greater than the weight of the object. Also, if the object is on an inclined plane, the normal force will always be less than the full weight of the object.
- Some problems will contain various physical quantities, such as forces, acceleration, velocity, or position. You can apply concepts from kinematics and dynamics in order to solve these problems of motion.

3.2 Further Applications of Newton's Laws of Motion

- Newton's laws of motion can be applied in numerous situations to solve problems of motion.
- Some problems will contain multiple force vectors acting in different directions on an object. Be sure to draw diagrams, resolve all force vectors into horizontal and vertical components, and draw a free-body diagram. Always analyze the

direction in which an object accelerates so that you can determine whether $F_{\text{net}} = ma$ or $F_{\text{net}} = 0$.

- The normal force on an object is not always equal in magnitude to the weight of the object. If an object is accelerating, the normal force will be less than or greater than the weight of the object. Also, if the object is on an inclined plane, the normal force will always be less than the full weight of the object.
- Some problems will contain various physical quantities, such as forces, acceleration, velocity, or position. You can apply concepts from kinematics and dynamics in order to solve these problems of motion.

Conceptual Questions

3.1 Problem Solving Strategy

1. To simulate the apparent weightlessness of space orbit, astronauts are trained in the hold of a cargo aircraft that is accelerating downward at g . Why will they appear to be weightless, as measured by standing on a bathroom scale, in this accelerated frame of reference? Is there any difference between their apparent weightlessness in orbit and in the aircraft?

2. A cartoon shows the toupee coming off the head of an elevator passenger when the elevator rapidly stops during an upward ride. Can this really happen without the person being tied to the floor of the elevator? Explain your answer.

3.2 Further Applications of Newton’s Laws of Motion

3. To simulate the apparent weightlessness of space orbit, astronauts are trained in the hold of a cargo aircraft that is accelerating downward at g . Why will they appear to be weightless, as measured by standing on a bathroom scale, in this accelerated frame of reference? Is there any difference between their apparent weightlessness in orbit and in the aircraft?

4. A cartoon shows the toupee coming off the head of an elevator passenger when the elevator rapidly stops during an upward ride. Can this really happen without the person being tied to the floor of the elevator? Explain your answer.

Problems & Exercises

3.1 Problem Solving Strategy

1. A flea jumps by exerting a force of $1.20\times10^{-5}\text{ N}$ straight down on the ground. A breeze blowing on the flea parallel to the ground exerts a force of $0.500\times10^{-6}\text{ N}$ on the flea. Find the direction and magnitude of the acceleration of the flea if its mass is $6.00\times10^{-7}\text{ kg}$. Do not neglect the gravitational force.

2. Two muscles in the back of the leg pull upward on the Achilles tendon, as shown in Figure 3.7. (These muscles are called the medial and lateral heads of the gastrocnemius muscle.) Find the magnitude and direction of the total force on the Achilles tendon. What type of movement could be caused by this force?

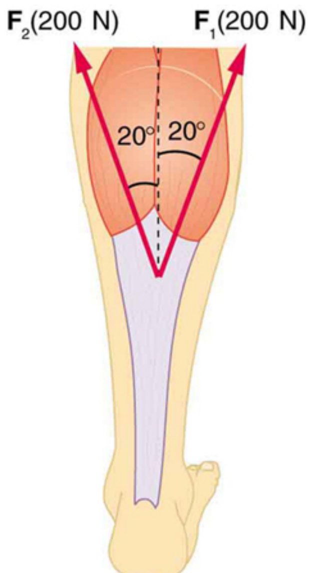


Figure 3.7 Achilles tendon

3. A 76.0-kg person is being pulled away from a burning building as shown in Figure 3.8. Calculate the tension in the two ropes if the person is momentarily motionless. Include a free-body diagram in your solution.

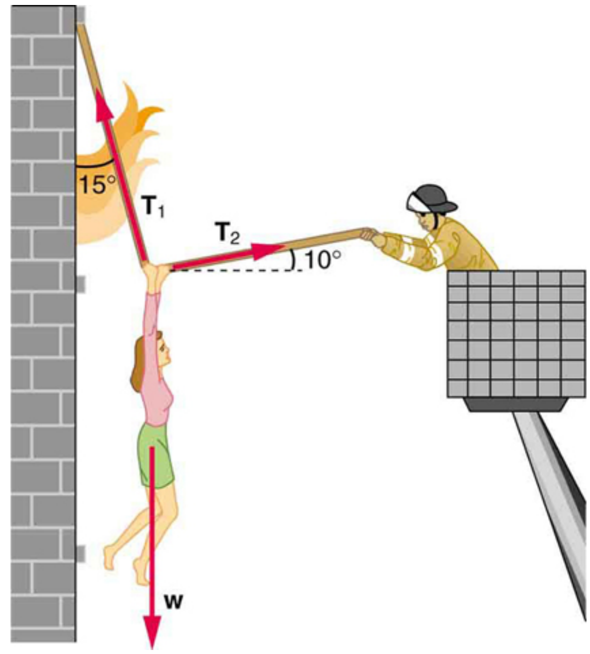


Figure 3.8 The force $\mathbf{T}_2$ needed to hold steady the person being rescued from the fire is less than her weight and less than the force $\mathbf{T}_1$ in the other rope, since the more vertical rope supports a greater part of her weight (a vertical force).

4. Integrated Concepts A 35.0-kg dolphin decelerates from 12.0 to 7.50 m/s in 2.30 s to join another dolphin in play. What average force was exerted to slow him if he was moving horizontally? (The gravitational force is balanced by the buoyant force of the water.)

5. Integrated Concepts When starting a foot race, a 70.0-kg sprinter exerts an average force of 650 N backward on the ground for 0.800 s. (a) What is his final speed? (b) How far does he travel?

6. Integrated Concepts A large rocket has a mass of $2.00\times10^{6}\text{ kg}$ at takeoff, and its engines produce a thrust of $3.50\times10^{7}\text{ N}$. (a) Find its initial acceleration if it takes off vertically. (b) How long does it take to reach a velocity of 120 km/h straight up, assuming constant mass and thrust? (c) In reality, the mass of a rocket decreases significantly as its fuel is consumed. Describe qualitatively how this affects the acceleration and time for this motion.

7. Integrated Concepts A basketball player jumps straight up for a ball. To do this, he lowers his body 0.300 m and then accelerates through this distance by forcefully straightening his legs. This player leaves the floor with a vertical velocity sufficient to carry him 0.900 m above the floor. (a) Calculate his velocity when he leaves the floor. (b) Calculate his acceleration while he is straightening his legs. He goes from zero to the velocity found in part (a) in a distance of 0.300 m. (c) Calculate the force he exerts on the floor to do this, given that his mass is 110 kg.

8. Integrated Concepts A 2.50-kg fireworks shell is fired straight up from a mortar and reaches a height of 110 m. (a) Neglecting air resistance (a poor assumption, but we will make it for this example), calculate the shell's velocity when it leaves the mortar. (b) The mortar itself is a tube 0.450 m long. Calculate the average acceleration of the shell in the tube as it goes from zero to the velocity found in (a). (c) What is the average force on the shell in the mortar? Express your answer in newtons and as a ratio to the weight of the shell.

9. Integrated Concepts Repeat **Exercise 3.8** for a shell fired at an angle 10.0° from the vertical.

10. Integrated Concepts An elevator filled with passengers has a mass of 1700 kg. (a) The elevator accelerates upward from rest at a rate of $1.20\ \text{m/s}^2$ for 1.50 s. Calculate the tension in the cable supporting the elevator. (b) The elevator continues upward at constant velocity for 8.50 s. What is the tension in the cable during this time? (c) The elevator decelerates at a rate of $0.600\ \text{m/s}^2$ for 3.00 s. What is the tension in the cable during deceleration? (d) How high has the elevator moved above its original starting point, and what is its final velocity?

11. Unreasonable Results (a) What is the final velocity of a car originally traveling at 50.0 km/h that decelerates at a rate of $0.400\ \text{m/s}^2$ for 50.0 s? (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

12. Unreasonable Results A 75.0-kg man stands on a bathroom scale in an elevator that accelerates from rest to 30.0 m/s in 2.00 s. (a) Calculate the scale reading in newtons and compare it with his weight. (The scale exerts an upward force on him equal to its reading.) (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

3.2 Further Applications of Newton's Laws of Motion

13. A flea jumps by exerting a force of $1.20\times10^{-5}\ \text{N}$ straight down on the ground. A breeze blowing on the flea parallel to the ground exerts a force of $0.500\times10^{-6}\ \text{N}$ on the flea. Find the direction and magnitude of the acceleration of the flea if its mass is $6.00\times10^{-7}\ \text{kg}$. Do not neglect the gravitational force.

14. Two muscles in the back of the leg pull upward on the Achilles tendon, as shown in **Figure 3.9**. (These muscles are called the medial and lateral heads of the gastrocnemius muscle.) Find the magnitude and direction of the total force on the Achilles tendon. What type of movement could be caused by this force?

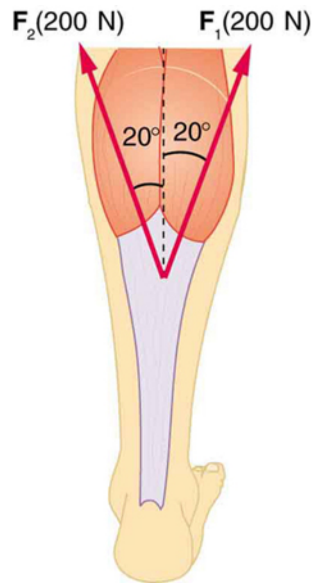


Figure 3.9 Achilles tendon

15. A 76.0-kg person is being pulled away from a burning building as shown in **Figure 3.10**. Calculate the tension in the two ropes if the person is momentarily motionless. Include a free-body diagram in your solution.

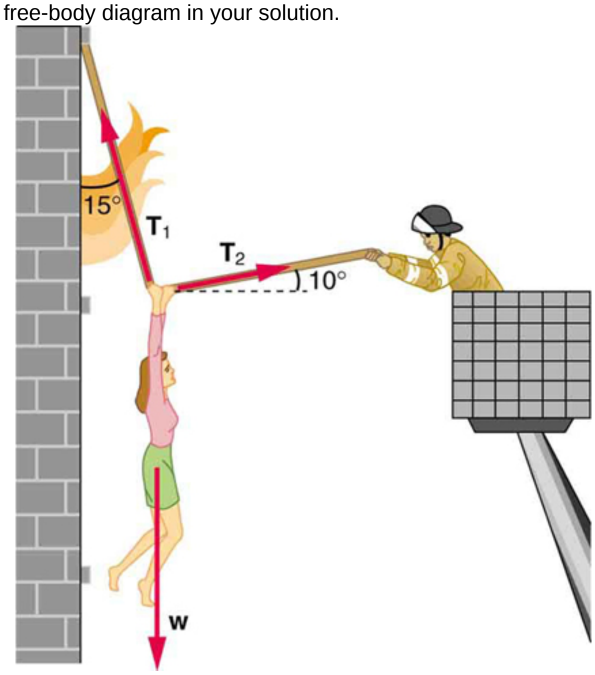


Figure 3.10 The force $\mathbf{T}_2$ needed to hold steady the person being rescued from the fire is less than her weight and less than the force $\mathbf{T}_1$ in the other rope, since the more vertical rope supports a greater part of her weight (a vertical force).

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21. Integrated Concepts Repeat **Exercise 3.20** for a shell fired at an angle 10.0° from the vertical.

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23. Unreasonable Results (a) What is the final velocity of a car originally traveling at 50.0 km/h that decelerates at a rate of 0.400 m/s^2 for 50.0 s? (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

24. Unreasonable Results A 75.0-kg man stands on a bathroom scale in an elevator that accelerates from rest to 30.0 m/s in 2.00 s. (a) Calculate the scale reading in newtons and compare it with his weight. (The scale exerts an upward force on him equal to its reading.) (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?

4 UNIFORM CIRCULAR MOTION AND GRAVITATION

Figure 4.1 This Australian Grand Prix Formula 1 race car moves in a circular path as it makes the turn. Its wheels also spin rapidly—the latter completing many revolutions, the former only part of one (a circular arc). The same physical principles are involved in each. (credit: Richard Munckton)

Chapter Outline

4.1. Rotation Angle and Angular Velocity

- Define arc length, rotation angle, radius of curvature and angular velocity.
- Calculate the angular velocity of a car wheel spin.

4.2. Centripetal Acceleration

- Establish the expression for centripetal acceleration.
- Explain the centrifuge.

4.3. Centripetal Force

By the end of the section, you will be able to:

- Explain the equation for centripetal acceleration
- Apply Newton's second law to develop the equation for centripetal force
- Use circular motion concepts in solving problems involving Newton's laws of motion

4.4. Fictitious Forces and Non-inertial Frames: The Coriolis Force

- Discuss the inertial frame of reference.
- Discuss the non-inertial frame of reference.
- Describe the effects of the Coriolis force.

4.5. Newton's Universal Law of Gravitation

- Explain Earth's gravitational force.
- Describe the gravitational effect of the Moon on Earth.
- Discuss weightlessness in space.
- Examine the Cavendish experiment

4.6. Satellites and Kepler's Laws: An Argument for Simplicity
- State Kepler's laws of planetary motion.
- Derive the third Kepler's law for circular orbits.
- Discuss the Ptolemaic model of the universe.

Introduction to Uniform Circular Motion and Gravitation

UMASS AMHERST Instructor's Notes

This content is not covered in this course, and is here solely for your information.

Many motions, such as the arc of a bird's flight or Earth's path around the Sun, are curved. Recall that Newton's first law tells us that motion is along a straight line at constant speed unless there is a net external force. We will therefore study not only motion along curves, but also the forces that cause it, including gravitational forces. In some ways, this chapter is a continuation of **Dynamics: Newton's Laws of Motion (https://legacy.cnx.org/content/m42129/latest/)** as we study more applications of Newton's laws of motion.

This chapter deals with the simplest form of curved motion, **uniform circular motion**, motion in a circular path at constant speed. Studying this topic illustrates most concepts associated with rotational motion and leads to the study of many new topics we group under the name *rotation*. Pure *rotational motion* occurs when points in an object move in circular paths centered on one point. Pure *translational motion* is motion with no rotation. Some motion combines both types, such as a rotating hockey puck moving along ice.

4.1 Rotation Angle and Angular Velocity

In **Kinematics (https://legacy.cnx.org/content/m42122/latest/)** , we studied motion along a straight line and introduced such concepts as displacement, velocity, and acceleration. **Two-Dimensional Kinematics (https://legacy.cnx.org/content/m42126/latest/)** dealt with motion in two dimensions. Projectile motion is a special case of two-dimensional kinematics in which the object is projected into the air, while being subject to the gravitational force, and lands a distance away. In this chapter, we consider situations where the object does not land but moves in a curve. We begin the study of uniform circular motion by defining two angular quantities needed to describe rotational motion.

Rotation Angle

When objects rotate about some axis—for example, when the CD (compact disc) in **Figure 4.2** rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each **pit** used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation and is analogous to linear distance. We define the **rotation angle** $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

$$\Delta\theta = \frac{\Delta s}{r}. \tag{4.1}$$

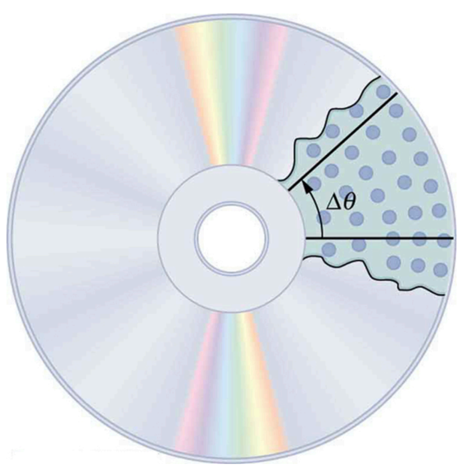


Figure 4.2 All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle $\Delta\theta$ in a time Δt .

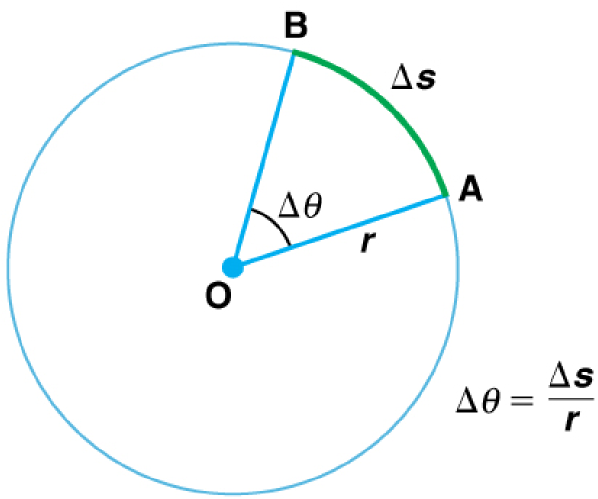


Figure 4.3 The radius of a circle is rotated through an angle $\Delta\theta$. The arc length Δs is described on the circumference.

The **arc length** Δs is the distance traveled along a circular path as shown in **Figure 4.3** Note that r is the **radius of curvature** of the circular path.

We know that for one complete revolution, the arc length is the circumference of a circle of radius r. The circumference of a circle is $2\pi r$. Thus for one complete revolution the rotation angle is

$$\Delta\theta = \frac{2\pi r}{r} = 2\pi. \tag{4.2}$$

This result is the basis for defining the units used to measure rotation angles, $\Delta\theta$ to be **radians** (rad), defined so that

$$2\pi \text{ rad} = 1 \text{ revolution.} \tag{4.3}$$

A comparison of some useful angles expressed in both degrees and radians is shown in **Table 4.1**.

Table 4.1 Comparison of Angular Units

Degree Measures	Radian Measure
30°	$\frac{\pi}{6}$
60°	$\frac{\pi}{3}$
90°	$\frac{\pi}{2}$
120°	$\frac{2\pi}{3}$
135°	$\frac{3\pi}{4}$
180°	π

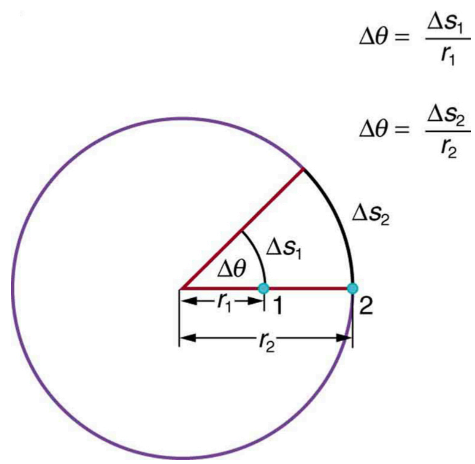


Figure 4.4 Points 1 and 2 rotate through the same angle ($\Delta\theta$), but point 2 moves through a greater arc length (Δs) because it is at a greater distance from the center of rotation (r) .

If $\Delta\theta = 2\pi$ rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are 360° in a circle or one revolution, the relationship between radians and degrees is thus

$$2\pi \text{ rad} = 360^{\circ} \tag{4.4}$$

so that

$$1 \text{ rad} = \frac{360^{\circ}}{2\pi} \approx 57.3^{\circ}. \tag{4.5}$$

Angular Velocity

How fast is an object rotating? We define **angular velocity** ω as the rate of change of an angle. In symbols, this is

$$\omega = \frac{\Delta\theta}{\Delta t}, \tag{4.6}$$

where an angular rotation $\Delta\theta$ takes place in a time Δt . The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity ω is analogous to linear velocity v . To get the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length Δs in a time Δt , and so it has a linear velocity

$$v = \frac{\Delta s}{\Delta t}. \tag{4.7}$$

From $\Delta\theta = \frac{\Delta s}{r}$ we see that $\Delta s = r\Delta\theta$. Substituting this into the expression for v gives

$$v = \frac{r\Delta\theta}{\Delta t} = r\omega. \tag{4.8}$$

We write this relationship in two different ways and gain two different insights:

$$v = r\omega \text{ or } \omega = \frac{v}{r}. \tag{4.9}$$

The first relationship in $v = r\omega \text{ or } \omega = \frac{v}{r}$ states that the linear velocity v is proportional to the distance from the center of rotation, thus, it is largest for a point on the rim (largest r), as you might expect. We can also call this linear speed v of a point on the rim the *tangential speed*. The second relationship in $v = r\omega \text{ or } \omega = \frac{v}{r}$ can be illustrated by considering the tire of a moving car. Note that the speed of a point on the rim of the tire is the same as the speed v of the car. See **Figure 4.5**. So the faster the car moves, the faster the tire spins—large v means a large ω , because $v = r\omega$. Similarly, a larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (v) for the car.

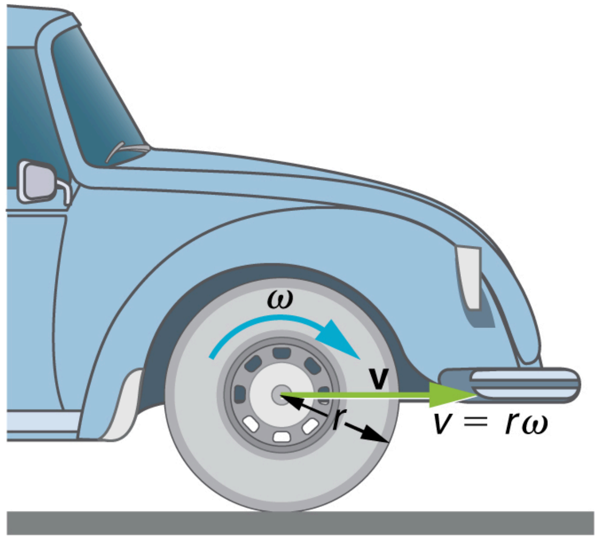


Figure 4.5 A car moving at a velocity v to the right has a tire rotating with an angular velocity ω .The speed of the tread of the tire relative to the axle is v , the same as if the car were jacked up. Thus the car moves forward at linear velocity $v = r\omega$, where r is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

Example 4.1 How Fast Does a Car Tire Spin?

Calculate the angular velocity of a 0.300 m radius car tire when the car travels at 15.0 m/s (about 54 km/h). See **Figure 4.5**.

Strategy

Because the linear speed of the tire rim is the same as the speed of the car, we have $v = 15.0\text{ m/s.}$ The radius of the tire is given to be $r = 0.300\text{ m.}$ Knowing v and r , we can use the second relationship in $v = r\omega, \omega = \frac{v}{r}$ to calculate the angular velocity.

Solution

To calculate the angular velocity, we will use the following relationship:

$$\omega = \frac{v}{r}. \tag{4.10}$$

Substituting the knowns,

$$\omega = \frac{15.0\text{ m/s}}{0.300\text{ m}} = 50.0\text{ rad/s.} \tag{4.11}$$

Discussion

When we cancel units in the above calculation, we get 50.0/s. But the angular velocity must have units of rad/s. Because radians are actually unitless (radians are defined as a ratio of distance), we can simply insert them into the answer for the angular velocity. Also note that if an earth mover with much larger tires, say 1.20 m in radius, were moving at the same speed of 15.0 m/s, its tires would rotate more slowly. They would have an angular velocity

$$\omega = (15.0\text{ m/s}) / (1.20\text{ m}) = 12.5\text{ rad/s.} \tag{4.12}$$

Both ω and v have directions (hence they are angular and linear *velocities*, respectively). Angular velocity has only two directions with respect to the axis of rotation—it is either clockwise or counterclockwise. Linear velocity is tangent to the path, as illustrated in **Figure 4.6**.

Take-Home Experiment

Tie an object to the end of a string and swing it around in a horizontal circle above your head (swing at your wrist). Maintain uniform speed as the object swings and measure the angular velocity of the motion. What is the approximate speed of the object? Identify a point close to your hand and take appropriate measurements to calculate the linear speed at this point. Identify other circular motions and measure their angular velocities.

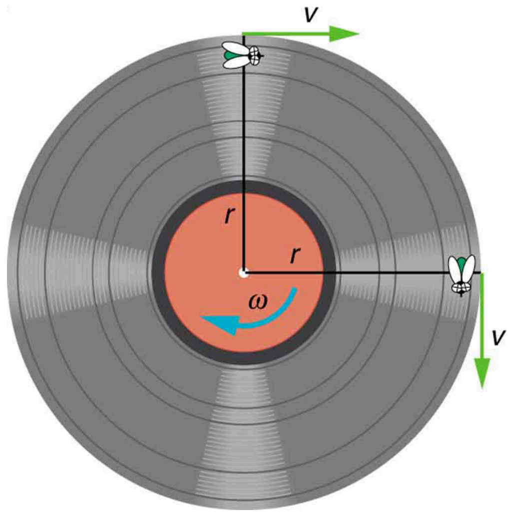


Figure 4.6 As an object moves in a circle, here a fly on the edge of an old-fashioned vinyl record, its instantaneous velocity is always tangent to the circle. The direction of the angular velocity is clockwise in this case.

PhET Explorations: Ladybug Revolution

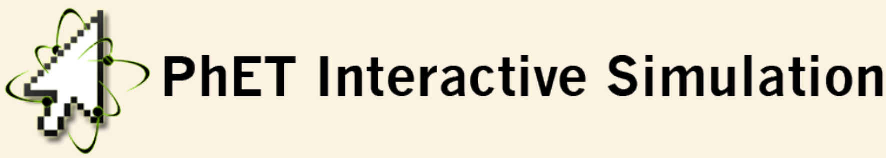


Figure 4.7 Ladybug Revolution (http://legacy.cnx.org/content/m42083/1.7/rotation_en.jar)

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle, or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's x,y position, velocity, and acceleration using vectors or graphs.

4.2 Centripetal Acceleration

We know from kinematics that acceleration is a change in velocity, either in its magnitude or in its direction, or both. In uniform circular motion, the direction of the velocity changes constantly, so there is always an associated acceleration, even though the magnitude of the velocity might be constant. You experience this acceleration yourself when you turn a corner in your car. (If you hold the wheel steady during a turn and move at constant speed, you are in uniform circular motion.) What you notice is a sideways acceleration because you and the car are changing direction. The sharper the curve and the greater your speed, the more noticeable this acceleration will become. In this section we examine the direction and magnitude of that acceleration.

Figure 4.8 shows an object moving in a circular path at constant speed. The direction of the instantaneous velocity is shown at two points along the path. Acceleration is in the direction of the change in velocity, which points directly toward the center of rotation (the center of the circular path). This pointing is shown with the vector diagram in the figure. We call the acceleration of an object moving in uniform circular motion (resulting from a net external force) the **centripetal acceleration**(a_c); centripetal means “toward the center” or “center seeking.”

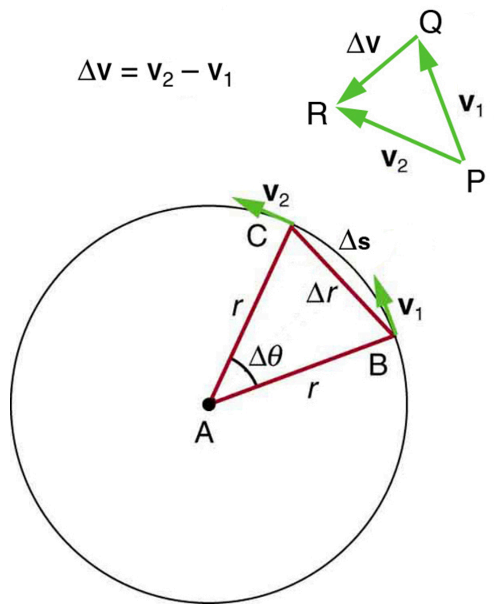


Figure 4.8 The directions of the velocity of an object at two different points are shown, and the change in velocity $\Delta\mathbf{v}$ is seen to point directly toward the center of curvature. (See small inset.) Because $\mathbf{a}_\text{c} = \Delta\mathbf{v} / \Delta t$, the acceleration is also toward the center; $\mathbf{a}_c$ is called centripetal acceleration. (Because $\Delta\theta$ is very small, the arc length Δs is equal to the chord length Δr for small time differences.)

The direction of centripetal acceleration is toward the center of curvature, but what is its magnitude? Note that the triangle formed by the velocity vectors and the one formed by the radii r and Δs are similar. Both the triangles ABC and PQR are isosceles triangles (two equal sides). The two equal sides of the velocity vector triangle are the speeds $v_1 = v_2 = v$. Using the properties of two similar triangles, we obtain

$$\frac{\Delta v}{v} = \frac{\Delta s}{r}. \tag{4.13}$$

Acceleration is $\Delta v / \Delta t$, and so we first solve this expression for Δv:

$$\Delta v = \frac{v}{r}\Delta s. \tag{4.14}$$

Then we divide this by Δt, yielding

$$\frac{\Delta v}{\Delta t} = \frac{v}{r} \times \frac{\Delta s}{\Delta t}. \tag{4.15}$$

Finally, noting that $\Delta v / \Delta t = a_\text{c}$ and that $\Delta s / \Delta t = v$, the linear or tangential speed, we see that the magnitude of the centripetal acceleration is

$$a_\text{c} = \frac{v^2}{r}, \tag{4.16}$$

which is the acceleration of an object in a circle of radius r at a speed v. So, centripetal acceleration is greater at high speeds and in sharp curves (smaller radius), as you have noticed when driving a car. But it is a bit surprising that a_c is proportional to speed squared, implying, for example, that it is four times as hard to take a curve at 100 km/h than at 50 km/h. A sharp corner has a small radius, so that a_c is greater for tighter turns, as you have probably noticed.

It is also useful to express a_c in terms of angular velocity. Substituting $v = r\omega$ into the above expression, we find $a_\text{c} = (r\omega)^2 / r = r\omega^2$. We can express the magnitude of centripetal acceleration using either of two equations:

$$a_\text{c} = \frac{v^2}{r};\ a_\text{c} = r\omega^2. \tag{4.17}$$

Recall that the direction of a_c is toward the center. You may use whichever expression is more convenient, as illustrated in examples below.

A **centrifuge** (see Figure 4.9b) is a rotating device used to separate specimens of different densities. High centripetal acceleration significantly decreases the time it takes for separation to occur, and makes separation possible with small samples. Centrifuges are used in a variety of applications in science and medicine, including the separation of single cell suspensions such as bacteria, viruses, and blood cells from a liquid medium and the separation of macromolecules, such as DNA and protein, from a solution. Centrifuges are often rated in terms of their centripetal acceleration relative to acceleration due to gravity (g); maximum centripetal acceleration of several hundred thousand g is possible in a vacuum. Human centrifuges, extremely large centrifuges, have been used to test the tolerance of astronauts to the effects of accelerations larger than that of Earth's gravity.

Example 4.2 How Does the Centripetal Acceleration of a Car Around a Curve Compare with That Due to Gravity?

What is the magnitude of the centripetal acceleration of a car following a curve of radius 500 m at a speed of 25.0 m/s (about 90 km/h)? Compare the acceleration with that due to gravity for this fairly gentle curve taken at highway speed. See **Figure 4.9**(a).

Strategy

Because v and r are given, the first expression in $a_\text{c} = \frac{v^2}{r}; a_\text{c} = r\omega^2$ is the most convenient to use.

Solution

Entering the given values of $v = 25.0\text{ m/s}$ and $r = 500\text{ m}$ into the first expression for a_c gives

$$a_\text{c} = \frac{v^2}{r} = \frac{(25.0\text{ m/s})^2}{500\text{ m}} = 1.25\text{ m/s}^2. \tag{4.18}$$

Discussion

To compare this with the acceleration due to gravity $(g = 9.80\text{ m/s}^2)$, we take the ratio of $a_\text{c}/g = \left(1.25\text{ m/s}^2\right)/\left(9.80\text{ m/s}^2\right) = 0.128$. Thus, $a_\text{c} = 0.128\text{ g}$ and is noticeable especially if you were not wearing a seat belt.

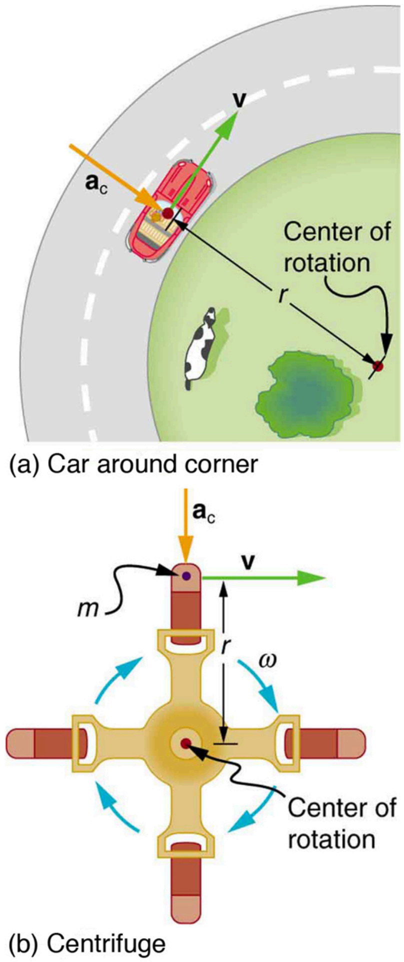


Figure 4.9 (a) The car following a circular path at constant speed is accelerated perpendicular to its velocity, as shown. The magnitude of this centripetal acceleration is found in **Example 4.2**. (b) A particle of mass in a centrifuge is rotating at constant angular velocity . It must be accelerated perpendicular to its velocity or it would continue in a straight line. The magnitude of the necessary acceleration is found in **Example 4.3**.

Example 4.3 How Big Is the Centripetal Acceleration in an Ultracentrifuge?

Calculate the centripetal acceleration of a point 7.50 cm from the axis of an **ultracentrifuge** spinning at 7.5×10^4 rev/min. Determine the ratio of this acceleration to that due to gravity. See **Figure 4.9**(b).

Strategy

The term rev/min stands for revolutions per minute. By converting this to radians per second, we obtain the angular velocity ω. Because r is given, we can use the second expression in the equation $a_c = \frac{v^2}{r}; a_c = r\omega^2$ to calculate the centripetal acceleration.

Solution

To convert $7.50{\times}10^4$ rev / min to radians per second, we use the facts that one revolution is $2\pi\text{rad}$ and one minute is 60.0 s. Thus,

$$\omega = 7.50{\times}10^4 \ \frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{1 \text{ rev}} \times \frac{1 \text{ min}}{60.0 \text{ s}} = 7854 \text{ rad/s}. \tag{4.19}$$

Now the centripetal acceleration is given by the second expression in $a_c = \frac{v^2}{r}; a_c = r\omega^2$ as

$$a_c = r\omega^2. \tag{4.20}$$

Converting 7.50 cm to meters and substituting known values gives

$$a_c = (0.0750 \text{ m})(7854 \text{ rad/s})^2 = 4.63{\times}10^6 \text{ m/s}^2. \tag{4.21}$$

Note that the unitless radians are discarded in order to get the correct units for centripetal acceleration. Taking the ratio of a_c to g yields

$$\frac{a_c}{g} = \frac{4.63 \times 10^6}{9.80} = 4.72 \times 10^5. \tag{4.22}$$

Discussion

This last result means that the centripetal acceleration is 472,000 times as strong as g. It is no wonder that such high ω centrifuges are called ultracentrifuges. The extremely large accelerations involved greatly decrease the time needed to cause the sedimentation of blood cells or other materials.

Of course, a net external force is needed to cause any acceleration, just as Newton proposed in his second law of motion. So a net external force is needed to cause a centripetal acceleration. In **Centripetal Force (https://legacy.cnx.org/content/m42086/latest/)** , we will consider the forces involved in circular motion.

PhET Explorations: Ladybug Motion 2D

Learn about position, velocity and acceleration vectors. Move the ladybug by setting the position, velocity or acceleration, and see how the vectors change. Choose linear, circular or elliptical motion, and record and playback the motion to analyze the behavior.

PhET Interactive Simulation

Figure 4.10 Ladybug Motion 2D (http://legacy.cnx.org/content/m42084/1.9/ladybug-motion-2d_en.jar)

4.3 Centripetal Force

In **Motion in Two and Three Dimensions (https://legacy.cnx.org/content/m58288/latest/)** , we examined the basic concepts of circular motion. An object undergoing circular motion, like one of the race cars shown at the beginning of this chapter, must be accelerating because it is changing the direction of its velocity. We proved that this centrally directed acceleration, called centripetal acceleration, is given by the formula

$$a_c = \frac{v^2}{r} \tag{4.23}$$

where *v* is the velocity of the object, directed along a tangent line to the curve at any instant. If we know the angular velocity ω , then we can use

$$a_c = r\omega^2. \tag{4.24}$$

Angular velocity gives the rate at which the object is turning through the curve, in units of rad/s. This acceleration acts along the radius of the curved path and is thus also referred to as a radial acceleration.

An acceleration must be produced by a force. Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth’s gravity on the Moon, friction between roller skates and a rink floor, a banked roadway’s force on a car, and forces on the tube of a spinning centrifuge. Any net force causing uniform circular motion is called a **centripetal force**. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton’s second law of motion, net force is mass times acceleration: $F_{net} = ma$. For uniform circular motion, the acceleration is the centripetal acceleration:. $a = a_c$. Thus, the magnitude of centripetal force F_c is

$$F_c = ma_c. \tag{4.25}$$

By substituting the expressions for centripetal acceleration a_c $\left(a_c = \frac{v^2}{r};\ a_c = r\omega^2\right)$, we get two expressions for the centripetal force F_c in terms of mass, velocity, angular velocity, and radius of curvature:

$$F_c = m\frac{v^2}{r}; \quad F_c = mr\omega^2. \tag{4.26}$$

You may use whichever expression for centripetal force is more convenient. Centripetal force $\vec{\mathbf{F}}_c$ is always perpendicular to the path and points to the center of curvature, because $\vec{\mathbf{a}}_c$ is perpendicular to the velocity and points to the center of curvature. Note that if you solve the first expression for *r*, you get

$$r = \frac{mv^2}{F_c}. \tag{4.27}$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature—that is, a tight curve, as in **Figure 4.11**.

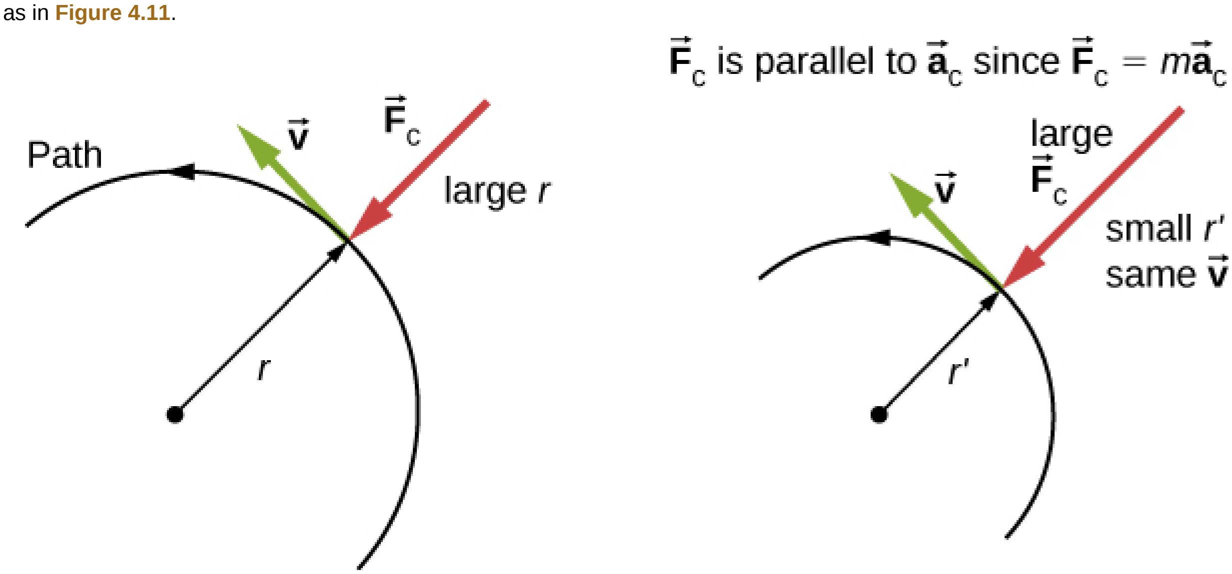


Figure 4.11 The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c, the smaller the radius of curvature *r* and the sharper the curve. The second curve has the same *v*, but a larger F_c produces a smaller *r*′.

Example 4.4

What Coefficient of Friction Do Cars Need on a Flat Curve?

(a) Calculate the centripetal force exerted on a 900.0-kg car that negotiates a 500.0-m radius curve at 25.00 m/s. (b) Assuming an unbanked curve, find the minimum static coefficient of friction between the tires and the road, static friction being the reason that keeps the car from slipping (**Figure 4.12**).

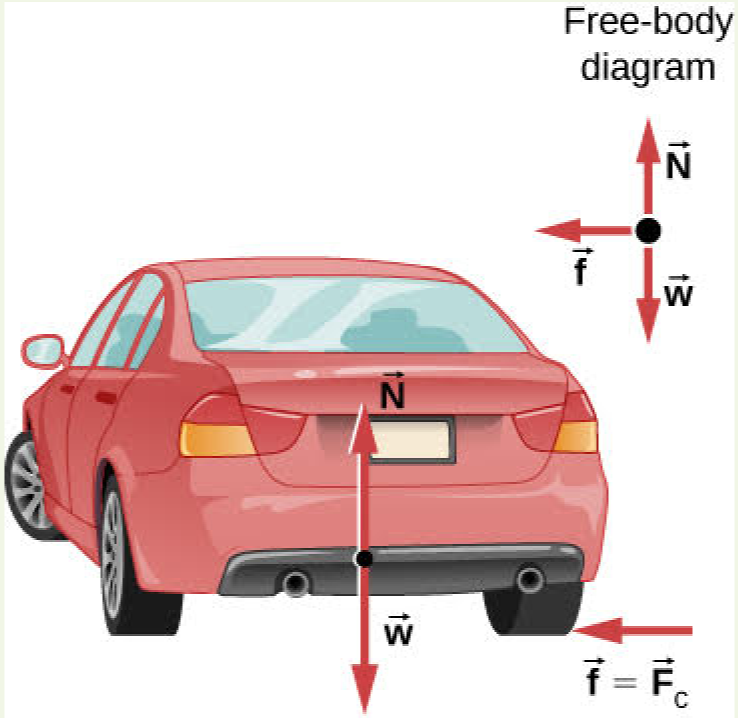


Figure 4.12 This car on level ground is moving away and turning to the left. The centripetal force causing the car to turn in a circular path is due to friction between the tires and the road. A minimum coefficient of friction is needed, or the car will move in a larger-radius curve and leave the roadway.

Strategy

a. We know that $F_c = \frac{mv^2}{r}$. Thus,

$$F_c = \frac{mv^2}{r} = \frac{(900.0\text{ kg})(25.00\text{ m/s})^2}{(500.0\text{ m})} = 1125\text{ N}. \tag{4.28}$$

b. **Figure 4.12** shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is $\mu_s N$, where μ_s is the static coefficient of friction and *N* is the normal force. The normal force equals the car's weight on level ground, so $N = mg$. Thus the centripetal force in this situation is

$$F_c \equiv f = \mu_s N = \mu_s mg. \tag{4.29}$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the equation

$$F_c = m\frac{v^2}{r}, \tag{4.30}$$

we obtain

$$m\frac{v^2}{r} = \mu_s mg. \tag{4.31}$$

We solve this for μ_s, noting that mass cancels, and obtain

$$\mu_s = \frac{v^2}{rg}. \tag{4.32}$$

Substituting the knowns,

$$\mu_s = \frac{(25.00\text{ m/s})^2}{(500.0\text{ m})(9.80\text{ m/s}^2)} = 0.13. \tag{4.33}$$

(Because coefficients of friction are approximate, the answer is given to only two digits.)

Significance

The coefficient of friction found in **Figure 4.12**(b) is much smaller than is typically found between tires and roads. The car still negotiates the curve if the coefficient is greater than 0.13, because static friction is a responsive force, able to assume a value less than but no more than $\mu_s N.$ A higher coefficient would also allow the car to negotiate the curve at a higher speed, but if the coefficient of friction is less, the safe speed would be less than 25 m/s. Note that mass cancels, implying that, in this example, it does not matter how heavily loaded the car is to negotiate the turn. Mass cancels because friction is assumed proportional to the normal force, which in turn is proportional to mass. If the surface of the road were banked, the normal force would be less, as discussed next.

Exercise 4.1

Check Your Understanding A car moving at 96.8 km/h travels around a circular curve of radius 182.9 m on a flat country road. What must be the minimum coefficient of static friction to keep the car from slipping?

Solution

0.40

Banked Curves

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve (**Figure 4.13**). The greater the angle θ, the faster you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an "ideally banked curve," the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve and consider an example related to it.

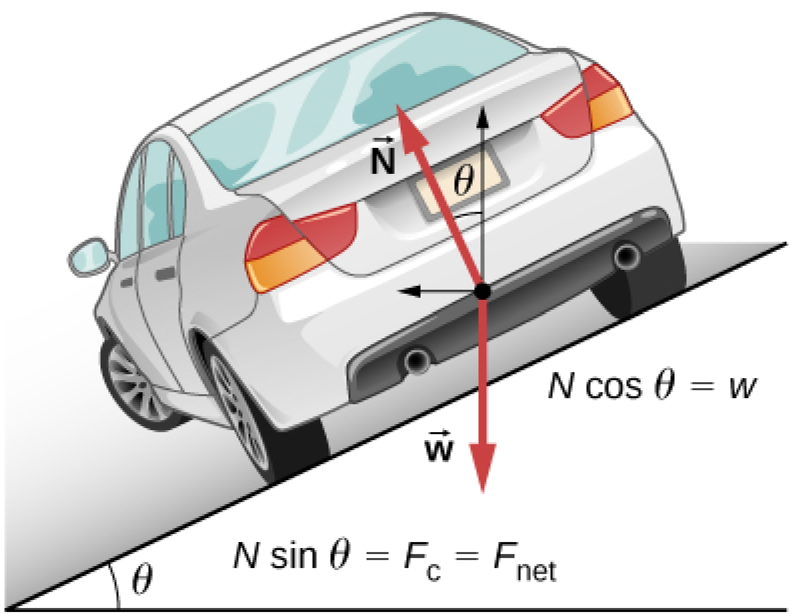


Figure 4.13 The car on this banked curve is moving away and turning to the left.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force *N* in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.

Figure 4.13 shows a free-body diagram for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius, then the net external force equals the necessary centripetal force. The only two external forces acting on the car are its weight $\vec{w}$ and the normal force of the road $\vec{N}$. (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude $mv^2/r.$ Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, so this must equal the centripetal force, that is,

$$N \sin \theta = \frac{mv^2}{r}. \tag{4.34}$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From **Figure 4.13**, we see that the vertical component of the normal force is $N \cos \theta,$ and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

$$N \cos \theta = mg. \tag{4.35}$$

Now we can combine these two equations to eliminate *N* and get an expression for θ, as desired. Solving the second equation for $N = mg/(cos\theta)$ and substituting this into the first yields

$$\begin{aligned} mg\frac{\sin \theta}{\cos \theta} &= \frac{mv^2}{r} \\ mg \tan \theta &= \frac{mv^2}{r} \\ \tan \theta &= \frac{v^2}{rg}. \end{aligned} \tag{4.36}$$

Taking the inverse tangent gives

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right). \tag{4.37}$$

This expression can be understood by considering how θ depends on *v* and *r*. A large θ is obtained for a large *v* and a small *r*. That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve were frictionless. Note that θ does not depend on the mass of the vehicle.

Example 4.5

What Is the Ideal Speed to Take a Steeply Banked Tight Curve?

Curves on some test tracks and race courses, such as Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100.0-m radius curve banked at $31.0°$ should be driven if the road were frictionless.

Strategy

We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

Solution

Starting with

$$\tan \theta = \frac{v^2}{rg}, \tag{4.38}$$

we get

$$v = \sqrt{rg \tan \theta}. \tag{4.39}$$

Noting that $\tan 31.0° = 0.609,$ we obtain

$$v = \sqrt{(100.0 \text{ m})(9.80 \text{ m/s}^2)(0.609)} = 24.4 \text{ m/s}. \tag{4.40}$$

Significance

This is just about 165 km/h, consistent with a very steeply banked and rather sharp curve. Tire friction enables a vehicle to take the curve at significantly higher speeds.

Airplanes also make turns by banking. The lift force, due to the force of the air on the wing, acts at right angles to the wing. When the airplane banks, the pilot is obtaining greater lift than necessary for level flight. The vertical component of lift balances the airplane's weight, and the horizontal component accelerates the plane. The banking angle shown in **Figure 4.14** is given by θ. We analyze the forces in the same way we treat the case of the car rounding a banked curve.

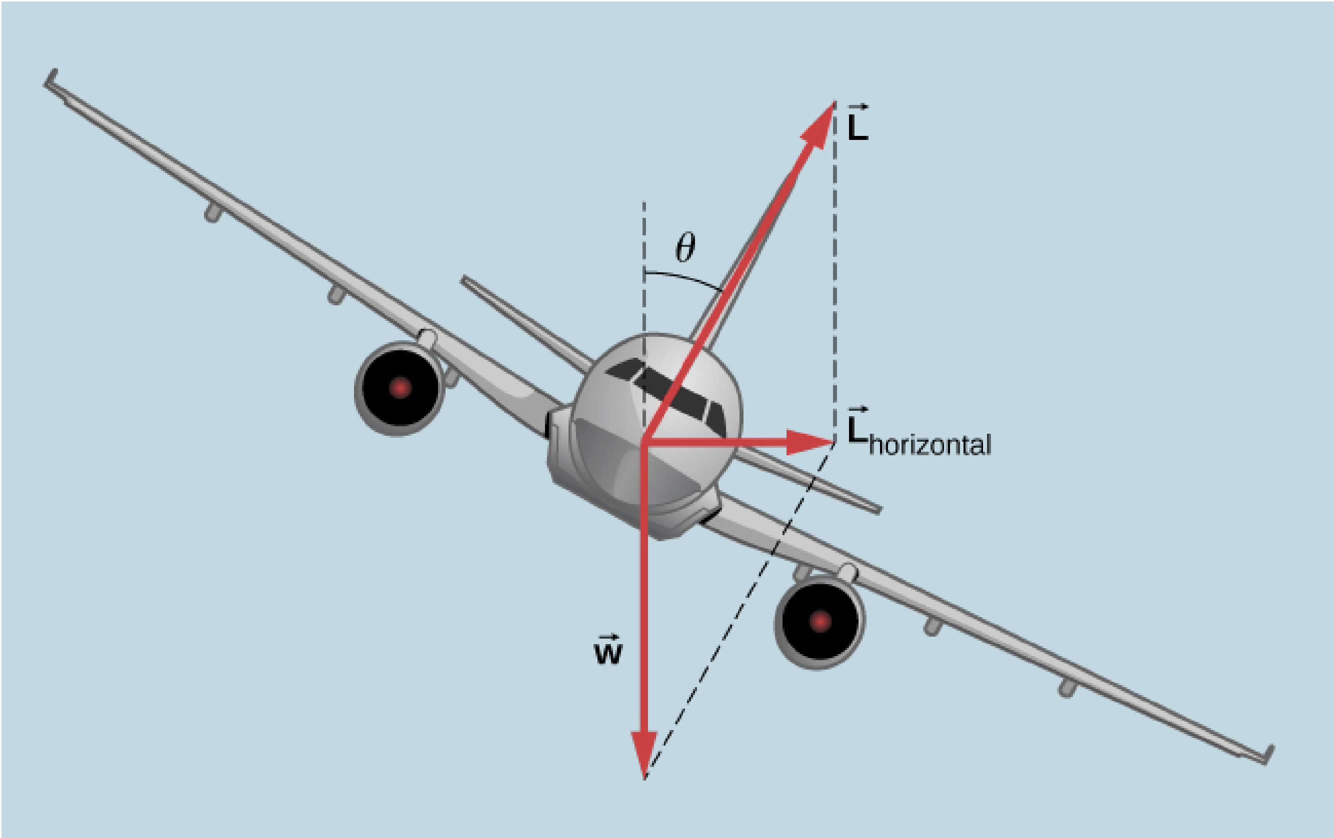


Figure 4.14 In a banked turn, the horizontal component of lift is unbalanced and accelerates the plane. The normal component of lift balances the plane's weight. The banking angle is given by θ . Compare the vector diagram with that shown in **Figure 4.13**.

Join the **ladybug (https://openstaxcollege.org/l/21ladybug)** in an exploration of rotational motion. Rotate the merry-go-round to change its angle or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's *xy*-position, velocity, and acceleration using vectors or graphs.

A circular motion requires a force, the so-called centripetal force, which is directed to the axis of rotation. This simplified **model of a carousel (https://openstaxcollege.org/l/21carousel)** demonstrates this force.

Inertial Forces and Noninertial (Accelerated) Frames: The Coriolis Force

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits inertial forces—forces that merely seem to arise from motion, because the observer's frame of reference is accelerating or rotating. When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that *you* tend to remain stationary while the *seat* pushes forward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right (**Figure 4.15**). You feel as if you are thrown (that is, *forced*) toward the left relative to the car. Again, a physicist would say that *you* are going in a straight line (recall Newton's first law) but the *car* moves to the right, not that you are experiencing a force from the left.

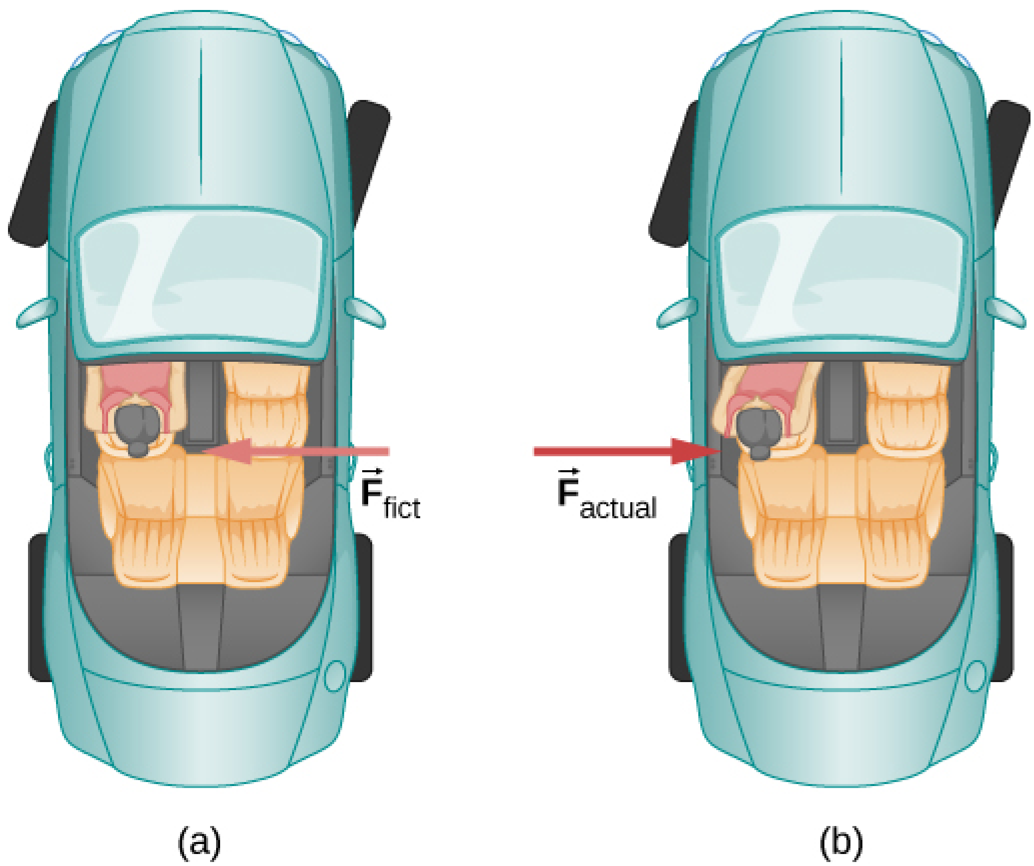


Figure 4.15 (a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is an inertial force arising from the use of the car as a frame of reference. (b) In Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no force to the left on the driver relative to Earth. Instead, there is a force to the right on the car to make it turn.

We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, whereas a physicist might use Earth. The physicist might make this choice because Earth is nearly an inertial frame of reference, in which all forces have an identifiable physical origin. In such a frame of reference, Newton's laws of motion take the form given in **Newton's Laws of Motion (https://legacy.cnx.org/content/m58294/latest/)** . The car is a **noninertial frame of reference** because it is accelerated to the side. The force to the left sensed by car passengers is an **inertial force** having no physical origin (it is due purely to the inertia of the passenger, not to some physical cause such as tension, friction, or gravitation). The car, as well as the driver, is actually accelerating to the right. This inertial force is said to be an inertial force because it does not have a physical origin, such as gravity.

A physicist will choose whatever reference frame is most convenient for the situation being analyzed. There is no problem to a physicist in including inertial forces and Newton's second law, as usual, if that is more convenient, for example, on a merry-go-round or on a rotating planet. Noninertial (accelerated) frames of reference are used when it is useful to do so. Different frames of reference must be considered in discussing the motion of an astronaut in a spacecraft traveling at speeds near the speed of light, as you will appreciate in the study of the special theory of relativity.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round (**Figure 4.16**). You take the merry-go-round to be your frame of reference because you rotate together. When rotating in that noninertial frame of reference, you feel an inertial force that tends to throw you off; this is often referred to as a *centrifugal force* (not to be confused with centripetal force). Centrifugal force is a commonly used term, but it does not actually exist. You must hang on tightly to counteract your inertia (which people often refer to as centrifugal force). In Earth's frame of reference, there is no force trying to throw you off; we emphasize that centrifugal force is a fiction. You must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round, in keeping with Newton's first law. But the force you exert acts toward the center of the circle.

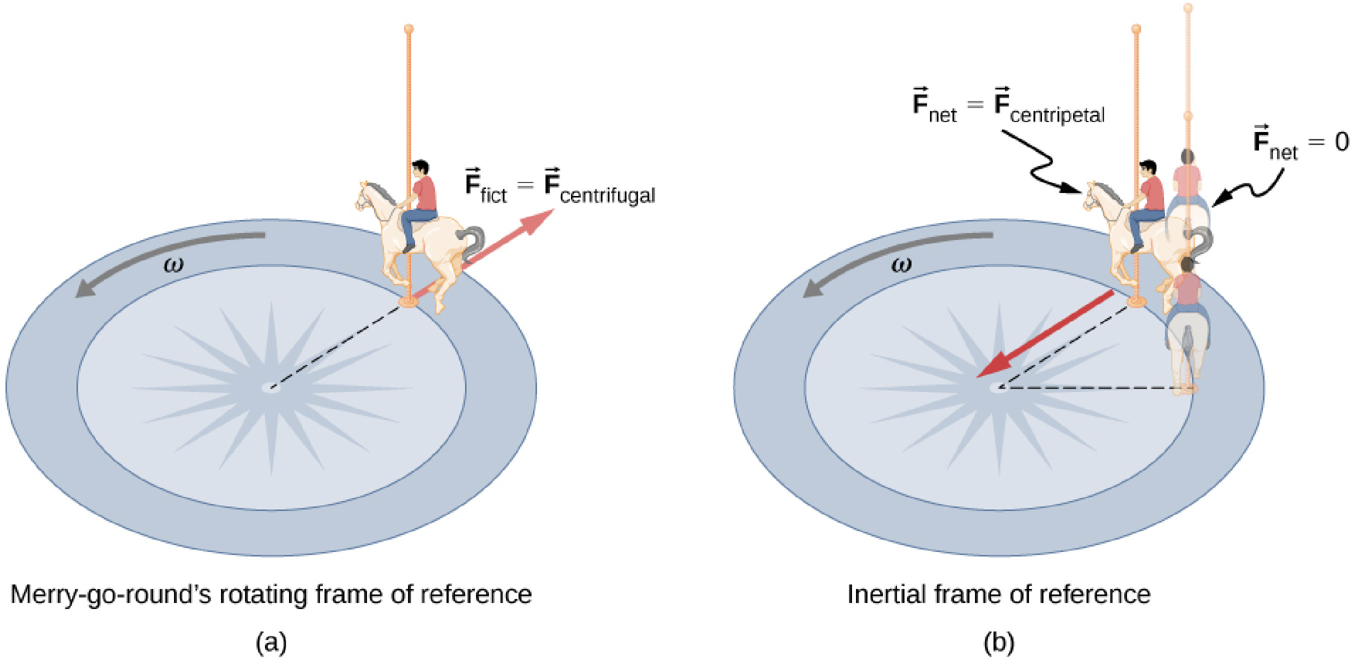


Figure 4.16 (a) A rider on a merry-go-round feels as if he is being thrown off. This inertial force is sometimes mistakenly called the centrifugal force in an effort to explain the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off (the unshaded rider has $F_{\text{net}} = 0$ and heads in a straight line). A force, $F_{\text{centripetal}}$, is needed to cause a circular path.

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (**Figure 4.17**). A centrifuge spins a sample very rapidly, as mentioned earlier in this chapter. Viewed from the rotating frame of reference, the inertial force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.

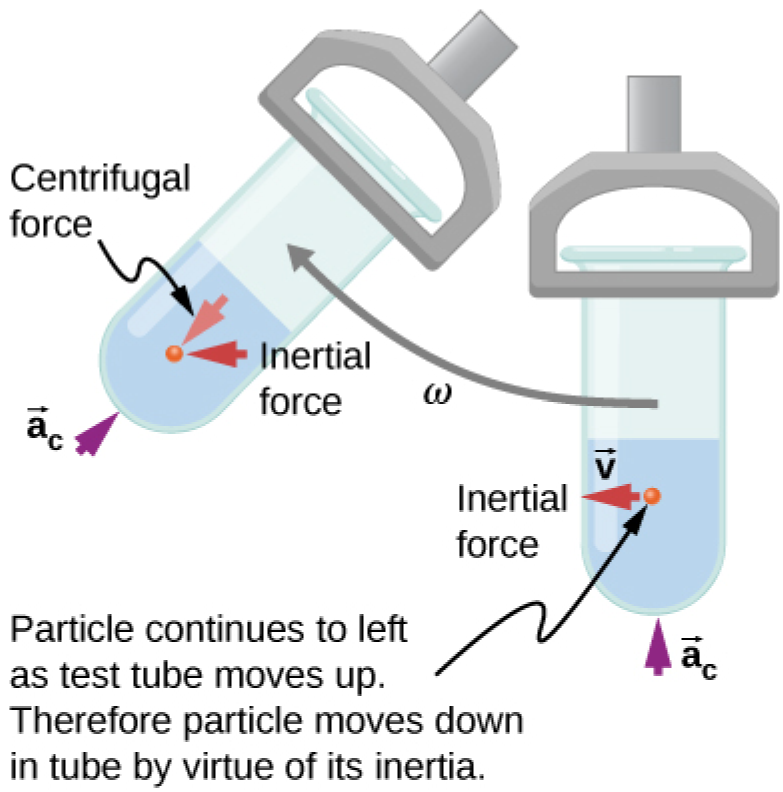


Figure 4.17 Centrifuges use inertia to perform their task. Particles in the fluid sediment settle out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles come into contact with the test tube walls, which then supply the centripetal force needed to make them move in a circle of constant radius.

Let us now consider what happens if something moves in a rotating frame of reference. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in **Figure 4.18**? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using an inertial force, called the **Coriolis force**, which causes the ball to curve to the right. The Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's laws in noninertial frames of reference.

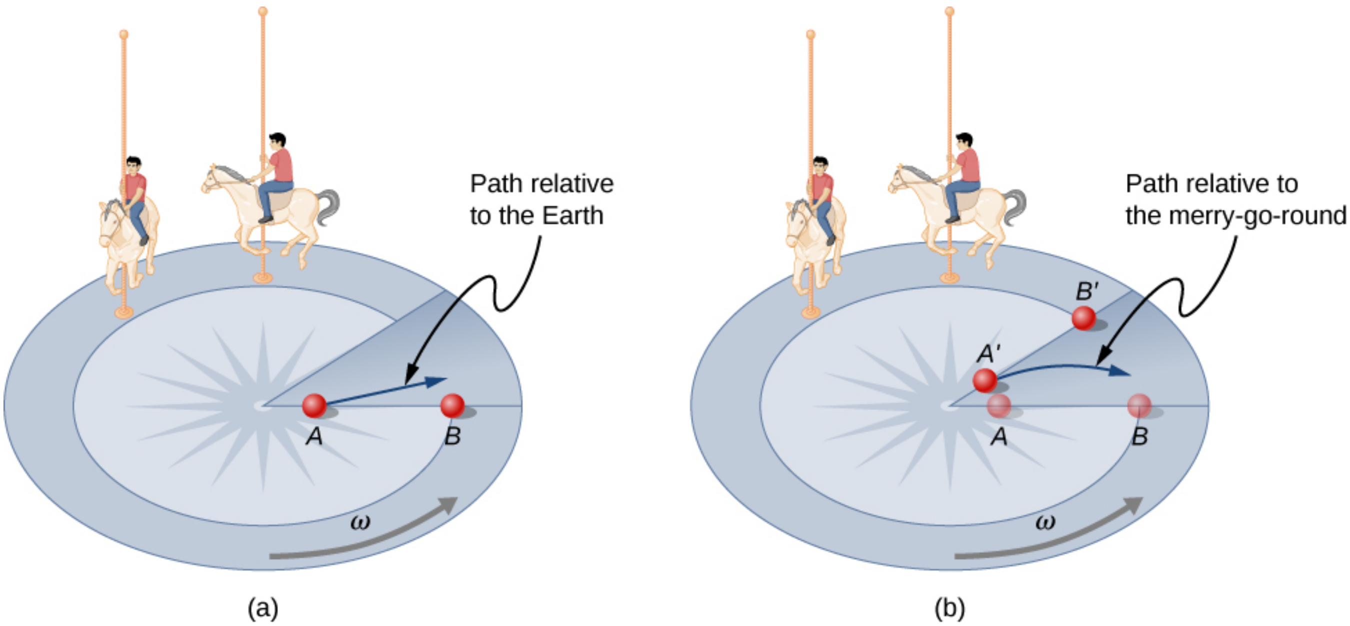


Figure 4.18 Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point *B*, starting at point *A*. Both points rotate to the shaded positions (*A*' and *B*') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.

Up until now, we have considered Earth to be an inertial frame of reference with little or no worry about effects due to its rotation. Yet such effects *do* exist—in the rotation of weather systems, for example. Most consequences of Earth's rotation can be qualitatively understood by analogy with the merry-go-round. Viewed from above the North Pole, Earth rotates counterclockwise, as does the merry-go-round in **Figure 4.18**. As on the merry-go-round, any motion in Earth's Northern Hemisphere experiences a Coriolis force to the right. Just the opposite occurs in the Southern Hemisphere; there, the force is to the left. Because Earth's angular velocity is small, the Coriolis force is usually negligible, but for large-scale motions, such as wind patterns, it has substantial effects.

The Coriolis force causes hurricanes in the Northern Hemisphere to rotate in the counterclockwise direction, whereas tropical cyclones in the Southern Hemisphere rotate in the clockwise direction. (The terms hurricane, typhoon, and tropical storm are regionally specific names for cyclones, which are storm systems characterized by low pressure centers, strong winds, and heavy rains.) **Figure 4.19** helps show how these rotations take place. Air flows toward any region of low pressure, and tropical cyclones contain particularly low pressures. Thus winds flow toward the center of a tropical cyclone or a low-pressure weather system at the surface. In the Northern Hemisphere, these inward winds are deflected to the right, as shown in the figure, producing a counterclockwise circulation at the surface for low-pressure zones of any type. Low pressure at the surface is associated with rising air, which also produces cooling and cloud formation, making low-pressure patterns quite visible from space. Conversely, wind circulation around high-pressure zones is clockwise in the Southern Hemisphere but is less visible because high pressure is associated with sinking air, producing clear skies.

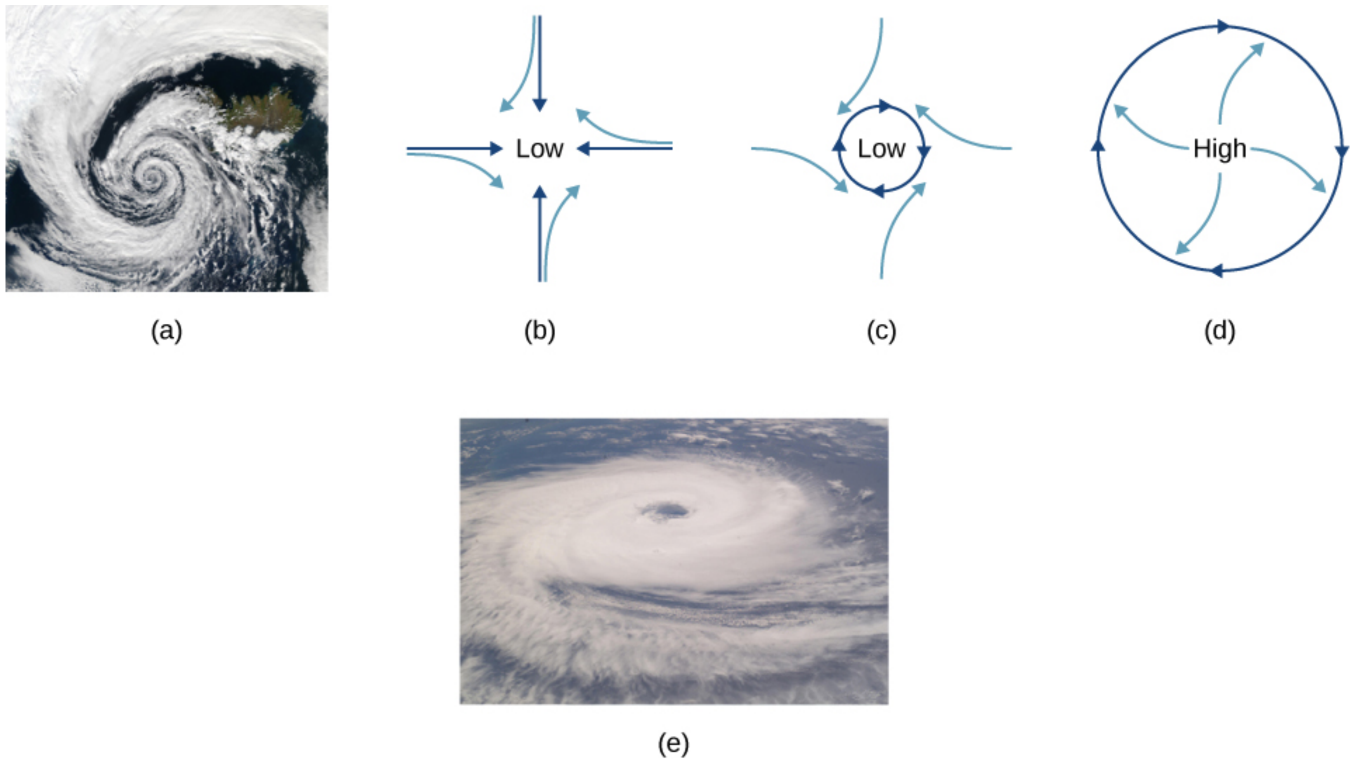


Figure 4.19 (a) The counterclockwise rotation of this Northern Hemisphere hurricane is a major consequence of the Coriolis force. (b) Without the Coriolis force, air would flow straight into a low-pressure zone, such as that found in tropical cyclones. (c) The Coriolis force deflects the winds to the right, producing a counterclockwise rotation. (d) Wind flowing away from a high-pressure zone is also deflected to the right, producing a clockwise rotation. (e) The opposite direction of rotation is produced by the Coriolis force in the Southern Hemisphere, leading to tropical cyclones. (credit a and credit e: modifications of work by NASA)

The rotation of tropical cyclones and the path of a ball on a merry-go-round can just as well be explained by inertia and the rotation of the system underneath. When noninertial frames are used, inertial forces, such as the Coriolis force, must be invented to explain the curved path. There is no identifiable physical source for these inertial forces. In an inertial frame, inertia explains the path, and no force is found to be without an identifiable source. Either view allows us to describe nature, but a view in an inertial frame is the simplest in the sense that all forces have origins and explanations.

Summary

- Centripetal force $\vec{\mathbf{F}}_\text{c}$ is a “center-seeking” force that always points toward the center of rotation. It is perpendicular to linear velocity and has the magnitude

$$F_\text{c} = ma_\text{c}. \tag{4.41}$$

- Rotating and accelerated frames of reference are noninertial. Inertial forces, such as the Coriolis force, are needed to explain motion in such frames.

Conceptual Questions

Exercise 4.2

If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.

Exercise 4.3

Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?

Solution

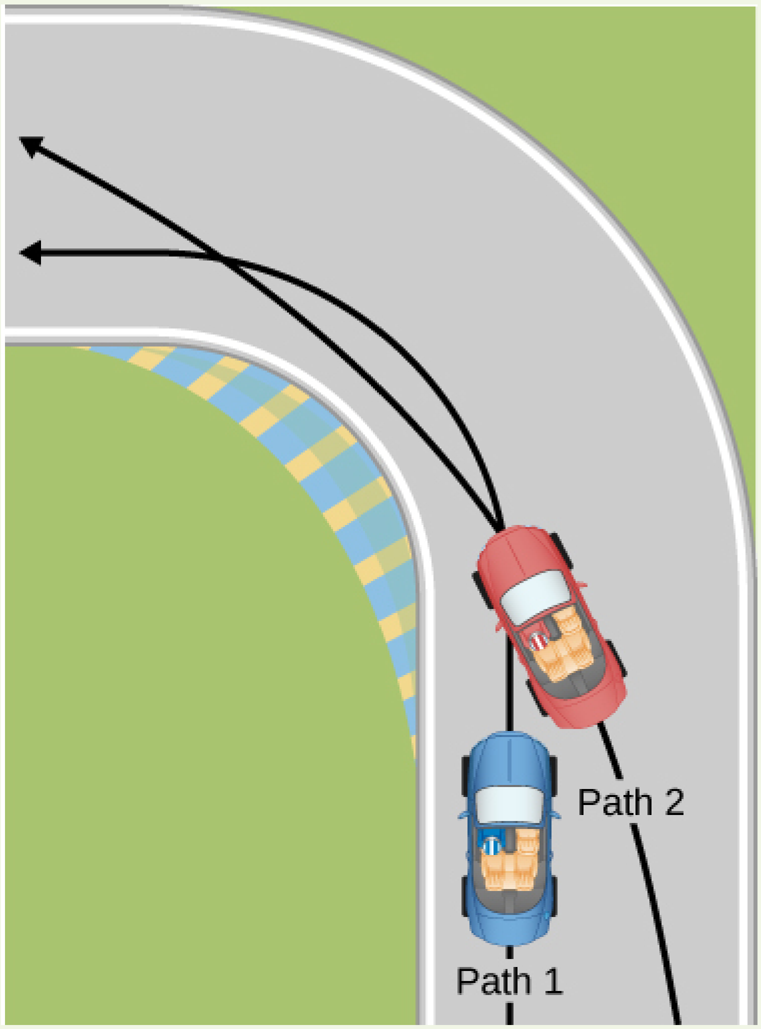
Centripetal force is defined as any net force causing uniform circular motion. The centripetal force is not a new kind of force. The label “centripetal” refers to *any* force that keeps something turning in a circle. That force could be tension, gravity, friction, electrical attraction, the normal force, or any other force. Any combination of these could be the source of centripetal force, for example, the centripetal force at the top of the path of a tetherball swung through a vertical circle is the result of both tension and gravity.

Exercise 4.4

If centripetal force is directed toward the center, why do you feel that you are ‘thrown’ away from the center as a car goes around a curve? Explain.

Exercise 4.5

Race car drivers routinely cut corners, as shown below (Path 2). Explain how this allows the curve to be taken at the greatest speed.



Solution

The driver who cuts the corner (on Path 2) has a more gradual curve, with a larger radius. That one will be the better racing line. If the driver goes too fast around a corner using a racing line, he will still slide off the track; the key is to stay at the maximum value of static friction. So, the driver wants maximum possible speed and maximum friction. Consider the equation for centripetal force: $F_c = m\frac{v^2}{r}$ where *v* is speed and *r* is the radius of curvature. So by decreasing the curvature (1/*r*) of the path that the car takes, we reduce the amount of force the tires have to exert on the road, meaning we can now increase the speed, *v*. Looking at this from the point of view of the driver on Path 1, we can reason this way: the sharper the turn, the smaller the turning circle; the smaller the turning circle, the larger is the required centripetal force. If this centripetal force is not exerted, the result is a skid.

Exercise 4.6

Many amusement parks have rides that make vertical loops like the one shown below. For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:

(a) The car goes over the top at faster than this speed?

(b) The car goes over the top at slower than this speed?

Exercise 4.7

What causes water to be removed from clothes in a spin-dryer?

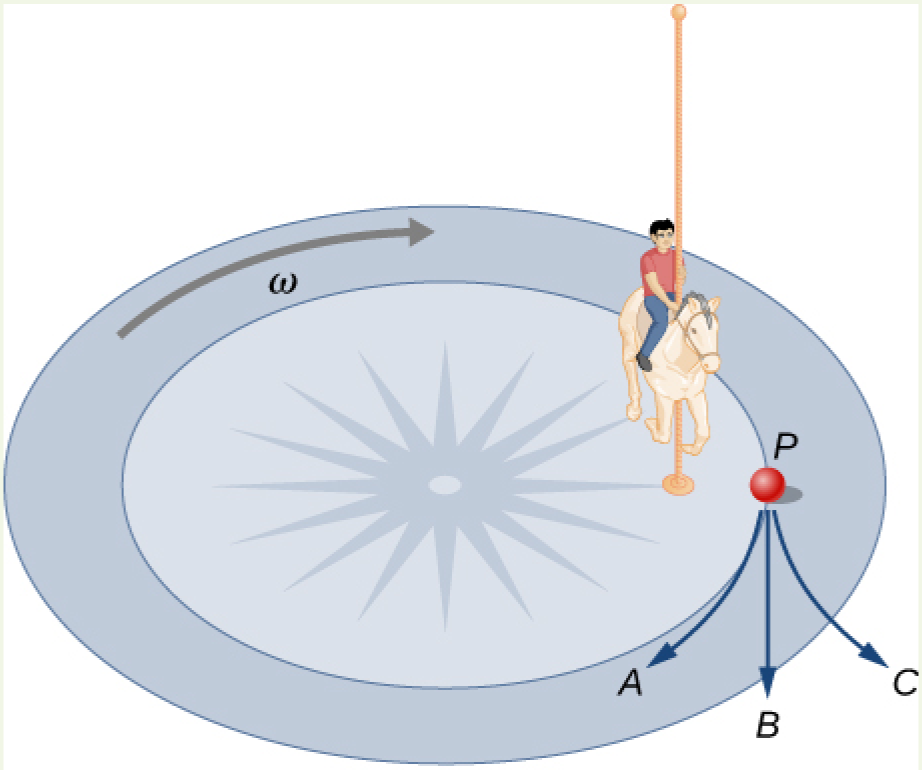
Solution
The barrel of the dryer provides a centripetal force on the clothes (including the water droplets) to keep them moving in a circular path. As a water droplet comes to one of the holes in the barrel, it will move in a path tangent to the circle.

Exercise 4.8

As a skater forms a circle, what force is responsible for making his turn? Use a free-body diagram in your answer.

Exercise 4.9

Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown below will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.



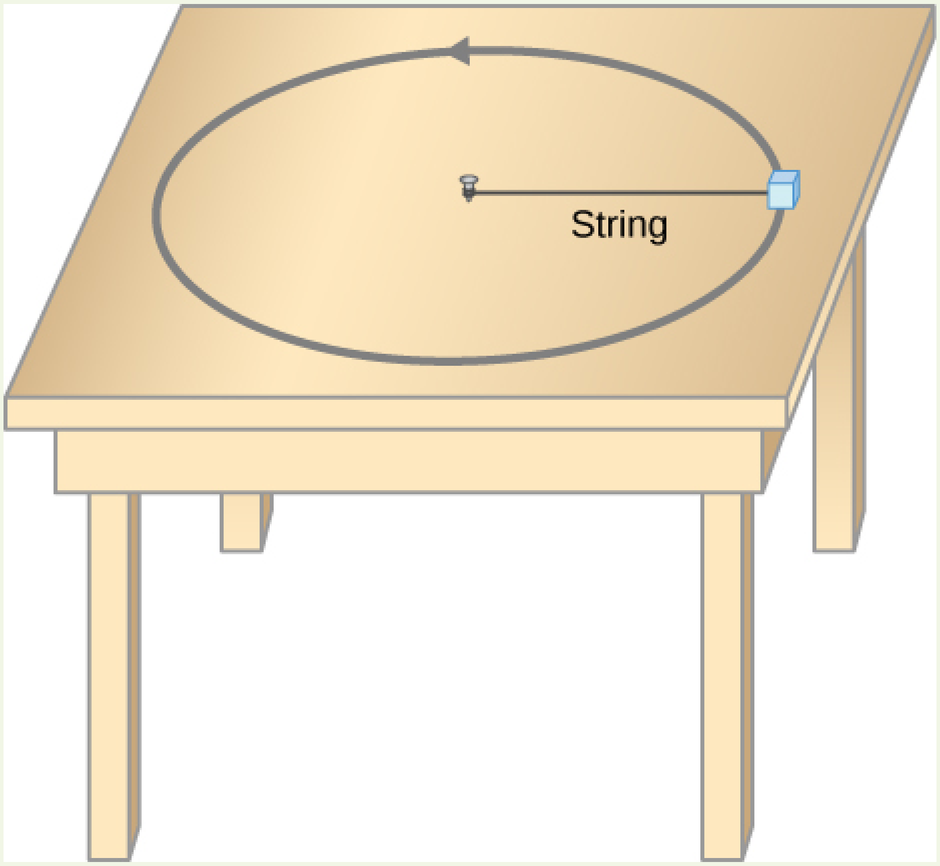
Solution
If there is no friction, then there is no centripetal force. This means that the lunch box will move along a path tangent to the circle, and thus follows path *B*. The dust trail will be straight. This is a result of Newton's first law of motion.

Exercise 4.10

Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car's speed? What is the direction of the force exerted on you by the car seat?

Exercise 4.11

Suppose a mass is moving in a circular path on a frictionless table as shown below. In Earth's frame of reference, there is no centrifugal force pulling the mass away from the center of rotation, yet there is a force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton's third law, explain what force stretches the string, identifying its physical origin.



Solution
There must be a centripetal force to maintain the circular motion; this is provided by the nail at the center. Newton's third law explains the phenomenon. The action force is the force of the string on the mass; the reaction force is the force of the mass on the string. This reaction force causes the string to stretch.

Exercise 4.12

When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the Northern Hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?

Exercise 4.13

A car rounds a curve and encounters a patch of ice with a very low coefficient of kinetic fiction. The car slides off the road. Describe the path of the car as it leaves the road.

Solution
Since the radial friction with the tires supplies the centripetal force, and friction is nearly 0 when the car encounters the ice, the car will obey Newton's first law and go off the road in a straight line path, tangent to the curve. A common misconception is that the car will follow a curved path off the road.

Exercise 4.14

In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is an inertial force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all forces acting on them.

Exercise 4.15

Two friends are having a conversation. Anna says a satellite in orbit is in free fall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in free fall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?

Solution
Anna is correct. The satellite is freely falling toward Earth due to gravity, even though gravity is weaker at the altitude of the satellite, and *g* is not 9.80 m/s^2. Free fall does not depend on the value of *g*; that is, you could experience free fall on Mars if you jumped off Olympus Mons (the tallest volcano in the solar system).

Exercise 4.16

A nonrotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

Problems

Exercise 4.17

(a) A 22.0-kg child is riding a playground merry-go-round that is rotating at 40.0 rev/min. What centripetal force is exerted if he is 1.25 m from its center? (b) What centripetal force is exerted if the merry-go-round rotates at 3.00 rev/min and he is 8.00 m from its center? (c) Compare each force with his weight.

Solution
a. 483 N; b. 17.4 N; c. 2.24, 0.0807

Exercise 4.18

Calculate the centripetal force on the end of a 100-m (radius) wind turbine blade that is rotating at 0.5 rev/s. Assume the mass is 4 kg.

Exercise 4.19

What is the ideal banking angle for a gentle turn of 1.20-km radius on a highway with a 105 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit?

Solution
4.14°

Exercise 4.20

What is the ideal speed to take a 100.0-m-radius curve banked at a 20.0° angle?

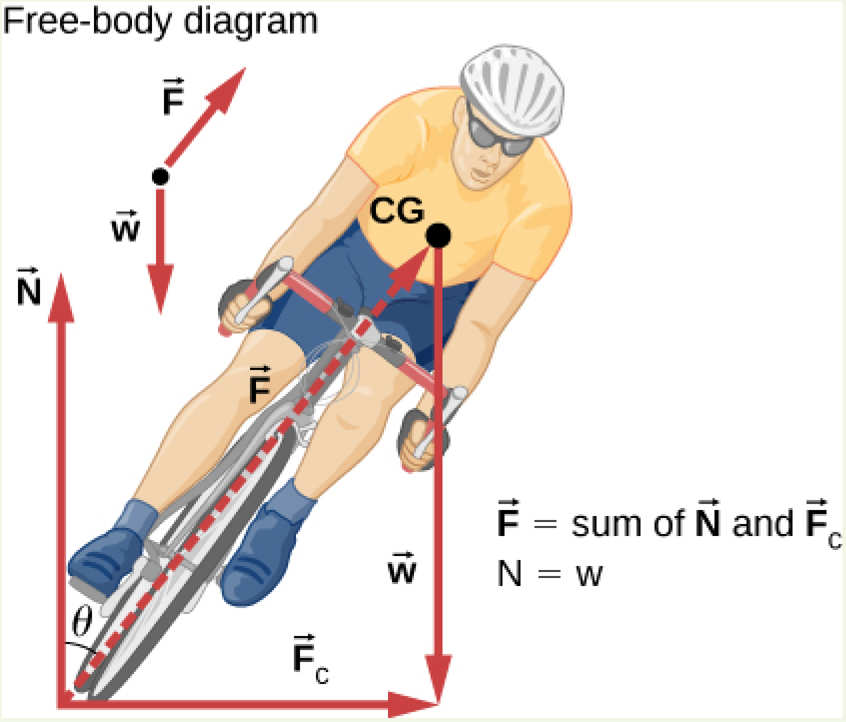
Exercise 4.21

(a) What is the radius of a bobsled turn banked at 75.0° and taken at 30.0 m/s, assuming it is ideally banked? (b) Calculate the centripetal acceleration. (c) Does this acceleration seem large to you?

Solution
a. 24.6 m; b. 36.6 m/s^2; c. 3.73 times *g*

Exercise 4.22

Part of riding a bicycle involves leaning at the correct angle when making a turn, as seen below. To be stable, the force exerted by the ground must be on a line going through the center of gravity. The force on the bicycle wheel can be resolved into two perpendicular components—friction parallel to the road (this must supply the centripetal force) and the vertical normal force (which must equal the system's weight). (a) Show that θ (as defined as shown) is related to the speed *v* and radius of curvature *r* of the turn in the same way as for an ideally banked roadway—that is, $\theta = \tan^{-1}(v^2/rg)$. (b) Calculate θ for a 12.0-m/s turn of radius 30.0 m (as in a race).



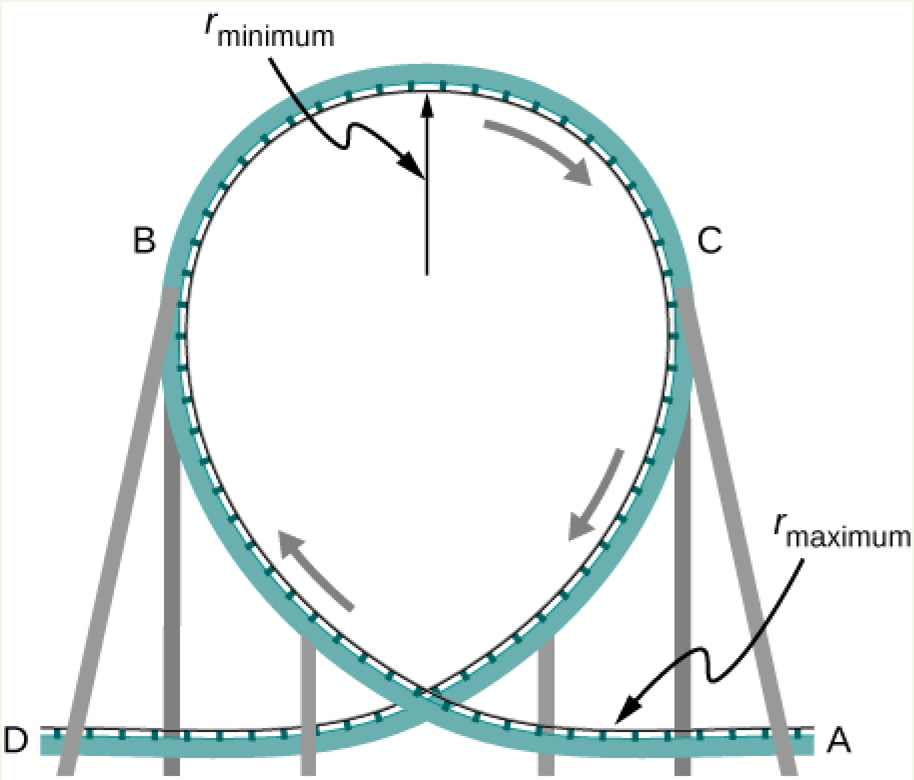
Exercise 4.23

If a car takes a banked curve at less than the ideal speed, friction is needed to keep it from sliding toward the inside of the curve (a problem on icy mountain roads). (a) Calculate the ideal speed to take a 100.0 m radius curve banked at 15.0°. (b) What is the minimum coefficient of friction needed for a frightened driver to take the same curve at 20.0 km/h?

Solution
a. 16.2 m/s; b. 0.234

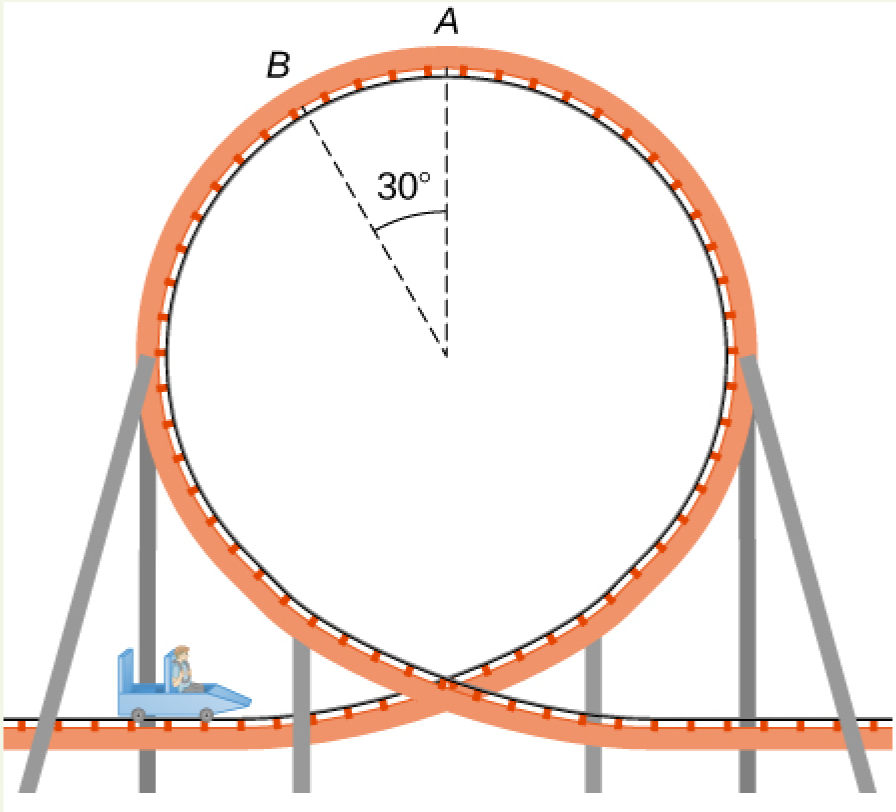
Exercise 4.24

Modern roller coasters have vertical loops like the one shown here. The radius of curvature is smaller at the top than on the sides so that the downward centripetal acceleration at the top will be greater than the acceleration due to gravity, keeping the passengers pressed firmly into their seats. (a) What is the speed of the roller coaster at the top of the loop if the radius of curvature there is 15.0 m and the downward acceleration of the car is 1.50 *g*? (b) How high above the top of the loop must the roller coaster start from rest, assuming negligible friction? (c) If it actually starts 5.00 m higher than your answer to (b), how much energy did it lose to friction? Its mass is 1.50×10^3 kg.



Exercise 4.25

A child of mass 40.0 kg is in a roller coaster car that travels in a loop of radius 7.00 m. At point A the speed of the car is 10.0 m/s, and at point B, the speed is 10.5 m/s. Assume the child is not holding on and does not wear a seat belt. (a) What is the force of the car seat on the child at point A? (b) What is the force of the car seat on the child at point B? (c) What minimum speed is required to keep the child in his seat at point A?



Solution
a. 179 N; b. 290 N; c. 8.3 m/s

Exercise 4.26

In the simple Bohr model of the ground state of the hydrogen atom, the electron travels in a circular orbit around a fixed proton. The radius of the orbit is 5.28×10^{-11} m, and the speed of the electron is 2.18×10^{6} m/s. The mass of an electron is 9.11×10^{-31} kg . What is the force on the electron?

Exercise 4.27

Railroad tracks follow a circular curve of radius 500.0 m and are banked at an angle of $5.0°$. For trains of what speed are these tracks designed?

Solution
20.7 m/s

Exercise 4.28

The CERN particle accelerator is circular with a circumference of 7.0 km. (a) What is the acceleration of the protons $(m = 1.67 \times 10^{-27}\ \text{kg})$ that move around the accelerator at 5% of the speed of light? (The speed of light is $v = 3.00 \times 10^{8}$ m/s.) (b) What is the force on the protons?

Exercise 4.29

A car rounds an unbanked curve of radius 65 m. If the coefficient of static friction between the road and car is 0.70, what is the maximum speed at which the car traverse the curve without slipping?

Solution
21 m/s

Exercise 4.30

A banked highway is designed for traffic moving at 90.0 km/h. The radius of the curve is 310 m. What is the angle of banking of the highway?

4.4 Fictitious Forces and Non-inertial Frames: The Coriolis Force

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits fictitious forces—unreal forces that arise from motion and may *seem* real, because the observer's frame of reference is accelerating or rotating.

When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that *you* tend to remain stationary while the *seat* pushes forward on you, and there is no real force backward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right. You feel as if you are thrown (that is, *forced*) toward the left relative to the car. Again, a physicist would say that *you* are going in a straight line but the *car* moves to the right, and there is no real force on you to the left. Recall Newton's first law.

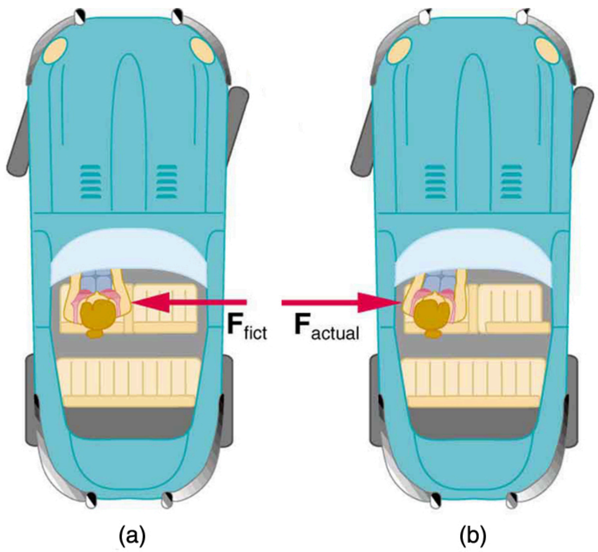


Figure 4.20 (a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is a fictitious force arising from the use of the car as a frame of reference. (b) In the Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no real force to the left on the driver relative to Earth. There is a real force to the right on the car to make it turn.

We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, while a physicist uses Earth. The physicist chooses Earth because it is very nearly an inertial frame of reference—one in which all forces are real (that is, in which all forces have an identifiable physical origin). In such a frame of reference, Newton's laws of motion take the form given in **Dynamics: Newton's Laws of Motion (https://legacy.cnx.org/content/m42129/latest/)** The car is a **non-inertial frame of reference** because it is accelerated to the side. The force to the left sensed by car passengers is a **fictitious force** having no physical origin. There is nothing real pushing them left—the car, as well as the driver, is actually accelerating to the right.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round. You take the merry-go-round to be your frame of reference because you rotate together. In that non-inertial frame, you feel a fictitious force, named **centrifugal force (**not to be confused with centripetal force**)**, trying to throw you off. You must hang on tightly to counteract the centrifugal force. In Earth's frame of reference, there is no force trying to throw you off. Rather you must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round.

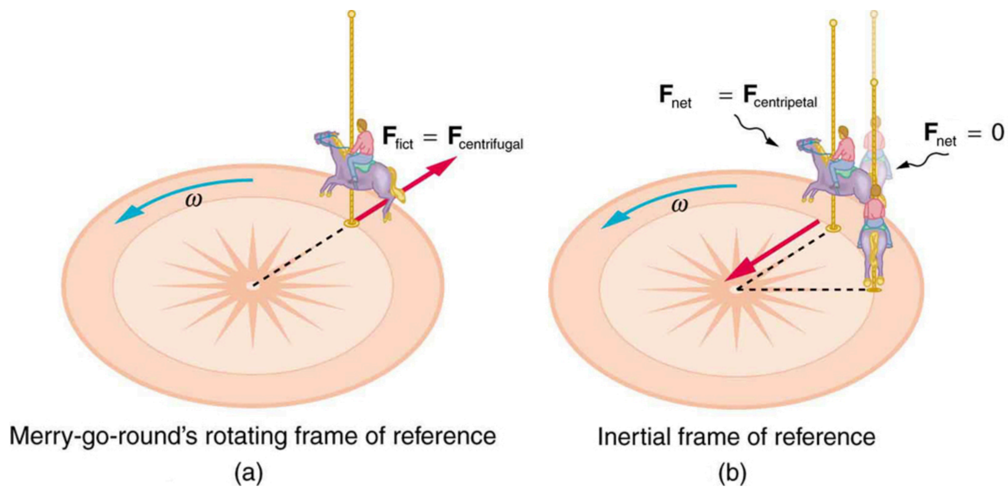


Figure 4.21 (a) A rider on a merry-go-round feels as if he is being thrown off. This fictitious force is called the centrifugal force—it explains the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off and not a real force (the unshaded rider has $F_{\text{net}} = 0$ and heads in a straight line). A real force, $F_{\text{centripetal}}$, is needed to cause a circular path.

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (see **Figure 4.22**). A centrifuge spins a sample very rapidly, as mentioned earlier in this chapter. Viewed from the rotating frame of reference, the fictitious centrifugal force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.

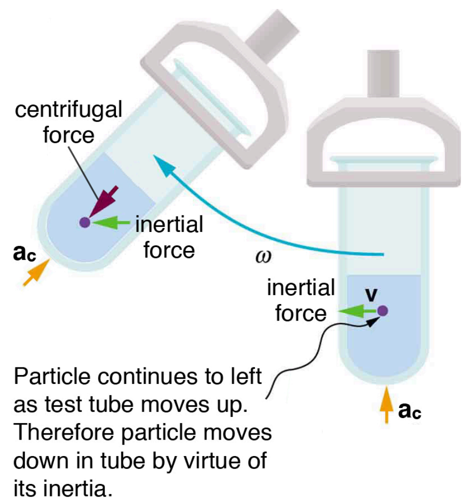


Figure 4.22 Centrifuges use inertia to perform their task. Particles in the fluid sediment come out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles will come into contact with the test tube walls, which will then supply the centripetal force needed to make them move in a circle of constant radius.

Let us now consider what happens if something moves in a frame of reference that rotates. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in **Figure 4.23**? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using a fictitious force, called the **Coriolis force**, that causes the ball to curve to the right. The fictitious Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's Laws in non-inertial frames of reference.

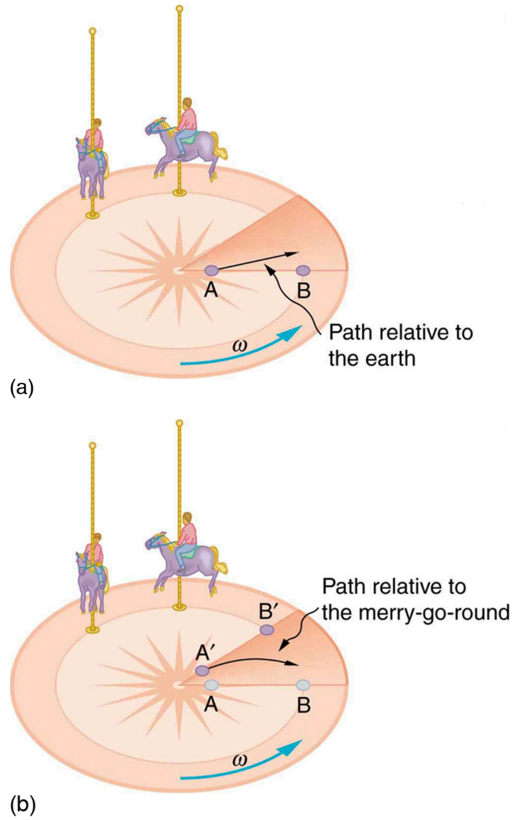


Figure 4.23 Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A' and B') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.

Up until now, we have considered Earth to be an inertial frame of reference with little or no worry about effects due to its rotation. Yet such effects *do* exist—in the rotation of weather systems, for example. Most consequences of Earth's rotation can be qualitatively understood by analogy with the merry-go-round. Viewed from above the North Pole, Earth rotates counterclockwise, as does the merry-go-round in **Figure 4.23**. As on the merry-go-round, any motion in Earth's northern hemisphere experiences a Coriolis force to the right. Just the opposite occurs in the southern hemisphere; there, the force is to the left. Because Earth's angular velocity is small, the Coriolis force is usually negligible, but for large-scale motions, such as wind patterns, it has substantial effects.

The Coriolis force causes hurricanes in the northern hemisphere to rotate in the counterclockwise direction, while the tropical cyclones (what hurricanes are called below the equator) in the southern hemisphere rotate in the clockwise direction. The terms hurricane, typhoon, and tropical storm are regionally-specific names for tropical cyclones, storm systems characterized by low pressure centers, strong winds, and heavy rains. **Figure 4.24** helps show how these rotations take place. Air flows toward any region of low pressure, and tropical cyclones contain particularly low pressures. Thus winds flow toward the center of a tropical cyclone or a low-pressure weather system at the surface. In the northern hemisphere, these inward winds are deflected to the right, as shown in the figure, producing a counterclockwise circulation at the surface for low-pressure zones of any type. Low pressure at the surface is associated with rising air, which also produces cooling and cloud formation, making low-pressure patterns quite visible from space. Conversely, wind circulation around high-pressure zones is clockwise in the northern hemisphere but is less visible because high pressure is associated with sinking air, producing clear skies.

The rotation of tropical cyclones and the path of a ball on a merry-go-round can just as well be explained by inertia and the rotation of the system underneath. When non-inertial frames are used, fictitious forces, such as the Coriolis force, must be invented to explain the curved path. There is no identifiable physical source for these fictitious forces. In an inertial frame, inertia explains the path, and no force is found to be without an identifiable source. Either view allows us to describe nature, but a view in an inertial frame is the simplest and truest, in the sense that all forces have real origins and explanations.

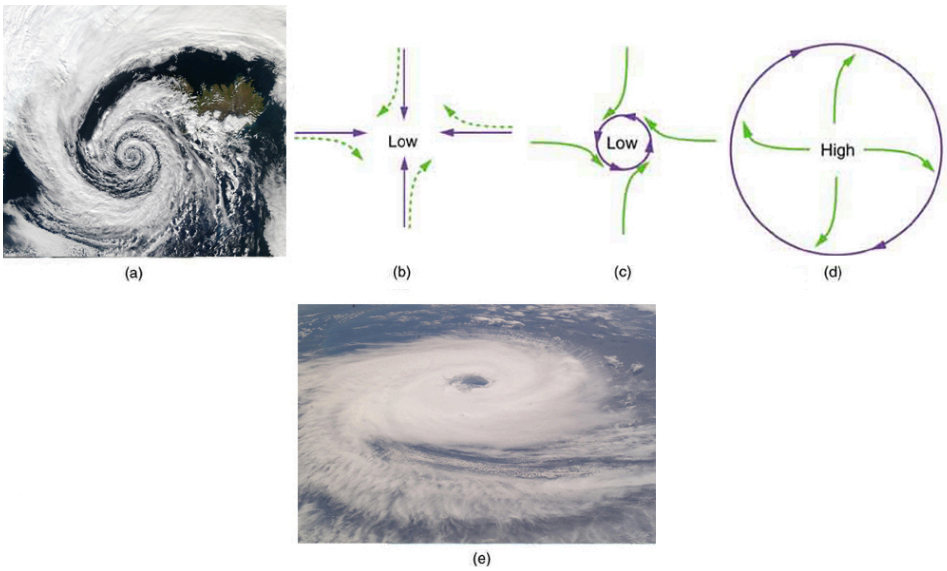


Figure 4.24 (a) The counterclockwise rotation of this northern hemisphere hurricane is a major consequence of the Coriolis force. (credit: NASA) (b) Without the Coriolis force, air would flow straight into a low-pressure zone, such as that found in tropical cyclones. (c) The Coriolis force deflects the winds to the right, producing a counterclockwise rotation. (d) Wind flowing away from a high-pressure zone is also deflected to the right, producing a clockwise rotation. (e) The opposite direction of rotation is produced by the Coriolis force in the southern hemisphere, leading to tropical cyclones. (credit: NASA)

4.5 Newton's Universal Law of Gravitation

What do aching feet, a falling apple, and the orbit of the Moon have in common? Each is caused by the gravitational force. Our feet are strained by supporting our weight—the force of Earth's gravity on us. An apple falls from a tree because of the same force acting a few meters above Earth's surface. And the Moon orbits Earth because gravity is able to supply the necessary centripetal force at a distance of hundreds of millions of meters. In fact, the same force causes planets to orbit the Sun, stars to orbit the center of the galaxy, and galaxies to cluster together. Gravity is another example of underlying simplicity in nature. It is the weakest of the four basic forces found in nature, and in some ways the least understood. It is a force that acts at a distance, without physical contact, and is expressed by a formula that is valid everywhere in the universe, for masses and distances that vary from the tiny to the immense.

Sir Isaac Newton was the first scientist to precisely define the gravitational force, and to show that it could explain both falling bodies and astronomical motions. See **Figure 4.25**. But Newton was not the first to suspect that the same force caused both our weight and the motion of planets. His forerunner Galileo Galilei had contended that falling bodies and planetary motions had the same cause. Some of Newton's contemporaries, such as Robert Hooke, Christopher Wren, and Edmund Halley, had also made some progress toward understanding gravitation. But Newton was the first to propose an exact mathematical form and to use that form to show that the motion of heavenly bodies should be conic sections—circles, ellipses, parabolas, and hyperbolas. This theoretical prediction was a major triumph—it had been known for some time that moons, planets, and comets follow such paths, but no one had been able to propose a mechanism that caused them to follow these paths and not others.

Figure 4.25 According to early accounts, Newton was inspired to make the connection between falling bodies and astronomical motions when he saw an apple fall from a tree and realized that if the gravitational force could extend above the ground to a tree, it might also reach the Sun. The inspiration of Newton's apple is a part of worldwide folklore and may even be based in fact. Great importance is attached to it because Newton's universal law of gravitation and his laws of motion answered very old questions about nature and gave tremendous support to the notion of underlying simplicity and unity in nature. Scientists still expect underlying simplicity to emerge from their ongoing inquiries into nature.

The gravitational force is relatively simple. It is always attractive, and it depends only on the masses involved and the distance between them. Stated in modern language, **Newton's universal law of gravitation** states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.

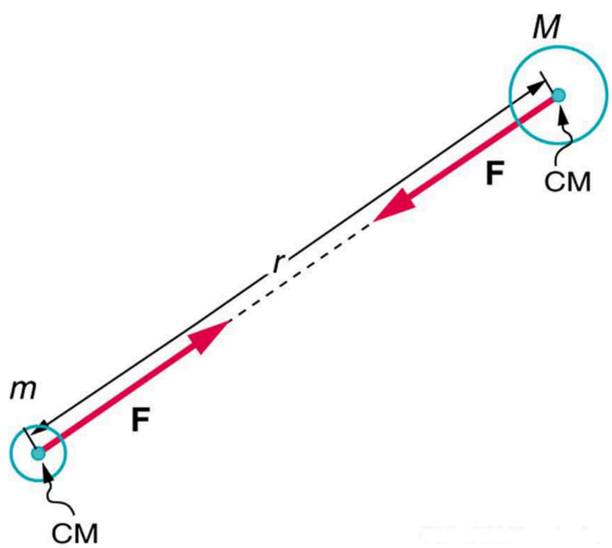


Figure 4.26 Gravitational attraction is along a line joining the centers of mass of these two bodies. The magnitude of the force is the same on each, consistent with Newton's third law.

Misconception Alert

The magnitude of the force on each object (one has larger mass than the other) is the same, consistent with Newton's third law.

The bodies we are dealing with tend to be large. To simplify the situation we assume that the body acts as if its entire mass is concentrated at one specific point called the **center of mass** (CM), which will be further explored in **Linear Momentum and**

Collisions (https://legacy.cnx.org/content/m42155/latest/) . For two bodies having masses m and M with a distance r between their centers of mass, the equation for Newton's universal law of gravitation is

$$F = G\frac{mM}{r^2}, \tag{4.42}$$

where F is the magnitude of the gravitational force and G is a proportionality factor called the **gravitational constant**. G is a universal gravitational constant—that is, it is thought to be the same everywhere in the universe. It has been measured experimentally to be

$$G = 6.674\times10^{-11}\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2} \tag{4.43}$$

in SI units. Note that the units of G are such that a force in newtons is obtained from $F = G\frac{mM}{r^2}$, when considering masses in kilograms and distance in meters. For example, two 1.000 kg masses separated by 1.000 m will experience a gravitational attraction of $6.674\times10^{-11}\text{ N}$. This is an extraordinarily small force. The small magnitude of the gravitational force is consistent with everyday experience. We are unaware that even large objects like mountains exert gravitational forces on us. In fact, our body weight is the force of attraction of the *entire Earth* on us with a mass of $6\times10^{24}\text{ kg}$.

Recall that the acceleration due to gravity g is about 9.80 m/s^2 on Earth. We can now determine why this is so. The weight of an object *mg* is the gravitational force between it and Earth. Substituting *mg* for F in Newton's universal law of gravitation gives

$$mg = G\frac{mM}{r^2}, \tag{4.44}$$

where m is the mass of the object, M is the mass of Earth, and r is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See **Figure 4.27**. The mass m of the object cancels, leaving an equation for g :

$$g = G\frac{M}{r^2}. \tag{4.45}$$

Substituting known values for Earth's mass and radius (to three significant figures),

$$g = \left(6.67\times10^{-11}\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\times\frac{5.98\times10^{24}\text{ kg}}{(6.38\times10^6\text{ m})^2}, \tag{4.46}$$

and we obtain a value for the acceleration of a falling body:

$$g = 9.80\text{ m/s}^2. \tag{4.47}$$

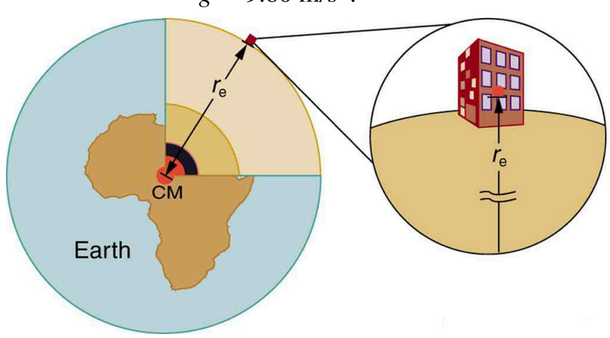


Figure 4.27 The distance between the centers of mass of Earth and an object on its surface is very nearly the same as the radius of Earth, because Earth is so much larger than the object.

This is the expected value *and is independent of the body's mass*. Newton's law of gravitation takes Galileo's observation that all masses fall with the same acceleration a step further, explaining the observation in terms of a force that causes objects to fall—in fact, in terms of a universally existing force of attraction between masses.

Take-Home Experiment

Take a marble, a ball, and a spoon and drop them from the same height. Do they hit the floor at the same time? If you drop a piece of paper as well, does it behave like the other objects? Explain your observations.

Making Connections

Attempts are still being made to understand the gravitational force. As we shall see in **Particle Physics (https://legacy.cnx.org/content/m42667/latest/)** , modern physics is exploring the connections of gravity to other forces, space, and time. General relativity alters our view of gravitation, leading us to think of gravitation as bending space and time.

In the following example, we make a comparison similar to one made by Newton himself. He noted that if the gravitational force caused the Moon to orbit Earth, then the acceleration due to gravity should equal the centripetal acceleration of the Moon in its orbit. Newton found that the two accelerations agreed “pretty nearly.”

Example 4.6 Earth’s Gravitational Force Is the Centripetal Force Making the Moon Move in a Curved Path

(a) Find the acceleration due to Earth’s gravity at the distance of the Moon.

(b) Calculate the centripetal acceleration needed to keep the Moon in its orbit (assuming a circular orbit about a fixed Earth), and compare it with the value of the acceleration due to Earth’s gravity that you have just found.

Strategy for (a)

This calculation is the same as the one finding the acceleration due to gravity at Earth’s surface, except that r is the distance from the center of Earth to the center of the Moon. The radius of the Moon’s nearly circular orbit is $3.84\times10^{8}\ \text{m}$.

Solution for (a)

Substituting known values into the expression for g found above, remembering that M is the mass of Earth not the Moon, yields

$$\begin{aligned} g &= G\frac{M}{r^2} = \left(6.67\times10^{-11}\frac{\text{N}\cdot\text{m}^2}{\text{kg}^2}\right)\times\frac{5.98\times10^{24}\ \text{kg}}{(3.84\times10^{8}\ \text{m})^2} \\ &= 2.70\times10^{-3}\ \text{m/s.}^2 \end{aligned} \tag{4.48}$$

Strategy for (b)

Centripetal acceleration can be calculated using either form of

$$\left.\begin{aligned} a_c &= \frac{v^2}{r} \\ a_c &= r\omega^2 \end{aligned}\right\}. \tag{4.49}$$

We choose to use the second form:

$$a_c = r\omega^2, \tag{4.50}$$

where ω is the angular velocity of the Moon about Earth.

Solution for (b)

Given that the period (the time it takes to make one complete rotation) of the Moon’s orbit is 27.3 days, (d) and using

$$1\ \text{d}\times24\frac{\text{hr}}{\text{d}}\times60\frac{\text{min}}{\text{hr}}\times60\frac{\text{s}}{\text{min}} = 86{,}400\ \text{s} \tag{4.51}$$

we see that

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi\ \text{rad}}{(27.3\ \text{d})(86{,}400\ \text{s/d})} = 2.66\times10^{-6}\frac{\text{rad}}{\text{s}}. \tag{4.52}$$

The centripetal acceleration is

$$\begin{aligned} a_c &= r\omega^2 = (3.84\times10^{8}\ \text{m})(2.66\times10^{-6}\ \text{rad/s})^2 \\ &= 2.72\times10^{-3}\ \text{m/s.}^2 \end{aligned} \tag{4.53}$$

The direction of the acceleration is toward the center of the Earth.

Discussion

The centripetal acceleration of the Moon found in (b) differs by less than 1% from the acceleration due to Earth’s gravity found in (a). This agreement is approximate because the Moon’s orbit is slightly elliptical, and Earth is not stationary (rather the Earth-Moon system rotates about its center of mass, which is located some 1700 km below Earth’s surface). The clear implication is that Earth’s gravitational force causes the Moon to orbit Earth.

Why does Earth not remain stationary as the Moon orbits it? This is because, as expected from Newton's third law, if Earth exerts a force on the Moon, then the Moon should exert an equal and opposite force on Earth (see **Figure 4.28**). We do not sense the Moon's effect on Earth's motion, because the Moon's gravity moves our bodies right along with Earth but there are other signs on Earth that clearly show the effect of the Moon's gravitational force as discussed in **Satellites and Kepler's Laws: An Argument for Simplicity**.

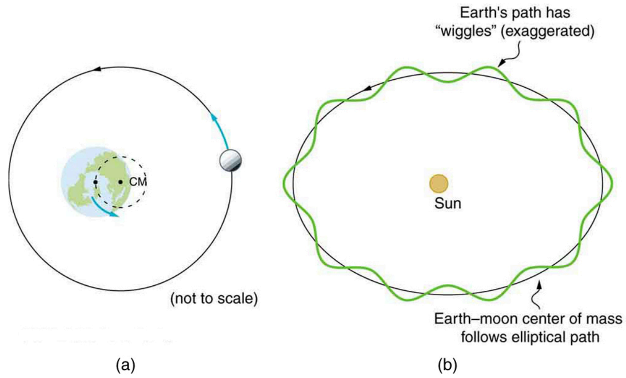


Figure 4.28 (a) Earth and the Moon rotate approximately once a month around their common center of mass. (b) Their center of mass orbits the Sun in an elliptical orbit, but Earth's path around the Sun has "wiggles" in it. Similar wiggles in the paths of stars have been observed and are considered direct evidence of planets orbiting those stars. This is important because the planets' reflected light is often too dim to be observed.

Tides

Ocean tides are one very observable result of the Moon's gravity acting on Earth. **Figure 4.29** is a simplified drawing of the Moon's position relative to the tides. Because water easily flows on Earth's surface, a high tide is created on the side of Earth nearest to the Moon, where the Moon's gravitational pull is strongest. Why is there also a high tide on the opposite side of Earth? The answer is that Earth is pulled toward the Moon more than the water on the far side, because Earth is closer to the Moon. So the water on the side of Earth closest to the Moon is pulled away from Earth, and Earth is pulled away from water on the far side. As Earth rotates, the tidal bulge (an effect of the tidal forces between an orbiting natural satellite and the primary planet that it orbits) keeps its orientation with the Moon. Thus there are two tides per day (the actual tidal period is about 12 hours and 25.2 minutes), because the Moon moves in its orbit each day as well).

Figure 4.29 The Moon causes ocean tides by attracting the water on the near side more than Earth, and by attracting Earth more than the water on the far side. The distances and sizes are not to scale. For this simplified representation of the Earth-Moon system, there are two high and two low tides per day at any location, because Earth rotates under the tidal bulge.

The Sun also affects tides, although it has about half the effect of the Moon. However, the largest tides, called spring tides, occur when Earth, the Moon, and the Sun are aligned. The smallest tides, called neap tides, occur when the Sun is at a 90° angle to the Earth-Moon alignment.

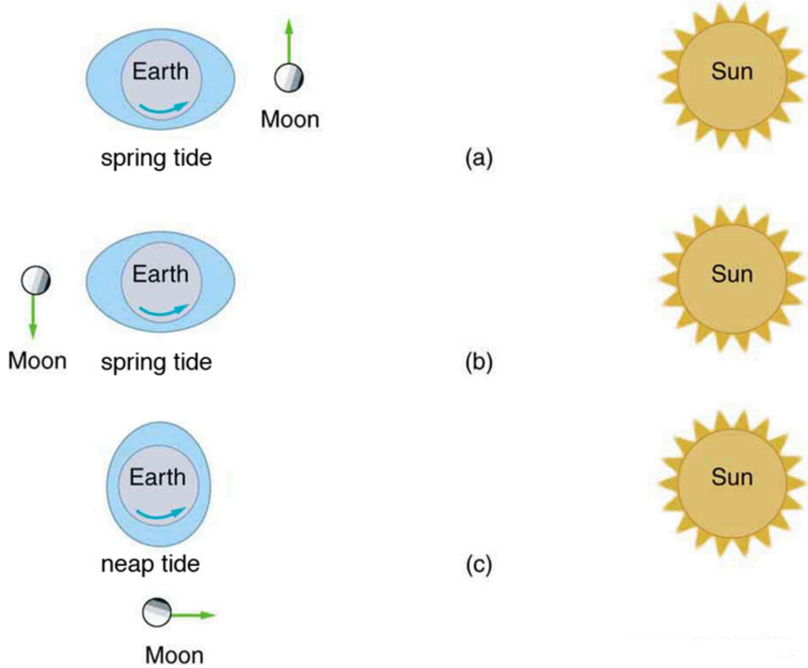


Figure 4.30 (a, b) Spring tides: The highest tides occur when Earth, the Moon, and the Sun are aligned. (c) Neap tide: The lowest tides occur when the Sun lies at 90° to the Earth-Moon alignment. Note that this figure is not drawn to scale.

Tides are not unique to Earth but occur in many astronomical systems. The most extreme tides occur where the gravitational force is the strongest and varies most rapidly, such as near black holes (see **Figure 4.31**). A few likely candidates for black holes have been observed in our galaxy. These have masses greater than the Sun but have diameters only a few kilometers across. The tidal forces near them are so great that they can actually tear matter from a companion star.

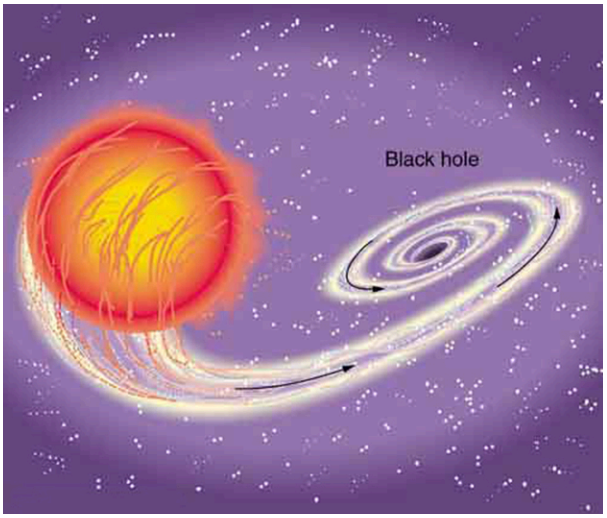


Figure 4.31 A black hole is an object with such strong gravity that not even light can escape it. This black hole was created by the supernova of one star in a two-star system. The tidal forces created by the black hole are so great that it tears matter from the companion star. This matter is compressed and heated as it is sucked into the black hole, creating light and X-rays observable from Earth.

"Weightlessness" and Microgravity

In contrast to the tremendous gravitational force near black holes is the apparent gravitational field experienced by astronauts orbiting Earth. What is the effect of "weightlessness" upon an astronaut who is in orbit for months? Or what about the effect of weightlessness upon plant growth? Weightlessness doesn't mean that an astronaut is not being acted upon by the gravitational force. There is no "zero gravity" in an astronaut's orbit. The term just means that the astronaut is in free-fall, accelerating with the acceleration due to gravity. If an elevator cable breaks, the passengers inside will be in free fall and will experience weightlessness. You can experience short periods of weightlessness in some rides in amusement parks.

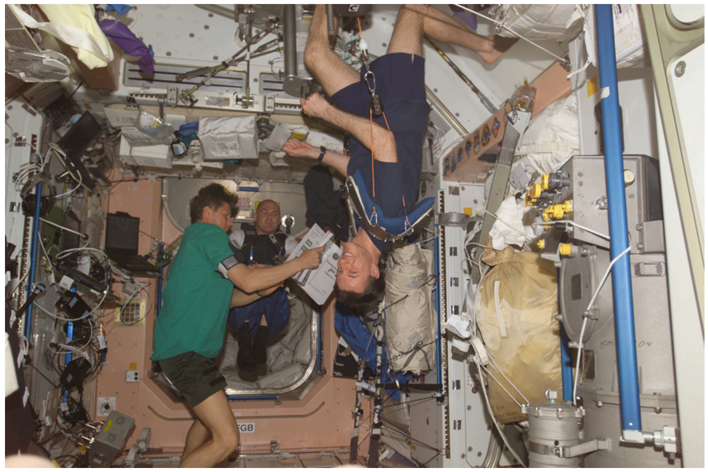


Figure 4.32 Astronauts experiencing weightlessness on board the International Space Station. (credit: NASA)

Microgravity refers to an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface. Many interesting biology and physics topics have been studied over the past three decades in the presence of microgravity. Of immediate concern is the effect on astronauts of extended times in outer space, such as at the International Space Station. Researchers have observed that muscles will atrophy (waste away) in this environment. There is also a corresponding loss of bone mass. Study continues on cardiovascular adaptation to space flight. On Earth, blood pressure is usually higher in the feet than in the head, because the higher column of blood exerts a downward force on it, due to gravity. When standing, 70% of your blood is below the level of the heart, while in a horizontal position, just the opposite occurs. What difference does the absence of this pressure differential have upon the heart?

Some findings in human physiology in space can be clinically important to the management of diseases back on Earth. On a somewhat negative note, spaceflight is known to affect the human immune system, possibly making the crew members more vulnerable to infectious diseases. Experiments flown in space also have shown that some bacteria grow faster in microgravity than they do on Earth. However, on a positive note, studies indicate that microbial antibiotic production can increase by a factor of two in space-grown cultures. One hopes to be able to understand these mechanisms so that similar successes can be achieved on the ground. In another area of physics space research, inorganic crystals and protein crystals have been grown in outer space that have much higher quality than any grown on Earth, so crystallography studies on their structure can yield much better results.

Plants have evolved with the stimulus of gravity and with gravity sensors. Roots grow downward and shoots grow upward. Plants might be able to provide a life support system for long duration space missions by regenerating the atmosphere, purifying water, and producing food. Some studies have indicated that plant growth and development are not affected by gravity, but there is still uncertainty about structural changes in plants grown in a microgravity environment.

The Cavendish Experiment: Then and Now

As previously noted, the universal gravitational constant G is determined experimentally. This definition was first done accurately by Henry Cavendish (1731–1810), an English scientist, in 1798, more than 100 years after Newton published his universal law of gravitation. The measurement of G is very basic and important because it determines the strength of one of the four forces in nature. Cavendish's experiment was very difficult because he measured the tiny gravitational attraction between two ordinary-sized masses (tens of kilograms at most), using apparatus like that in Figure 4.33. Remarkably, his value for G differs by less than 1% from the best modern value.

One important consequence of knowing G was that an accurate value for Earth's mass could finally be obtained. This was done by measuring the acceleration due to gravity as accurately as possible and then calculating the mass of Earth M from the relationship Newton's universal law of gravitation gives

$$mg = G\frac{mM}{r^2}, \tag{4.54}$$

where m is the mass of the object, M is the mass of Earth, and r is the distance to the center of Earth (the distance between the centers of mass of the object and Earth). See Figure 4.26. The mass m of the object cancels, leaving an equation for g :

$$g = G\frac{M}{r^2}. \tag{4.55}$$

Rearranging to solve for M yields

$$M = \frac{gr^2}{G}. \tag{4.56}$$

So M can be calculated because all quantities on the right, including the radius of Earth r, are known from direct measurements. We shall see in **Satellites and Kepler's Laws: An Argument for Simplicity** that knowing G also allows for the determination of astronomical masses. Interestingly, of all the fundamental constants in physics, G is by far the least well determined.

The Cavendish experiment is also used to explore other aspects of gravity. One of the most interesting questions is whether the gravitational force depends on substance as well as mass—for example, whether one kilogram of lead exerts the same gravitational pull as one kilogram of water. A Hungarian scientist named Roland von Eötvös pioneered this inquiry early in the 20th century. He found, with an accuracy of five parts per billion, that the gravitational force does not depend on the substance. Such experiments continue today, and have improved upon Eötvös' measurements. Cavendish-type experiments such as those of Eric Adelberger and others at the University of Washington, have also put severe limits on the possibility of a fifth force and have verified a major prediction of general relativity—that gravitational energy contributes to rest mass. Ongoing measurements there use a torsion balance and a parallel plate (not spheres, as Cavendish used) to examine how Newton's law of gravitation works over sub-millimeter distances. On this small-scale, do gravitational effects depart from the inverse square law? So far, no deviation has been observed.

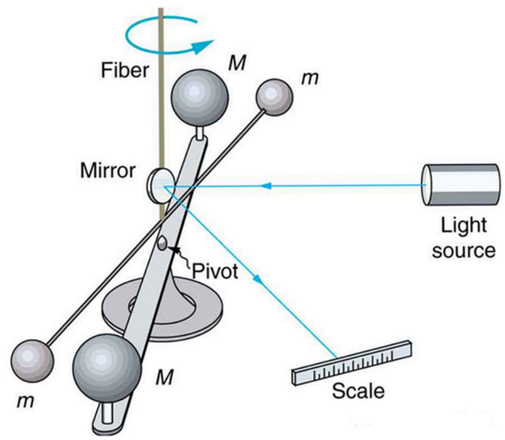


Figure 4.33 Cavendish used an apparatus like this to measure the gravitational attraction between the two suspended spheres (m) and the two on the stand (M) by observing the amount of torsion (twisting) created in the fiber. Distance between the masses can be varied to check the dependence of the force on distance. Modern experiments of this type continue to explore gravity.

4.6 Satellites and Kepler's Laws: An Argument for Simplicity

Examples of gravitational orbits abound. Hundreds of artificial satellites orbit Earth together with thousands of pieces of debris. The Moon's orbit about Earth has intrigued humans from time immemorial. The orbits of planets, asteroids, meteors, and comets about the Sun are no less interesting. If we look further, we see almost unimaginable numbers of stars, galaxies, and other celestial objects orbiting one another and interacting through gravity.

All these motions are governed by gravitational force, and it is possible to describe them to various degrees of precision. Precise descriptions of complex systems must be made with large computers. However, we can describe an important class of orbits without the use of computers, and we shall find it instructive to study them. These orbits have the following characteristics:

1. *A small mass m orbits a much larger mass M*. This allows us to view the motion as if M were stationary—in fact, as if from an inertial frame of reference placed on M —without significant error. Mass m is the satellite of M, if the orbit is gravitationally bound.
2. *The system is isolated from other masses*. This allows us to neglect any small effects due to outside masses.

The conditions are satisfied, to good approximation, by Earth's satellites (including the Moon), by objects orbiting the Sun, and by the satellites of other planets. Historically, planets were studied first, and there is a classical set of three laws, called Kepler's laws of planetary motion, that describe the orbits of all bodies satisfying the two previous conditions (not just planets in our solar system). These descriptive laws are named for the German astronomer Johannes Kepler (1571–1630), who devised them after careful study (over some 20 years) of a large amount of meticulously recorded observations of planetary motion done by Tycho Brahe (1546–1601). Such careful collection and detailed recording of methods and data are hallmarks of good science. Data constitute the evidence from which new interpretations and meanings can be constructed.

Kepler's Laws of Planetary Motion

Kepler's First Law

The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

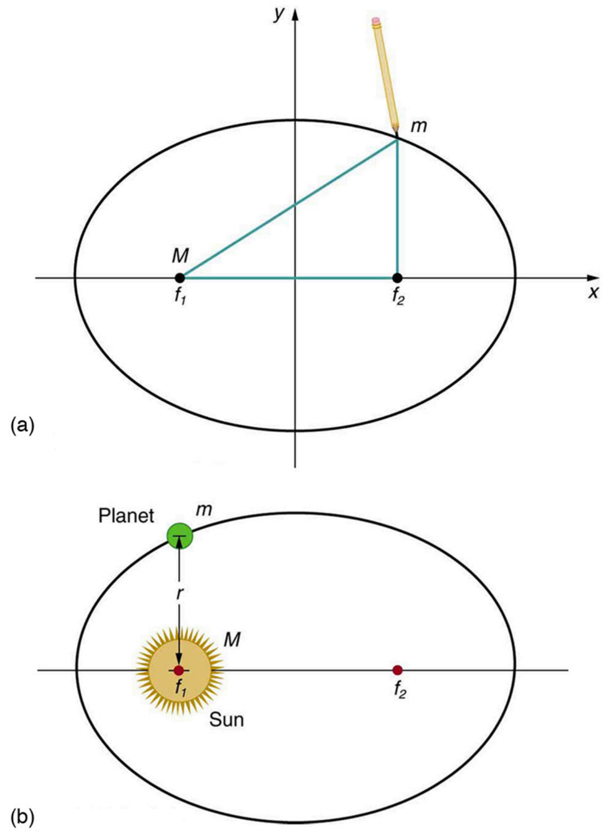


Figure 4.34 (a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci (f_1 and f_2) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the two foci coincide (thus any point on the circle is the same distance from the center). (b) For any closed gravitational orbit, m follows an elliptical path with M at one focus. Kepler's first law states this fact for planets orbiting the Sun.

Kepler's Second Law

Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times (see **Figure 4.35**).

Kepler's Third Law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun. In equation form, this is

$$\frac{T_1^{\,2}}{T_2^{\,2}} = \frac{r_1^{\,3}}{r_2^{\,3}}, \tag{4.57}$$

where T is the period (time for one orbit) and r is the average radius. This equation is valid only for comparing two small masses orbiting the same large one. Most importantly, this is a descriptive equation only, giving no information as to the cause of the equality.

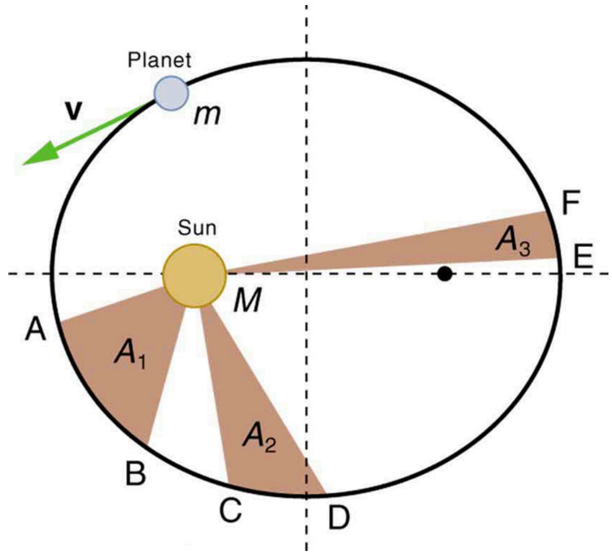


Figure 4.35 The shaded regions have equal areas. It takes equal times for m to go from A to B, from C to D, and from E to F. The mass m moves fastest when it is closest to M. Kepler's second law was originally devised for planets orbiting the Sun, but it has broader validity.

Note again that while, for historical reasons, Kepler's laws are stated for planets orbiting the Sun, they are actually valid for all bodies satisfying the two previously stated conditions.

Example 4.7 Find the Time for One Orbit of an Earth Satellite

Given that the Moon orbits Earth each 27.3 d and that it is an average distance of $3.84\times10^8\text{ m}$ from the center of Earth, calculate the period of an artificial satellite orbiting at an average altitude of 1500 km above Earth's surface.

Strategy

The period, or time for one orbit, is related to the radius of the orbit by Kepler's third law, given in mathematical form in $\frac{T_1^2}{T_2^2}=\frac{r_1^3}{r_2^3}$. Let us use the subscript 1 for the Moon and the subscript 2 for the satellite. We are asked to find T_2. The given information tells us that the orbital radius of the Moon is $r_1=3.84\times10^8\text{ m}$, and that the period of the Moon is $T_1=27.3\text{ d}$. The height of the artificial satellite above Earth's surface is given, and so we must add the radius of Earth (6380 km) to get $r_2=(1500+6380)\text{ km}=7880\text{ km}$. Now all quantities are known, and so T_2 can be found.

Solution

Kepler's third law is

$$\frac{T_1^2}{T_2^2}=\frac{r_1^3}{r_2^3}. \tag{4.58}$$

To solve for T_2, we cross-multiply and take the square root, yielding

$$T_2^2=T_1^2\left(\frac{r_2}{r_1}\right)^3 \tag{4.59}$$

$$T_2=T_1\left(\frac{r_2}{r_1}\right)^{3/2}. \tag{4.60}$$

Substituting known values yields

$$\begin{aligned} T_2 &= 27.3\text{ d}\times\frac{24.0\text{ h}}{\text{d}}\times\left(\frac{7880\text{ km}}{3.84\times10^5\text{ km}}\right)^{3/2} \\ &= 1.93\text{ h}. \end{aligned} \tag{4.61}$$

Discussion This is a reasonable period for a satellite in a fairly low orbit. It is interesting that any satellite at this altitude will orbit in the same amount of time. This fact is related to the condition that the satellite's mass is small compared with that of

Earth.

People immediately search for deeper meaning when broadly applicable laws, like Kepler's, are discovered. It was Newton who took the next giant step when he proposed the law of universal gravitation. While Kepler was able to discover *what* was happening, Newton discovered that gravitational force was the cause.

Derivation of Kepler's Third Law for Circular Orbits

We shall derive Kepler's third law, starting with Newton's laws of motion and his universal law of gravitation. The point is to demonstrate that the force of gravity is the cause for Kepler's laws (although we will only derive the third one).

Let us consider a circular orbit of a small mass m around a large mass M, satisfying the two conditions stated at the beginning of this section. Gravity supplies the centripetal force to mass m. Starting with Newton's second law applied to circular motion,

$$F_{\text{net}} = ma_{\text{c}} = m\frac{v^2}{r}. \tag{4.62}$$

The net external force on mass m is gravity, and so we substitute the force of gravity for F_{net}:

$$G\frac{mM}{r^2} = m\frac{v^2}{r}. \tag{4.63}$$

The mass m cancels, yielding

$$G\frac{M}{r} = v^2. \tag{4.64}$$

The fact that m cancels out is another aspect of the oft-noted fact that at a given location all masses fall with the same acceleration. Here we see that at a given orbital radius r, all masses orbit at the same speed. (This was implied by the result of the preceding worked example.) Now, to get at Kepler's third law, we must get the period T into the equation. By definition, period T is the time for one complete orbit. Now the average speed v is the circumference divided by the period—that is,

$$v = \frac{2\pi r}{T}. \tag{4.65}$$

Substituting this into the previous equation gives

$$G\frac{M}{r} = \frac{4\pi^2 r^2}{T^2}. \tag{4.66}$$

Solving for T^2 yields

$$T^2 = \frac{4\pi^2}{GM}r^3. \tag{4.67}$$

Using subscripts 1 and 2 to denote two different satellites, and taking the ratio of the last equation for satellite 1 to satellite 2 yields

$$\frac{T_1^{\,2}}{T_2^{\,2}} = \frac{r_1^{\,3}}{r_2^{\,3}}. \tag{4.68}$$

This is Kepler's third law. Note that Kepler's third law is valid only for comparing satellites of the same parent body, because only then does the mass of the parent body M cancel.

Now consider what we get if we solve $T^2 = \frac{4\pi^2}{GM}r^3$ for the ratio r^3 / T^2. We obtain a relationship that can be used to determine the mass M of a parent body from the orbits of its satellites:

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2}M. \tag{4.69}$$

If r and T are known for a satellite, then the mass M of the parent can be calculated. This principle has been used extensively to find the masses of heavenly bodies that have satellites. Furthermore, the ratio r^3 / T^2 should be a constant for all satellites of the same parent body (because $r^3 / T^2 = GM / 4\pi^2$). (See **Table 4.2**).

It is clear from **Table 4.2** that the ratio of r^3 / T^2 is constant, at least to the third digit, for all listed satellites of the Sun, and for those of Jupiter. Small variations in that ratio have two causes—uncertainties in the r and T data, and perturbations of the orbits due to other bodies. Interestingly, those perturbations can be—and have been—used to predict the location of new planets and moons. This is another verification of Newton's universal law of gravitation.

Making Connections

Newton's universal law of gravitation is modified by Einstein's general theory of relativity, as we shall see in **Particle Physics (https://legacy.cnx.org/content/m42667/latest/)** . Newton's gravity is not seriously in error—it was and still is an extremely good approximation for most situations. Einstein's modification is most noticeable in extremely large gravitational fields, such as near black holes. However, general relativity also explains such phenomena as small but long-known deviations of the orbit of the planet Mercury from classical predictions.

The Case for Simplicity

The development of the universal law of gravitation by Newton played a pivotal role in the history of ideas. While it is beyond the scope of this text to cover that history in any detail, we note some important points. The definition of planet set in 2006 by the International Astronomical Union (IAU) states that in the solar system, a planet is a celestial body that:

1. is in orbit around the Sun,
2. has sufficient mass to assume hydrostatic equilibrium and
3. has cleared the neighborhood around its orbit.

A non-satellite body fulfilling only the first two of the above criteria is classified as "dwarf planet."

In 2006, Pluto was demoted to a 'dwarf planet' after scientists revised their definition of what constitutes a "true" planet.

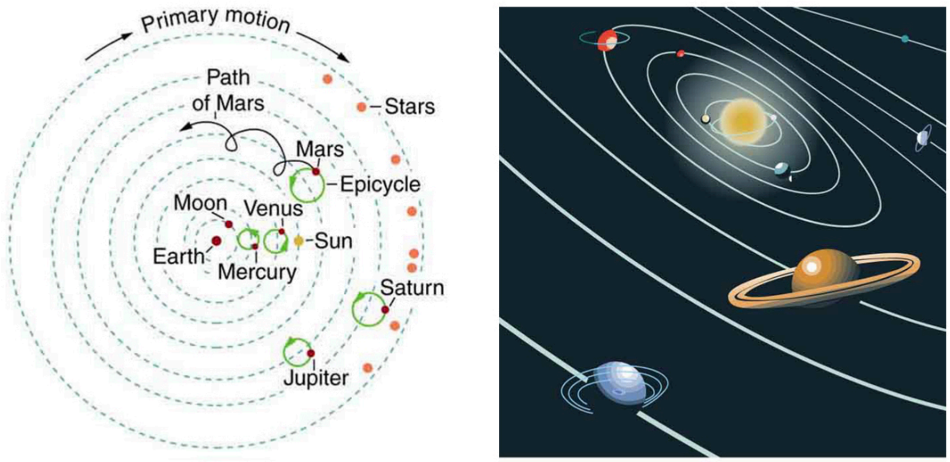
Table 4.2 Orbital Data and Kepler's Third Law

Parent	Satellite	Average orbital radius r(km)	Period $T(y)$	r^3 / T^2 (km^3 / y^2)
Earth	Moon	3.84×10^5	0.07481	1.01×10^{19}
Sun	Mercury	5.79×10^7	0.2409	3.34×10^{24}
	Venus	1.082×10^8	0.6150	3.35×10^{24}
	Earth	1.496×10^8	1.000	3.35×10^{24}
	Mars	2.279×10^8	1.881	3.35×10^{24}
	Jupiter	7.783×10^8	11.86	3.35×10^{24}
	Saturn	1.427×10^9	29.46	3.35×10^{24}
	Neptune	4.497×10^9	164.8	3.35×10^{24}
	Pluto	5.90×10^9	248.3	3.33×10^{24}
Jupiter	Io	4.22×10^5	0.00485 (1.77 d)	3.19×10^{21}
	Europa	6.71×10^5	0.00972 (3.55 d)	3.20×10^{21}
	Ganymede	1.07×10^6	0.0196 (7.16 d)	3.19×10^{21}
	Callisto	1.88×10^6	0.0457 (16.19 d)	3.20×10^{21}

The universal law of gravitation is a good example of a physical principle that is very broadly applicable. That single equation for the gravitational force describes all situations in which gravity acts. It gives a cause for a vast number of effects, such as the orbits of the planets and moons in the solar system. It epitomizes the underlying unity and simplicity of physics.

Before the discoveries of Kepler, Copernicus, Galileo, Newton, and others, the solar system was thought to revolve around Earth as shown in **Figure 4.36**(a). This is called the Ptolemaic view, for the Greek philosopher who lived in the second century AD. This model is characterized by a list of facts for the motions of planets with no cause and effect explanation. There tended to be a different rule for each heavenly body and a general lack of simplicity.

Figure 4.36(b) represents the modern or Copernican model. In this model, a small set of rules and a single underlying force explain not only all motions in the solar system, but all other situations involving gravity. The breadth and simplicity of the laws of physics are compelling. As our knowledge of nature has grown, the basic simplicity of its laws has become ever more evident.



Ptolemaic view
(a)

Copernican view
(b)

Figure 4.36 (a) The Ptolemaic model of the universe has Earth at the center with the Moon, the planets, the Sun, and the stars revolving about it in complex superpositions of circular paths. This geocentric model, which can be made progressively more accurate by adding more circles, is purely descriptive, containing no hints as to what are the causes of these motions. (b) The Copernican model has the Sun at the center of the solar system. It is fully explained by a small number of laws of physics, including Newton's universal law of gravitation.

Glossary

angular velocity: ω , the rate of change of the angle with which an object moves on a circular path

arc length: Δs , the distance traveled by an object along a circular path

banked curve: curve in a road that is sloping in a manner that helps a vehicle negotiate the curve

center of mass: the point where the entire mass of an object can be thought to be concentrated

centrifugal force: a fictitious force that tends to throw an object off when the object is rotating in a non-inertial frame of reference

centripetal acceleration: the acceleration of an object moving in a circle, directed toward the center

centripetal force: any net force causing uniform circular motion

Coriolis force: inertial force causing the apparent deflection of moving objects when viewed in a rotating frame of reference

Coriolis force: the fictitious force causing the apparent deflection of moving objects when viewed in a rotating frame of reference

fictitious force: a force having no physical origin

gravitational constant, *G*: a proportionality factor used in the equation for Newton's universal law of gravitation; it is a universal constant—that is, it is thought to be the same everywhere in the universe

ideal banking: sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction

inertial force: force that has no physical origin

microgravity: an environment in which the apparent net acceleration of a body is small compared with that produced by Earth at its surface

Newton's universal law of gravitation: every particle in the universe attracts every other particle with a force along a line joining them; the force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them

non-inertial frame of reference: an accelerated frame of reference

noninertial frame of reference: accelerated frame of reference

pit: a tiny indentation on the spiral track moulded into the top of the polycarbonate layer of CD

radians: a unit of angle measurement

radius of curvature: radius of a circular path

rotation angle: the ratio of the arc length to the radius of curvature on a circular path:

$$\Delta\theta = \frac{\Delta s}{r}$$

ultracentrifuge: a centrifuge optimized for spinning a rotor at very high speeds

uniform circular motion: the motion of an object in a circular path at constant speed

Section Summary

4.1 Rotation Angle and Angular Velocity

- Uniform circular motion is motion in a circle at constant speed. The rotation angle $\Delta\theta$ is defined as the ratio of the arc length to the radius of curvature:
$$\Delta\theta = \frac{\Delta s}{r},$$
where arc length Δs is distance traveled along a circular path and r is the radius of curvature of the circular path. The quantity $\Delta\theta$ is measured in units of radians (rad), for which
$$2\pi \text{ rad} = 360^{\circ} = 1 \text{ revolution.}$$
- The conversion between radians and degrees is $1 \text{ rad} = 57.3^{\circ}$.
- Angular velocity ω is the rate of change of an angle,
$$\omega = \frac{\Delta\theta}{\Delta t},$$
where a rotation $\Delta\theta$ takes place in a time Δt. The units of angular velocity are radians per second (rad/s). Linear velocity v and angular velocity ω are related by
$$v = r\omega \text{ or } \omega = \frac{v}{r}.$$

4.2 Centripetal Acceleration

- Centripetal acceleration a_{c} is the acceleration experienced while in uniform circular motion. It always points toward the center of rotation. It is perpendicular to the linear velocity v and has the magnitude
$$a_{\text{c}} = \frac{v^2}{r};\ a_{\text{c}} = r\omega^2.$$
- The unit of centripetal acceleration is m/s^2.

4.4 Fictitious Forces and Non-inertial Frames: The Coriolis Force

- Rotating and accelerated frames of reference are non-inertial.
- Fictitious forces, such as the Coriolis force, are needed to explain motion in such frames.

4.5 Newton's Universal Law of Gravitation

- Newton's universal law of gravitation: Every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. In equation form, this is
$$F = G\frac{mM}{r^2},$$
where F is the magnitude of the gravitational force. G is the gravitational constant, given by $G = 6.674\times10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$.
- Newton's law of gravitation applies universally.

4.6 Satellites and Kepler's Laws: An Argument for Simplicity

- Kepler's laws are stated for a small mass m orbiting a larger mass M in near-isolation. Kepler's laws of planetary motion are then as follows:
Kepler's first law

  The orbit of each planet about the Sun is an ellipse with the Sun at one focus.

  Kepler's second law

  Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times.

Kepler's third law

The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun:

$$\frac{T_1^2}{T_2^2} = \frac{r_1^3}{r_2^3},$$

where T is the period (time for one orbit) and r is the average radius of the orbit.

- The period and radius of a satellite's orbit about a larger body M are related by

$$T^2 = \frac{4\pi^2}{GM}r^3$$

or

$$\frac{r^3}{T^2} = \frac{G}{4\pi^2}M.$$

Conceptual Questions

4.1 Rotation Angle and Angular Velocity

1. There is an analogy between rotational and linear physical quantities. What rotational quantities are analogous to distance and velocity?

4.2 Centripetal Acceleration

2. Can centripetal acceleration change the speed of circular motion? Explain.

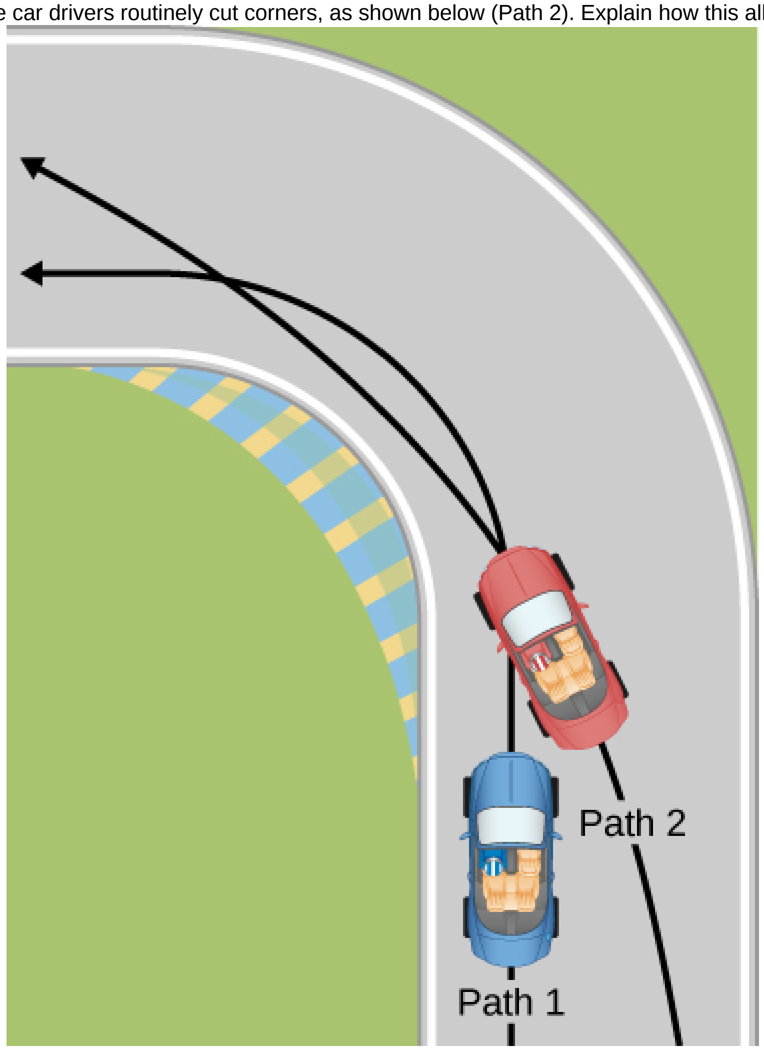
4.3 Centripetal Force

3. If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.

4. Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?

5. If centripetal force is directed toward the center, why do you feel that you are 'thrown' away from the center as a car goes around a curve? Explain.

6. Race car drivers routinely cut corners, as shown below (Path 2). Explain how this allows the curve to be taken at the greatest speed.



7. Many amusement parks have rides that make vertical loops like the one shown below. For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:

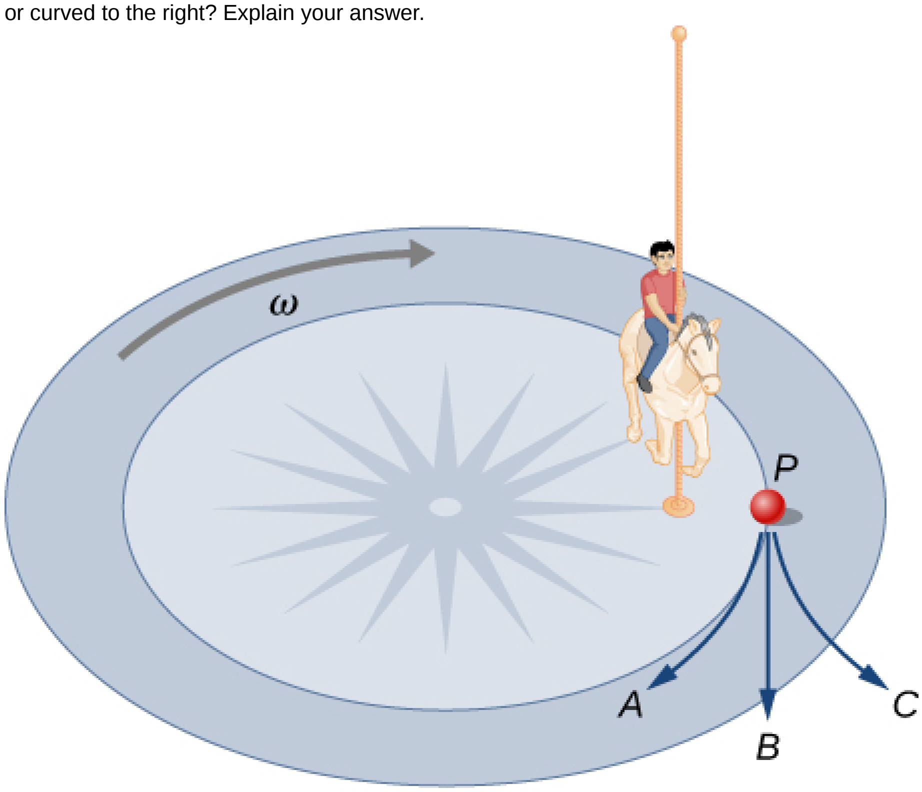
(a) The car goes over the top at faster than this speed?

(b) The car goes over the top at slower than this speed?

8. What causes water to be removed from clothes in a spin-dryer?

9. As a skater forms a circle, what force is responsible for making his turn? Use a free-body diagram in your answer.

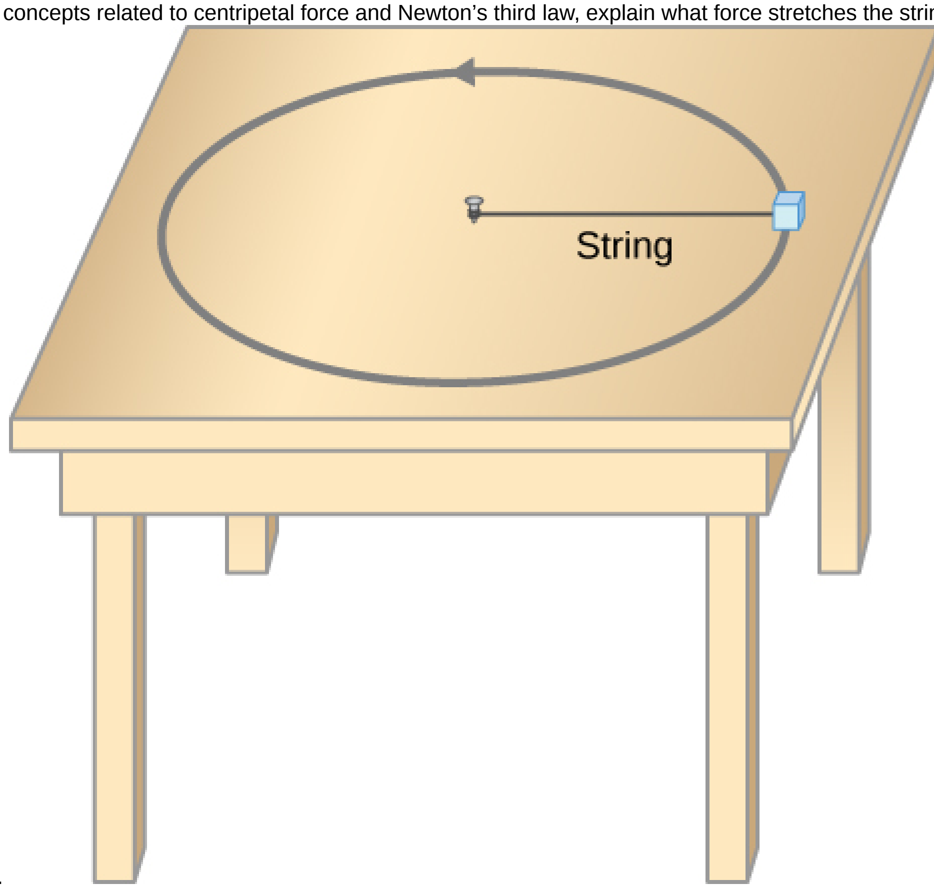
10. Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown below will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.



Merry-go-round's rotating frame of reference

11. Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car's speed? What is the direction of the force exerted on you by the car seat?

12. Suppose a mass is moving in a circular path on a frictionless table as shown below. In Earth's frame of reference, there is no centrifugal force pulling the mass away from the center of rotation, yet there is a force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton's third law, explain what force stretches the string, identifying its



physical origin.

13. When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the Northern Hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?

14. A car rounds a curve and encounters a patch of ice with a very low coefficient of kinetic fiction. The car slides off the road. Describe the path of the car as it leaves the road.

15. In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is an inertial force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all forces acting on them.

16. Two friends are having a conversation. Anna says a satellite in orbit is in free fall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in free fall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?

17. A nonrotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

4.4 Fictitious Forces and Non-inertial Frames: The Coriolis Force

18. When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the northern hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?

19. Is there a real force that throws water from clothes during the spin cycle of a washing machine? Explain how the water is removed.

20. In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is a fictitious force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all of the real forces acting on them.

21. Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

22. Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?

23. A non-rotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

4.5 Newton's Universal Law of Gravitation

24. Action at a distance, such as is the case for gravity, was once thought to be illogical and therefore untrue. What is the ultimate determinant of the truth in physics, and why was this action ultimately accepted?

25. Two friends are having a conversation. Anna says a satellite in orbit is in freefall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in freefall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?

26. Draw a free body diagram for a satellite in an elliptical orbit showing why its speed increases as it approaches its parent body and decreases as it moves away.

27. Newton's laws of motion and gravity were among the first to convincingly demonstrate the underlying simplicity and unity in nature. Many other examples have since been discovered, and we now expect to find such underlying order in complex situations. Is there proof that such order will always be found in new explorations?

4.6 Satellites and Kepler's Laws: An Argument for Simplicity

28. In what frame(s) of reference are Kepler's laws valid? Are Kepler's laws purely descriptive, or do they contain causal information?

Problems & Exercises

4.1 Rotation Angle and Angular Velocity

1. Semi-trailer trucks have an odometer on one hub of a trailer wheel. The hub is weighted so that it does not rotate, but it contains gears to count the number of wheel revolutions—it then calculates the distance traveled. If the wheel has a 1.15 m diameter and goes through 200,000 rotations, how many kilometers should the odometer read?

2. Microwave ovens rotate at a rate of about 6 rev/min. What is this in revolutions per second? What is the angular velocity in radians per second?

3. An automobile with 0.260 m radius tires travels 80,000 km before wearing them out. How many revolutions do the tires make, neglecting any backing up and any change in radius due to wear?

4. (a) What is the period of rotation of Earth in seconds? (b) What is the angular velocity of Earth? (c) Given that Earth has a radius of $6.4\times10^6\text{ m}$ at its equator, what is the linear velocity at Earth's surface?

5. A baseball pitcher brings his arm forward during a pitch, rotating the forearm about the elbow. If the velocity of the ball in the pitcher's hand is 35.0 m/s and the ball is 0.300 m from the elbow joint, what is the angular velocity of the forearm?

6. In lacrosse, a ball is thrown from a net on the end of a stick by rotating the stick and forearm about the elbow. If the angular velocity of the ball about the elbow joint is 30.0 rad/s and the ball is 1.30 m from the elbow joint, what is the velocity of the ball?

7. A truck with 0.420-m-radius tires travels at 32.0 m/s. What is the angular velocity of the rotating tires in radians per second? What is this in rev/min?

8. Integrated Concepts When kicking a football, the kicker rotates his leg about the hip joint.

(a) If the velocity of the tip of the kicker's shoe is 35.0 m/s and the hip joint is 1.05 m from the tip of the shoe, what is the shoe tip's angular velocity?

(b) The shoe is in contact with the initially stationary 0.500 kg football for 20.0 ms. What average force is exerted on the football to give it a velocity of 20.0 m/s?

(c) Find the maximum range of the football, neglecting air resistance.

9. Construct Your Own Problem

Consider an amusement park ride in which participants are rotated about a vertical axis in a cylinder with vertical walls. Once the angular velocity reaches its full value, the floor drops away and friction between the walls and the riders prevents them from sliding down. Construct a problem in which you calculate the necessary angular velocity that assures the riders will not slide down the wall. Include a free body diagram of a single rider. Among the variables to consider are the radius of the cylinder and the coefficients of friction between the riders' clothing and the wall.

4.2 Centripetal Acceleration

10. A fairground ride spins its occupants inside a flying saucer-shaped container. If the horizontal circular path the riders follow has an 8.00 m radius, at how many revolutions per minute will the riders be subjected to a centripetal acceleration whose magnitude is 1.50 times that due to gravity?

11. A runner taking part in the 200 m dash must run around the end of a track that has a circular arc with a radius of curvature of 30 m. If he completes the 200 m dash in 23.2 s and runs at constant speed throughout the race, what is the magnitude of his centripetal acceleration as he runs the curved portion of the track?

12. Taking the age of Earth to be about 4×10^9 years and assuming its orbital radius of 1.5×10^{11} m has not changed and is circular, calculate the approximate total distance Earth has traveled since its birth (in a frame of reference stationary with respect to the Sun).

13. The propeller of a World War II fighter plane is 2.30 m in diameter.

(a) What is its angular velocity in radians per second if it spins at 1200 rev/min?

(b) What is the linear speed of its tip at this angular velocity if the plane is stationary on the tarmac?

(c) What is the centripetal acceleration of the propeller tip under these conditions? Calculate it in meters per second squared and convert to multiples of g.

14. An ordinary workshop grindstone has a radius of 7.50 cm and rotates at 6500 rev/min.

(a) Calculate the magnitude of the centripetal acceleration at its edge in meters per second squared and convert it to multiples of g.

(b) What is the linear speed of a point on its edge?

15. Helicopter blades withstand tremendous stresses. In addition to supporting the weight of a helicopter, they are spun at rapid rates and experience large centripetal accelerations, especially at the tip.

(a) Calculate the magnitude of the centripetal acceleration at the tip of a 4.00 m long helicopter blade that rotates at 300 rev/min.

(b) Compare the linear speed of the tip with the speed of sound (taken to be 340 m/s).

16. Olympic ice skaters are able to spin at about 5 rev/s.

(a) What is their angular velocity in radians per second?

(b) What is the centripetal acceleration of the skater's nose if it is 0.120 m from the axis of rotation?

(c) An exceptional skater named Dick Button was able to spin much faster in the 1950s than anyone since—at about 9 rev/s. What was the centripetal acceleration of the tip of his nose, assuming it is at 0.120 m radius?

(d) Comment on the magnitudes of the accelerations found. It is reputed that Button ruptured small blood vessels during his spins.

17. What percentage of the acceleration at Earth's surface is the acceleration due to gravity at the position of a satellite located 300 km above Earth?

18. Verify that the linear speed of an ultracentrifuge is about 0.50 km/s, and Earth in its orbit is about 30 km/s by calculating:

(a) The linear speed of a point on an ultracentrifuge 0.100 m from its center, rotating at 50,000 rev/min.

(b) The linear speed of Earth in its orbit about the Sun (use data from the text on the radius of Earth's orbit and approximate it as being circular).

19. A rotating space station is said to create "artificial gravity"—a loosely-defined term used for an acceleration that would be crudely similar to gravity. The outer wall of the rotating space station would become a floor for the astronauts, and centripetal acceleration supplied by the floor would allow astronauts to exercise and maintain muscle and bone strength more naturally than in non-rotating space environments. If the space station is 200 m in diameter, what angular velocity would produce an "artificial gravity" of $9.80\ \text{m/s}^2$ at the rim?

20. At takeoff, a commercial jet has a 60.0 m/s speed. Its tires have a diameter of 0.850 m.

(a) At how many rev/min are the tires rotating?

(b) What is the centripetal acceleration at the edge of the tire?

(c) With what force must a determined $1.00\times10^{-15}\ \text{kg}$ bacterium cling to the rim?

(d) Take the ratio of this force to the bacterium's weight.

21. Integrated Concepts

Riders in an amusement park ride shaped like a Viking ship hung from a large pivot are rotated back and forth like a rigid pendulum. Sometime near the middle of the ride, the ship is momentarily motionless at the top of its circular arc. The ship then swings down under the influence of gravity.

(a) Assuming negligible friction, find the speed of the riders at the bottom of its arc, given the system's center of mass travels in an arc having a radius of 14.0 m and the riders are near the center of mass.

(b) What is the centripetal acceleration at the bottom of the arc?

(c) Draw a free body diagram of the forces acting on a rider at the bottom of the arc.

(d) Find the force exerted by the ride on a 60.0 kg rider and compare it to her weight.

(e) Discuss whether the answer seems reasonable.

22. Unreasonable Results

A mother pushes her child on a swing so that his speed is 9.00 m/s at the lowest point of his path. The swing is suspended 2.00 m above the child's center of mass.

(a) What is the magnitude of the centripetal acceleration of the child at the low point?

(b) What is the magnitude of the force the child exerts on the seat if his mass is 18.0 kg?

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent?

4.5 Newton's Universal Law of Gravitation

23. (a) Calculate Earth's mass given the acceleration due to gravity at the North Pole is $9.830\ \text{m/s}^2$ and the radius of the Earth is 6371 km from center to pole.

(b) Compare this with the accepted value of $5.979\times10^{24}\ \text{kg}$.

24. (a) Calculate the magnitude of the acceleration due to gravity on the surface of Earth due to the Moon.

(b) Calculate the magnitude of the acceleration due to gravity at Earth due to the Sun.

(c) Take the ratio of the Moon's acceleration to the Sun's and comment on why the tides are predominantly due to the Moon in spite of this number.

25. (a) What is the acceleration due to gravity on the surface of the Moon?

(b) On the surface of Mars? The mass of Mars is $6.418\times10^{23}\ \text{kg}$ and its radius is $3.38\times10^{6}\ \text{m}$.

26. (a) Calculate the acceleration due to gravity on the surface of the Sun.

(b) By what factor would your weight increase if you could stand on the Sun? (Never mind that you cannot.)

27. The Moon and Earth rotate about their common center of mass, which is located about 4700 km from the center of Earth. (This is 1690 km below the surface.)

(a) Calculate the magnitude of the acceleration due to the Moon's gravity at that point.

(b) Calculate the magnitude of the centripetal acceleration of the center of Earth as it rotates about that point once each lunar month (about 27.3 d) and compare it with the acceleration found in part (a). Comment on whether or not they are equal and why they should or should not be.

28. Solve part (b) of **Example 4.6** using $a_c = v^2/r$.

29. Astrology, that unlikely and vague pseudoscience, makes much of the position of the planets at the moment of one's birth. The only known force a planet exerts on Earth is gravitational.

(a) Calculate the magnitude of the gravitational force exerted on a 4.20 kg baby by a 100 kg father 0.200 m away at birth (he is assisting, so he is close to the child).

(b) Calculate the magnitude of the force on the baby due to Jupiter if it is at its closest distance to Earth, some $6.29\times10^{11}\ \text{m}$ away. How does the force of Jupiter on the baby compare to the force of the father on the baby? Other objects in the room and the hospital building also exert similar gravitational forces. (Of course, there could be an unknown force acting, but scientists first need to be convinced that there is even an effect, much less that an unknown force causes it.)

30. The existence of the dwarf planet Pluto was proposed based on irregularities in Neptune's orbit. Pluto was subsequently discovered near its predicted position. But it now appears that the discovery was fortuitous, because Pluto is small and the irregularities in Neptune's orbit were not well known. To illustrate that Pluto has a minor effect on the orbit of Neptune compared with the closest planet to Neptune:

(a) Calculate the acceleration due to gravity at Neptune due to Pluto when they are $4.50\times10^{12}\text{ m}$ apart, as they are at present. The mass of Pluto is $1.4\times10^{22}\text{ kg}$.

(b) Calculate the acceleration due to gravity at Neptune due to Uranus, presently about $2.50\times10^{12}\text{ m}$ apart, and compare it with that due to Pluto. The mass of Uranus is $8.62\times10^{25}\text{ kg}$.

31. (a) The Sun orbits the Milky Way galaxy once each $2.60\text{ x }10^8\text{ y}$, with a roughly circular orbit averaging $3.00\text{ x }10^4$ light years in radius. (A light year is the distance traveled by light in 1 y.) Calculate the centripetal acceleration of the Sun in its galactic orbit. Does your result support the contention that a nearly inertial frame of reference can be located at the Sun?

(b) Calculate the average speed of the Sun in its galactic orbit. Does the answer surprise you?

32. Unreasonable Result

A mountain 10.0 km from a person exerts a gravitational force on him equal to 2.00% of his weight.

(a) Calculate the mass of the mountain.

(b) Compare the mountain's mass with that of Earth.

(c) What is unreasonable about these results?

(d) Which premises are unreasonable or inconsistent? (Note that accurate gravitational measurements can easily detect the effect of nearby mountains and variations in local geology.)

4.6 Satellites and Kepler's Laws: An Argument for Simplicity

33. A geosynchronous Earth satellite is one that has an orbital period of precisely 1 day. Such orbits are useful for communication and weather observation because the satellite remains above the same point on Earth (provided it orbits in the equatorial plane in the same direction as Earth's rotation). Calculate the radius of such an orbit based on the data for the moon in **Table 4.2**.

34. Calculate the mass of the Sun based on data for Earth's orbit and compare the value obtained with the Sun's actual mass.

35. Find the mass of Jupiter based on data for the orbit of one of its moons, and compare your result with its actual mass.

36. Find the ratio of the mass of Jupiter to that of Earth based on data in **Table 4.2**.

37. Astronomical observations of our Milky Way galaxy indicate that it has a mass of about 8.0×10^{11} solar masses. A star orbiting on the galaxy's periphery is about 6.0×10^4 light years from its center. (a) What should the orbital period of that star be? (b) If its period is 6.0×10^7 instead, what is the mass of the galaxy? Such calculations are used to imply the existence of "dark matter" in the universe and have indicated, for example, the existence of very massive black holes at the centers of some galaxies.

38. Integrated Concepts

Space debris left from old satellites and their launchers is becoming a hazard to other satellites. (a) Calculate the speed of a satellite in an orbit 900 km above Earth's surface. (b) Suppose a loose rivet is in an orbit of the same radius that intersects the satellite's orbit at an angle of 90° relative to Earth. What is the velocity of the rivet relative to the satellite just before striking it? (c) Given the rivet is 3.00 mm in size, how long will its collision with the satellite last? (d) If its mass is 0.500 g, what is the average force it exerts on the satellite? (e) How much energy in joules is generated by the collision? (The satellite's velocity does not change appreciably, because its mass is much greater than the rivet's.)

39. Unreasonable Results

(a) Based on Kepler's laws and information on the orbital characteristics of the Moon, calculate the orbital radius for an Earth satellite having a period of 1.00 h. (b) What is unreasonable about this result? (c) What is unreasonable or inconsistent about the premise of a 1.00 h orbit?

40. Construct Your Own Problem

On February 14, 2000, the NEAR spacecraft was successfully inserted into orbit around Eros, becoming the first artificial satellite of an asteroid. Construct a problem in which you determine the orbital speed for a satellite near Eros. You will need to find the mass of the asteroid and consider such things as a safe distance for the orbit. Although Eros is not spherical, calculate the acceleration due to gravity on its surface at a point an average distance from its center of mass. Your instructor may also wish to have you calculate the escape velocity from this point on Eros.

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Unit III

Forces and…

Photo credit: Hirth, Lion. *UMass at Amherst, Picture Taken from the W.E.B. Dubois Library*, October 2006. Own work. https://commons.wikimedia.org/wiki/File:UMass_Amherst_Pond.jpg.

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UNIT 3 OVERVIEW

UMASS AMHERST Instructor's Notes

This overview is also available as a video **here (https://www.youtube.com/watch?v=rzGwxTZtnrA)** .

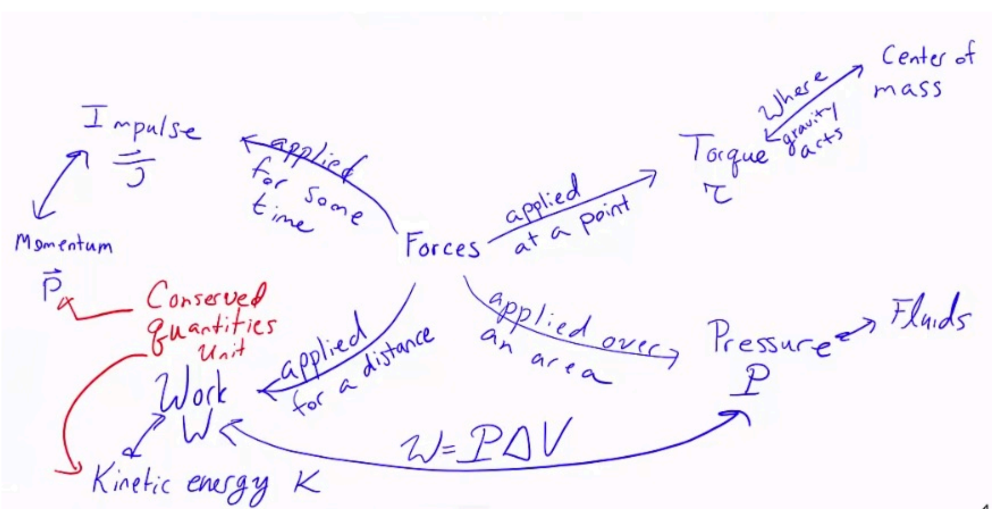
What is the meaning of the title of this unit? Well in our last unit, we introduced the ideas of forces and Newton's laws, and while forces are sufficient to determine the acceleration, there are clearly other quantities that are interesting if we stop and think about it. For example, it is easier to open a door by pushing on the door on the far side from the hinge where the knob is located than by pushing on the door near the hinge. If you've never really thought about this while opening a door, I encourage you to go try it. Similarly, if I apply a force to a ball for a short time, I get a different result than if I apply that same force for a long amount of time. If I apply the force for a long amount of time, the final velocity of the ball will be larger. For these reasons, we will be exploring in this unit forces in conjunction with other quantities. All our principles from the last unit still work, and many of these situations could be analyzed solely within the context of Newton's laws if you need it to. However, the new ideas we're going to introduce in this unit are often simpler to think about and to work with. However, with all the new concepts, choosing which concept to apply in each situation becomes its own unique challenge.

So, let's do a quick overview of the different concepts we're going to talk about in this unit. The first unit we will discuss is torque, and this is the fact that where forces are applied can matter. So, this goes back to the door; applying a force near the hinge results in a different experience than applying a force far away from the hinge at the knob. The next quantity is impulse; how long we apply a force also matters. We will also introduce the idea of pressure. Pressure is the fact the area over which the force is applied can matter. This quantity is particularly relevant when we discuss fluids, which, remember, include both gases and liquids. The final quantity we will discuss in this unit is work. Work is the fact that the distance over which the forces applied matters. If I apply a force over a long distance, I can get a different result than if I apply that same force over a short distance. Work can also be expressed in terms of pressure to talk about the work done on, or by, fluids.

There are some new symbols to learn when discussing these quantities. Torque is represented by the τ. Impulse is represented by $\vec{J}$; note that it is a vector quantity. Pressure, represented by the capital P, and work, I will use the capital W. Along the way in discussing these concepts, we will meet some other important ideas.

When we discuss torque, it will be important to introduce the idea of center of gravity. Torque is when you're interested about where the forces are being applied. If you're interested in where the forces are being applied, then you need to think about where does the force of gravity act. This is the idea behind center of gravity. When we discuss impulse, we will introduce the quantity momentum which uses a lowercase p, and you can see is also a vector. Momentum is a quantity connected to impulse, which we will revisit in greater detail in our unit on conserved quantities. The final quantity we will introduce in this unit is the quantity of kinetic energy represented by a capital K. Kinetic energy is connected to work, and again, we will revisit this quantity in our unit on conserved quantities.

Because of all the different quantities we're introducing in this unit, a nice way to organize them might be map.



So, we can think of the idea of forces from the last unit. So, a force applied at a point takes us to a torque, which we represent by the Greek τ, and the idea of torque is going to be connected to the idea of center of mass, as the center of mass dictates where gravity acts. We can also talk about forces being applied over an area, and this brings us to the idea of pressure, which again we represent by a capital P, and pressure will connect to our study of fluids. We can talk about forces being applied for some amount of time, and this brings us to the idea of impulse, J, which is connected to the idea of momentum as we'll see in this prep, which is represented by the lowercase p. And we can talk about forces being applied for a distance, bringing us to work, W, which is related to the idea of kinetic energy, K, and both will be related to our conserved quantities unit. Also, during this prep, you will see how work connects to the idea of pressure as work being pressure times the change in volume. This idea of a map to help you sort of organize all the information is a great study tool when studying physics, and I encourage you to maybe build your own using this as a core as you go through this unit.

1 IMPULSE

1.1 Linear Momentum and Force

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Calculate the momentum for any object
- Recall that momentum is a vector
- From the change in momentum, compute the average force

Linear Momentum

The scientific definition of linear momentum is consistent with most people's intuitive understanding of momentum: a large, fast-moving object has greater momentum than a smaller, slower object. **Linear momentum** is defined as the product of a system's mass multiplied by its velocity. In symbols, linear momentum is expressed as

$$\mathbf{p} = m\mathbf{v}. \tag{1.1}$$

Momentum is directly proportional to the object's mass and also its velocity. Thus the greater an object's mass or the greater its velocity, the greater its momentum. Momentum $\mathbf{p}$ is a vector having the same direction as the velocity $\mathbf{v}$. The SI unit for momentum is $\text{kg} \cdot \text{m/s}$.

Linear Momentum

Linear momentum is defined as the product of a system's mass multiplied by its velocity:

$$\mathbf{p} = m\mathbf{v}. \tag{1.2}$$

UMASS AMHERST Instructor's Notes

The main focus for this section is the definition of momentum above, as well as the calculation of momentum. The following is a good example of what we expect you to be able to do with regards to the calculation. Also, pay attention to the discussion in the example as well, as it talks about how both mass and velocity can affect momentum.

Example 1.1 Calculating Momentum: A Football Player and a Football

(a) Calculate the momentum of a 110-kg football player running at 8.00 m/s. (b) Compare the player's momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s.

Strategy

No information is given regarding direction, and so we can calculate only the magnitude of the momentum, p. (As usual, a symbol that is in italics is a magnitude, whereas one that is italicized, boldfaced, and has an arrow is a vector.) In both parts of this example, the magnitude of momentum can be calculated directly from the definition of momentum given in the equation, which becomes

$$p = mv \tag{1.3}$$

when only magnitudes are considered.

Solution for (a)

To determine the momentum of the player, substitute the known values for the player's mass and speed into the equation.

$$p_{\text{player}} = (110\text{ kg})(8.00\text{ m/s}) = 880\text{ kg} \cdot \text{m/s} \tag{1.4}$$

Solution for (b)

To determine the momentum of the ball, substitute the known values for the ball's mass and speed into the equation.

$$p_{\text{ball}} = (0.410\text{ kg})(25.0\text{ m/s}) = 10.3\text{ kg}\cdot\text{m/s} \tag{1.5}$$

The ratio of the player’s momentum to that of the ball is

$$\frac{p_{\text{player}}}{p_{\text{ball}}} = \frac{880}{10.3} = 85.9. \tag{1.6}$$

Discussion

Although the ball has greater velocity, the player has a much greater mass. Thus the momentum of the player is much greater than the momentum of the football, as you might guess. As a result, the player’s motion is only slightly affected if he catches the ball. We shall quantify what happens in such collisions in terms of momentum in later sections.

UMASS AMHERST Instructor's Notes

The next part will tie into the next section, impulse, so be sure to pay attention to this part here as well.

Momentum and Newton’s Second Law

The importance of momentum, unlike the importance of energy, was recognized early in the development of classical physics. Momentum was deemed so important that it was called the “quantity of motion.” Newton actually stated his **second law of motion** in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}, \tag{1.7}$$

where $\mathbf{F}_{\text{net}}$ is the net external force, $\Delta \mathbf{p}$ is the change in momentum, and Δt is the change in time.

Newton’s Second Law of Motion in Terms of Momentum

The net external force equals the change in momentum of a system divided by the time over which it changes.

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t} \tag{1.8}$$

Making Connections: Force and Momentum

Force and momentum are intimately related. Force acting over time can change momentum, and Newton’s second law of motion, can be stated in its most broadly applicable form in terms of momentum. Momentum continues to be a key concept in the study of atomic and subatomic particles in quantum mechanics.

This statement of Newton’s second law of motion includes the more familiar $\mathbf{F}_{\text{net}} = m\mathbf{a}$ as a special case. We can derive this form as follows. First, note that the change in momentum $\Delta \mathbf{p}$ is given by

$$\Delta \mathbf{p} = \Delta(m\mathbf{v}). \tag{1.9}$$

If the mass of the system is constant, then

$$\Delta(m\mathbf{v}) = m\Delta\mathbf{v}. \tag{1.10}$$

So that for constant mass, Newton’s second law of motion becomes

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t} = \frac{m\Delta \mathbf{v}}{\Delta t}. \tag{1.11}$$

Because $\frac{\Delta \mathbf{v}}{\Delta t} = \mathbf{a}$, we get the familiar equation

$$\mathbf{F}_{\text{net}} = m\mathbf{a} \tag{1.12}$$

when the mass of the system is constant.

Newton’s second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass. We will consider systems with varying mass in some detail**;** however, the relationship between momentum and force remains useful when mass is constant, such as in the following example.

Example 1.2 Calculating Force: Venus Williams' Racquet

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet, assuming that the ball's speed just after impact is 58 m/s, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms (milliseconds)?

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}. \quad (1.13)$$

As noted above, when mass is constant, the change in momentum is given by

$$\Delta p = m\Delta v = m(v_{\text{f}} - v_{\text{i}}). \quad (1.14)$$

In this example, the velocity just after impact and the change in time are given; thus, once Δp is calculated, $F_{\text{net}} = \frac{\Delta p}{\Delta t}$ can be used to find the force.

Solution

To determine the change in momentum, substitute the values for the initial and final velocities into the equation above.

$$\begin{aligned} \Delta p &= m(v_{\text{f}} - v_{\text{i}}) \\ &= (0.057 \text{ kg})(58 \text{ m/s} - 0 \text{ m/s}) \\ &= 3.306 \text{ kg} \cdot \text{m/s} \approx 3.3 \text{ kg} \cdot \text{m/s} \end{aligned} \quad (1.15)$$

Now the magnitude of the net external force can determined by using $F_{\text{net}} = \frac{\Delta p}{\Delta t}$:

$$\begin{aligned} F_{\text{net}} &= \frac{\Delta p}{\Delta t} = \frac{3.306 \text{ kg} \cdot \text{m/s}}{5.0 \times 10^{-3} \text{ s}} \\ &= 661 \text{ N} \approx 660 \text{ N}, \end{aligned} \quad (1.16)$$

where we have retained only two significant figures in the final step.

Discussion

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.56-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using $F_{\text{net}} = ma$, but one additional step would be required compared with the strategy used in this example.

Section Summary

- Linear momentum (*momentum* for brevity) is defined as the product of a system's mass multiplied by its velocity.
- In symbols, linear momentum $\mathbf{p}$ is defined to be

$$\mathbf{p} = m\mathbf{v}, \quad (1.17)$$

  where m is the mass of the system and $\mathbf{v}$ is its velocity.
- The SI unit for momentum is $\text{kg} \cdot \text{m/s}$.
- Newton's second law of motion in terms of momentum states that the net external force equals the change in momentum of a system divided by the time over which it changes.
- In symbols, Newton's second law of motion is defined to be

$$\mathbf{F}_{\text{net}} = \frac{\Delta \mathbf{p}}{\Delta t}, \quad (1.18)$$

  $\mathbf{F}_{\text{net}}$ is the net external force, $\Delta \mathbf{p}$ is the change in momentum, and Δt is the change time.

Conceptual Questions

Exercise 1.1

An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?

Exercise 1.2

An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

Exercise 1.3

Professional Application

Football coaches advise players to block, hit, and tackle with their feet on the ground rather than by leaping through the air. Using the concepts of momentum, work, and energy, explain how a football player can be more effective with his feet on the ground.

Exercise 1.4

How can a small force impart the same momentum to an object as a large force?

Problems & Exercises

Exercise 1.5

(a) Calculate the momentum of a 2000-kg elephant charging a hunter at a speed of 7.50 m/s. (b) Compare the elephant's momentum with the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s. (c) What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

Solution

(a) $1.50\times10^4\text{ kg}\cdot\text{m/s}$

(b) 625 to 1

(c) $6.66\times10^2\text{ kg}\cdot\text{m/s}$

Exercise 1.6

(a) What is the mass of a large ship that has a momentum of $1.60\times10^9\text{ kg}\cdot\text{m/s}$, when the ship is moving at a speed of 48.0 km/h? (b) Compare the ship's momentum to the momentum of a 1100-kg artillery shell fired at a speed of 1200 m/s.

Exercise 1.7

(a) At what speed would a $2.00\times10^4\text{-kg}$ airplane have to fly to have a momentum of $1.60\times10^9\text{ kg}\cdot\text{m/s}$ (the same as the ship's momentum in the problem above)? (b) What is the plane's momentum when it is taking off at a speed of 60.0 m/s? (c) If the ship is an aircraft carrier that launches these airplanes with a catapult, discuss the implications of your answer to (b) as it relates to recoil effects of the catapult on the ship.

Solution

(a) $8.00\times10^4\text{ m/s}$

(b) $1.20\times10^{6}\ \text{kg}\cdot\text{m/s}$

(c) Because the momentum of the airplane is 3 orders of magnitude smaller than of the ship, the ship will not recoil very much. The recoil would be $-0.0100\ \text{m/s}$, which is probably not noticeable.

Exercise 1.8

(a) What is the momentum of a garbage truck that is $1.20\times10^{4}\ \text{kg}$ and is moving at $10.0\ \text{m/s}$? (b) At what speed would an 8.00-kg trash can have the same momentum as the truck?

Exercise 1.9

A runaway train car that has a mass of 15,000 kg travels at a speed of $5.4\ \text{m/s}$ down a track. Compute the time required for a force of 1500 N to bring the car to rest.

Solution
54 s

Exercise 1.10

The mass of Earth is $5.972\times10^{24}\ \text{kg}$ and its orbital radius is an average of $1.496\times10^{11}\ \text{m}$. Calculate its linear momentum.

1.2 Impulse

UMASS AMHERST Instructor's Notes

Your Quiz will Cover
- From an impulse, compute the change in momentum
- Identify which aspect of an F(t) graph represents impulse
- Compute the net change in momentum of an object using an F(t) graph

The effect of a force on an object depends on how long it acts, as well as how great the force is. In **m42156 (https://legacy.cnx.org/content/m42156/latest/#fs-id1356444)**, a very large force acting for a short time had a great effect on the momentum of the tennis ball. A small force could cause the same **change in momentum**, but it would have to act for a much longer time. For example, if the ball were thrown upward, the gravitational force (which is much smaller than the tennis racquet's force) would eventually reverse the momentum of the ball. Quantitatively, the effect we are talking about is the change in momentum $\Delta\mathbf{p}$.

By rearranging the equation $\mathbf{F}_{\text{net}}=\frac{\Delta\mathbf{p}}{\Delta t}$ to be

$$\Delta\mathbf{p}=\mathbf{F}_{\text{net}}\Delta t, \tag{1.19}$$

we can see how the change in momentum equals the average net external force multiplied by the time this force acts. The quantity $\mathbf{F}_{\text{net}}\Delta t$ is given the name **impulse**. Impulse is the same as the change in momentum.

Impulse: Change in Momentum

Change in momentum equals the average net external force multiplied by the time this force acts.

$$\Delta\mathbf{p}=\mathbf{F}_{\text{net}}\Delta t \tag{1.20}$$

The quantity $\mathbf{F}_{\text{net}}\Delta t$ is given the name impulse.

There are many ways in which an understanding of impulse can save lives, or at least limbs. The dashboard padding in a car, and certainly the airbags, allow the net force on the occupants in the car to act over a much longer time when there is a sudden stop. The momentum change is the same for an occupant, whether an air bag is deployed or not, but the force (to bring the occupant to a stop) will be much less if it acts over a larger time. Cars today have many plastic components. One advantage of plastics is their lighter weight, which results in better gas mileage. Another advantage is that a car will crumple in a collision, especially in the event of a head-on collision. A longer collision time means the force on the car will be less. Deaths during car races decreased dramatically when the rigid frames of racing cars were replaced with parts that could crumple or collapse in the event of an accident.

Bones in a body will fracture if the force on them is too large. If you jump onto the floor from a table, the force on your legs can be immense if you land stiff-legged on a hard surface. Rolling on the ground after jumping from the table, or landing with a parachute, extends the time over which the force (on you from the ground) acts.

Example 1.3 Calculating Magnitudes of Impulses: Two Billiard Balls Striking a Rigid Wall

Two identical billiard balls strike a rigid wall with the same speed, and are reflected without any change of speed. The first ball strikes perpendicular to the wall. The second ball strikes the wall at an angle of 30^o from the perpendicular, and bounces off at an angle of 30^o from perpendicular to the wall.

(a) Determine the direction of the force on the wall due to each ball.

(b) Calculate the ratio of the magnitudes of impulses on the two balls by the wall.

Strategy for (a)

In order to determine the force on the wall, consider the force on the ball due to the wall using Newton's second law and then apply Newton's third law to determine the direction. Assume the x-axis to be normal to the wall and to be positive in the initial direction of motion. Choose the y-axis to be along the wall in the plane of the second ball's motion. The momentum direction and the velocity direction are the same.

Solution for (a)

The first ball bounces directly into the wall and exerts a force on it in the $+x$ direction. Therefore the wall exerts a force on the ball in the $-x$ direction. The second ball continues with the same momentum component in the y direction, but reverses its x-component of momentum, as seen by sketching a diagram of the angles involved and keeping in mind the proportionality between velocity and momentum.

These changes mean the change in momentum for both balls is in the $-x$ direction, so the force of the wall on each ball is along the $-x$ direction.

Strategy for (b)

Calculate the change in momentum for each ball, which is equal to the impulse imparted to the ball.

Solution for (b)

Let u be the speed of each ball before and after collision with the wall, and m the mass of each ball. Choose the x-axis and y-axis as previously described, and consider the change in momentum of the first ball which strikes perpendicular to the wall.

$$p_{\text{xi}} = mu;\ p_{\text{yi}} = 0 \tag{1.21}$$

$$p_{\text{xf}} = -mu;\ p_{\text{yf}} = 0 \tag{1.22}$$

Impulse is the change in momentum vector. Therefore the x-component of impulse is equal to $-2mu$ and the y-component of impulse is equal to zero.

Now consider the change in momentum of the second ball.

$$p_{\text{xi}} = mu\cos 30^\text{o};\ p_{\text{yi}} = -mu\sin 30^\text{o} \tag{1.23}$$

$$p_{\text{xf}} = -mu\cos 30^\text{o};\ p_{\text{yf}} = -mu\sin 30^\text{o} \tag{1.24}$$

It should be noted here that while p_x changes sign after the collision, p_y does not. Therefore the x-component of impulse is equal to $-2mu\cos 30^\text{o}$ and the y-component of impulse is equal to zero.

The ratio of the magnitudes of the impulse imparted to the balls is

$$\frac{2mu}{2mu\cos 30^\text{o}} = \frac{2}{\sqrt{3}} = 1.155. \tag{1.25}$$

Discussion

The direction of impulse and force is the same as in the case of (a); it is normal to the wall and along the negative x -direction. Making use of Newton's third law, the force on the wall due to each ball is normal to the wall along the positive x -direction.

UMASS AMHERST Instructor's Notes

Pay attention to the example above, as it provides a good example of how impulse and momentum are connected, as well as working with momentum and impulse as vector quantities. Also, this next line brings up an important idea to impulse. Most of the time, the force is not constant over time, however, you can think of the average force instead.

Our definition of impulse includes an assumption that the force is constant over the time interval Δt . *Forces are usually not constant*. Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force F_{eff} that produces the same result as the corresponding time-varying force. **Figure 1.1** shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times t_1 and t_2 . That area is equal to the area inside the rectangle bounded by F_{eff} , t_1 , and t_2 . Thus the impulses and their effects are the same for both the actual and effective forces.

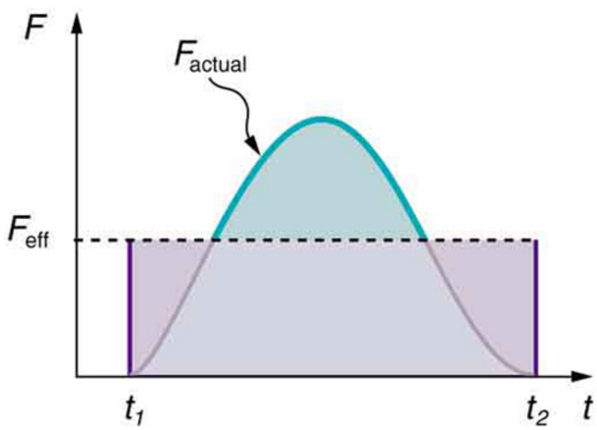


Figure 1.1 A graph of force versus time with time along the x -axis and force along the y -axis for an actual force and an equivalent effective force. The areas under the two curves are equal.

UMASS AMHERST Instructor's Notes

To get impulse from a graph, you can take the area under the curve, since impulse is force times time. The average force here is the force that makes a rectangle with the same area as the area under the curve, which would also result in the same impulse.

Making Connections: Take-Home Investigation—Hand Movement and Impulse

Try catching a ball while "giving" with the ball, pulling your hands toward your body. Then, try catching a ball while keeping your hands still. Hit water in a tub with your full palm. After the water has settled, hit the water again by diving your hand with your fingers first into the water. (Your full palm represents a swimmer doing a belly flop and your diving hand represents a swimmer doing a dive.) Explain what happens in each case and why. Which orientations would you advise people to avoid and why?

Making Connections: Constant Force and Constant Acceleration

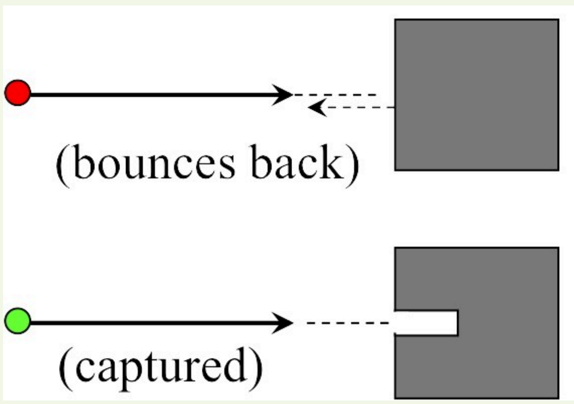
The assumption of a constant force in the definition of impulse is analogous to the assumption of a constant acceleration in kinematics. In both cases, nature is adequately described without the use of calculus.

UMASS AMHERST Instructor's Notes

Here's an additional example of interest. This example is from:
umdberg / Example: The impulse-momentum theorem. Available at: **http://umdberg.pbworks.com/w/page/98831066/Example%3A%20The%20impulse-momentum%20theorem. (http://umdberg.pbworks.com/w/page/98831066/Example%3A%20The%20impulse-momentum%20theorem)** (Accessed: 24th July 2017)

Example 1.4 UMDBerg / Example: The impulse-momentum theorem

A box rests on an air table and can slide freely without friction. If a small frictionless puck is slid towards the box consider two situations: it bounces straight back with about the same velocity or it is captured. If the interaction times between the box and the puck are the same, which puck exerts a greater force on the box?



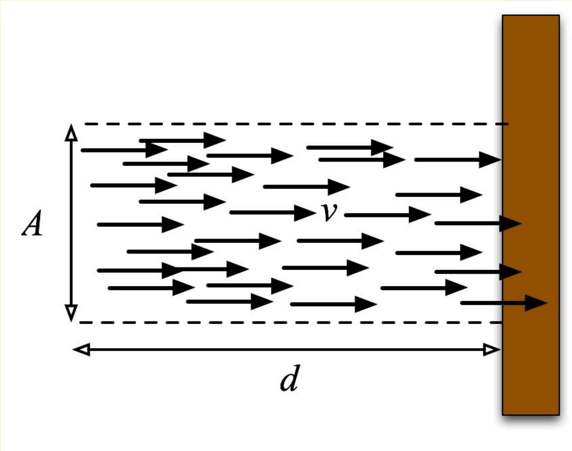
This is a rather trivial problem and doesn't seem very interest. We'll see however, in the next problem that it has interesting implications.

This is a qualitative problem, but we can still use an equation to solve it: the Impulse-momentum theorem. The change in momentum of the puck is equal to the impulse it receives from the box. The box seems much bigger than the puck so lets ignore the box's motion at first. If the puck has mass m and velocity v in the captured case the magnitude of the change in momentum is about mv -- it goes from mv to 0. In the bounces back case, the momentum of the puck goes from mv to $-mv$, so the magnitude of the change is $2mv$. (Remember that momentum is a vector quantity. It first decrease to 0, then decreases even further to negative values.) So the impulse received by the puck from the box is twice as big in the bounces back case as in the captured case. If the interaction times are the same (given) then the force the box exerts in the bounces back case is twice as big as in the captured case.

But that's the force of the box on the puck. What about the force of the puck on the box? Of course these two forces are related by **Newton's 3rd law (http://umdberg.pbworks.com/w/page/68390209/Newton's%203rd%20law%20(2013))** : In any interaction, the force that two objects exert on each other is equal and opposite. So if we know the force the box exerts on the puck, we know the force the puck exerts on the box.

This trivial case can be imbedded in a much more interesting case: molecules hitting a wall. Again, we will take only a simple case -- a stream of molecules in a vacuum. But we will see later that the same reasoning will allow us to understand how a gas exerts pressure and to extract the physical meaning of the ideal gas law in terms of molecules.

Suppose of stream of gas having cross sectional area A is traveling in a vacuum and is directed at a wall. If the density of molecules in the gas is n (number of molecules per cubic meter) and they are traveling with a speed v, what will be the average force that the molecules exert on the wall if (a) they stick to the wall, and (b) they bounce off the wall with the same speed they hit the wall with?



As is typical in any problem, there are assumptions hidden in the way the problem is stated and we have to figure out how to treat it. The wall is being bombarded by lots of little molecules. Each one that hits it will exert a sudden quick force on the wall and then so will the next, and the next, etc. So there will be lots of tiny little forces that vary quickly. The problem can't mean for us to calculate those -- there isn't enough information about the wall molecule interaction. But the fact that the problem uses a macroscopic word ("wall") and a microscopic word ("molecule") suggests that we might make some reasonable approximations.

So let's assume that we have lots of molecules in the gas and that they are moving fast. The word "average" suggests that we shouldn't focus on the individual fluctuations of the force but rather on the result of lots of molecules. Since "wall" implies much, much bigger than a molecule, let's assume that the wall doesn't move significantly when a molecule hits it. (A typical molecule has a mass on the order of 10^{-26} kg and a wall might have a mass of a few kgs.)

Each molecule that hits the wall changes its momentum. To get a force, we might use the impulse-momentum theorem. But that gives the force the wall exerts on the molecule. We want the force the molecule exerts on the wall! Of course these two forces are related by **Newton's 3rd law (http://umdberg.pbworks.com/w/page/68390209/Newton's%203rd%20law%20(2013))** : In any interaction, the force that two objects exert on each other is equal and opposite. So if we know the force the wall exerts on the molecule, we know the force the molecule exerts on the wall. Since the times during the interaction are equal, the impulse that the wall gives to the molecule must be equal and opposite to the impulse that the molecule gives to the wall.

This also resolves the time issue. On a time scale natural for the wall, lots of molecules will hit it. The impulse momentum theorem tells us the amount of impulse the wall must provide to a bunch of molecules in a certain time interval, Δ*t*. This will then tell us the amount of impulse the molecules provide to the wall in that time. Since we are told what happens to the velocities of the molecules, we can figure out their momentum change. Then we can calculate the average force the molecules exert on the wall.

$$\left\langle \vec{F}_{wall \to molecules} \right\rangle \Delta t = \Delta \vec{p}_{molecules}$$

$$\left\langle \vec{F}_{molecules \to wall} \right\rangle = \left\langle \vec{F}_{wall \to molecules} \right\rangle = \frac{\Delta \vec{p}_{molecules}}{\Delta t}$$

This is a rather standard way to use the Impulse-Momentum theorem. If you know the momentum change in a time interval, you can infer the impulse and therefore something about the average forces during that interval.

Let's see how that works here. Consider case (a): the molecules stick to the wall. In that case, the molecule initially had momentum mv and after the collision it basically stops. (Assuming that the wall doesn't recoil significantly. This of course depends on our assumptions about how big the wall is and how big the stream of gas is.) This means each molecule changes its momentum by an amount mv: from mv to 0.

Now let's consider a time interval in which many molecules will hit the wall. In a time interval, Δ*t,* how many will hit? To get this, look at the figure above. In a time interval, Δ*t*, a molecule will move a distance $d = v\Delta t$. If we take our distance *d* in the figure to be $v\Delta t$ then all the molecules in there will hit the wall and stick. How many is that? Well, we know the density and the volume of molecules hitting the wall is A x $d = Av\Delta t$. So the total number, *N*, hitting the wall in that time is the density times the volume or

N = number hitting the wall in time Δt = $nAv\Delta t$

So since each molecule changes its momentum by *mv*, the total change in momentum of the molecules in that time is *Nmv*, which gives a force

$$\left\langle \vec{F}_{molecules \to wall} \right\rangle = \left\langle \vec{F}_{wall \to molecules} \right\rangle = \frac{\Delta \vec{p}_{molecules}}{\Delta t} = \frac{mv(nAv\Delta t)}{\Delta t} = mnAv^2$$

For case (b), if each molecule bounces back with the same speed as it entered it changes its momentum from *mv* to *-mv*, a total change of 2*mv*. Therefore, the result will be twice as big as if the molecule stuck to the wall.

Joe Redish 8/6/15

Section Summary

- Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts:

$$\Delta \mathbf{p} = \mathbf{F}_{\text{net}} \Delta t. \quad (1.26)$$

- Forces are usually not constant over a period of time.

Conceptual Questions

Exercise 1.11

Professional Application

Explain in terms of impulse how padding reduces forces in a collision. State this in terms of a real example, such as the advantages of a carpeted vs. tile floor for a day care center.

Exercise 1.12

While jumping on a trampoline, sometimes you land on your back and other times on your feet. In which case can you reach a greater height and why?

Exercise 1.13

Professional Application

Tennis racquets have “sweet spots.” If the ball hits a sweet spot then the player's arm is not jarred as much as it would be otherwise. Explain why this is the case.

Problems & Exercises

Exercise 1.14

A bullet is accelerated down the barrel of a gun by hot gases produced in the combustion of gun powder. What is the average force exerted on a 0.0300-kg bullet to accelerate it to a speed of 600 m/s in a time of 2.00 ms (milliseconds)?

Solution

9.00×10^3 N

Exercise 1.15

Professional Application

A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seat belt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.

Exercise 1.16

A person slaps her leg with her hand, bringing her hand to rest in 2.50 milliseconds from an initial speed of 4.00 m/s. (a) What is the average force exerted on the leg, taking the effective mass of the hand and forearm to be 1.50 kg? (b) Would the force be any different if the woman clapped her hands together at the same speed and brought them to rest in the same time? Explain why or why not.

Solution

a) 2.40×10^3 N toward the leg

b) The force on each hand would have the same magnitude as that found in part (a) (but in opposite directions by Newton's third law) because the change in momentum and the time interval are the same.

Exercise 1.17

Professional Application

A professional boxer hits his opponent with a 1000-N horizontal blow that lasts for 0.150 s. (a) Calculate the impulse imparted by this blow. (b) What is the opponent's final velocity, if his mass is 105 kg and he is motionless in midair when struck near his center of mass? (c) Calculate the recoil velocity of the opponent's 10.0-kg head if hit in this manner, assuming the head does not initially transfer significant momentum to the boxer's body. (d) Discuss the implications of your answers for parts (b) and (c).

Exercise 1.18

Professional Application

Suppose a child drives a bumper car head on into the side rail, which exerts a force of 4000 N on the car for 0.200 s. (a) What impulse is imparted by this force? (b) Find the final velocity of the bumper car if its initial velocity was 2.80 m/s and the car plus driver have a mass of 200 kg. You may neglect friction between the car and floor.

Solution

a) $800 \text{ kg} \cdot \text{m/s}$ away from the wall

b) 1.20 m/s away from the wall

Exercise 1.19

Professional Application

One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of $4.00 \times 10^3 \text{ m/s}$, given the collision lasts $6.00 \times 10^{-8} \text{ s}$.

Exercise 1.20

Professional Application

A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment. (a) Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm. (b) Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.

Solution

(a) 1.50×10^6 N away from the dashboard

(b) 1.00×10^5 N away from the dashboard

Exercise 1.21

Professional Application

Military rifles have a mechanism for reducing the recoil forces of the gun on the person firing it. An internal part recoils over a relatively large distance and is stopped by damping mechanisms in the gun. The larger distance reduces the average force needed to stop the internal part. (a) Calculate the recoil velocity of a 1.00-kg plunger that directly interacts with a 0.0200-kg bullet fired at 600 m/s from the gun. (b) If this part is stopped over a distance of 20.0 cm, what average force is exerted upon it by the gun? (c) Compare this to the force exerted on the gun if the bullet is accelerated to its velocity in 10.0 ms (milliseconds).

Exercise 1.22

A cruise ship with a mass of 1.00×10^7 kg strikes a pier at a speed of 0.750 m/s. It comes to rest 6.00 m later, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (Hint: First calculate the time it took to bring the ship to rest.)

Solution

4.69×10^5 N in the boat's original direction of motion

Exercise 1.23

Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of 1.76×10^4 N for 5.50×10^{-2} s.

Exercise 1.24

Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the magnitude of the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.

Solution

2.10×10^3 N away from the wall

Exercise 1.25

A 0.450-kg hammer is moving horizontally at 7.00 m/s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?

Exercise 1.26

Starting with the definitions of momentum and kinetic energy, derive an equation for the kinetic energy of a particle expressed as a function of its momentum.

Solution

$$\begin{aligned} &\mathbf{p} = m\mathbf{v} \Rightarrow p^2 = m^2v^2 \Rightarrow \frac{p^2}{m} = mv^2 \\ &\Rightarrow \frac{p^2}{2m} = \frac{1}{2}mv^2 = KE \\ &KE = \frac{p^2}{2m} \end{aligned} \tag{1.27}$$

Exercise 1.27

A ball with an initial velocity of 10 m/s moves at an angle 60° above the $+x$-direction. The ball hits a vertical wall and bounces off so that it is moving 60° above the $-x$-direction with the same speed. What is the impulse delivered by the wall?

Exercise 1.28

When serving a tennis ball, a player hits the ball when its velocity is zero (at the highest point of a vertical toss). The racquet exerts a force of 540 N on the ball for 5.00 ms, giving it a final velocity of 45.0 m/s. Using these data, find the mass of the ball.

Solution

60.0 g

Exercise 1.29

A punter drops a ball from rest vertically 1 meter down onto his foot. The ball leaves the foot with a speed of 18 m/s at an angle 55° above the horizontal. What is the impulse delivered by the foot (magnitude and direction)?

Glossary

change in momentum: the difference between the final and initial momentum; the mass times the change in velocity

impulse: the average net external force times the time it acts; equal to the change in momentum

linear momentum: the product of mass and velocity

second law of motion: physical law that states that the net external force equals the change in momentum of a system divided by the time over which it changes

2 CONSERVATION OF MOMENTUM

2.1 Introduction

This chapter covers conservation of momentum. However, this is not something we will be covering in this course, so reading this chapter is not required, and has been included for your reference.

2.2 Conservation of Momentum

Momentum is an important quantity because it is conserved. Yet it was not conserved in the examples in **Impulse (https://legacy.cnx.org/content/m42159/latest/)** and **Linear Momentum and Force (https://legacy.cnx.org/content/m42156/latest/)** , where large changes in momentum were produced by forces acting on the system of interest. Under what circumstances is momentum conserved?

The answer to this question entails considering a sufficiently large system. It is always possible to find a larger system in which total momentum is constant, even if momentum changes for components of the system. If a football player runs into the goalpost in the end zone, there will be a force on him that causes him to bounce backward. However, the Earth also recoils —conserving momentum—because of the force applied to it through the goalpost. Because Earth is many orders of magnitude more massive than the player, its recoil is immeasurably small and can be neglected in any practical sense, but it is real nevertheless.

Consider what happens if the masses of two colliding objects are more similar than the masses of a football player and Earth—for example, one car bumping into another, as shown in **Figure 2.1**. Both cars are coasting in the same direction when the lead car (labeled m_2) is bumped by the trailing car (labeled m_1). The only unbalanced force on each car is the force of the collision. (Assume that the effects due to friction are negligible.) Car 1 slows down as a result of the collision, losing some momentum, while car 2 speeds up and gains some momentum. We shall now show that the total momentum of the two-car system remains constant.

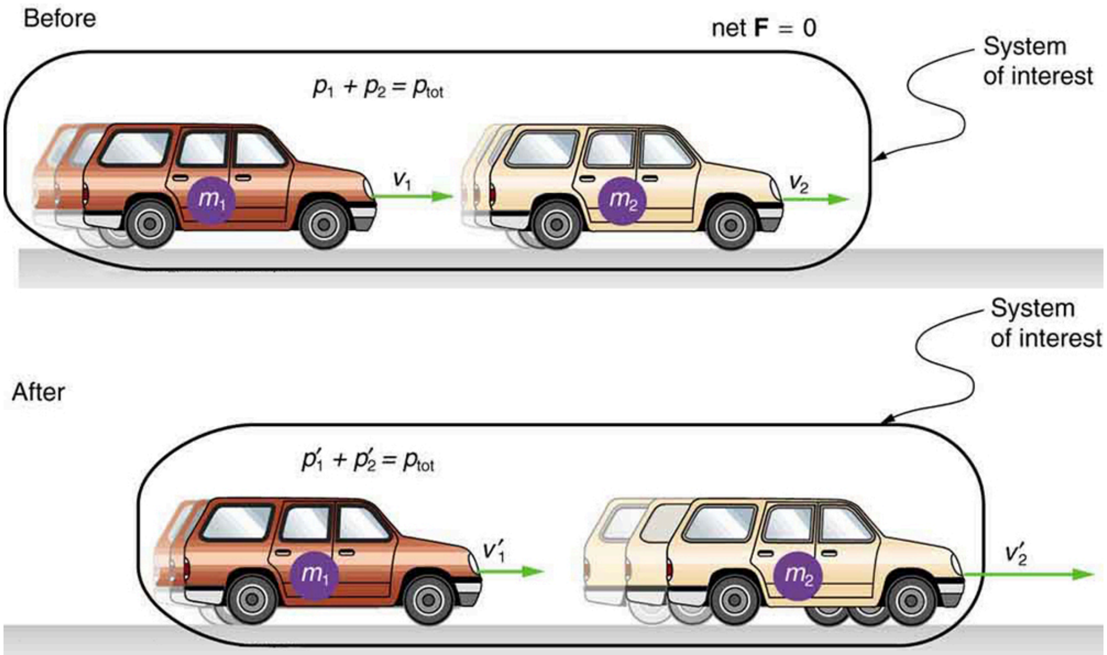


Figure 2.1 A car of mass m_1 moving with a velocity of v_1 bumps into another car of mass m_2 and velocity v_2 that it is following. As a result, the first car slows down to a velocity of v'_1 and the second speeds up to a velocity of v'_2. The momentum of each car is changed, but the total momentum p_{tot} of the two cars is the same before and after the collision (if you assume friction is negligible).

Using the definition of impulse, the change in momentum of car 1 is given by

$$\Delta p_1 = F_1 \Delta t, \tag{2.1}$$

where F_1 is the force on car 1 due to car 2, and Δt is the time the force acts (the duration of the collision). Intuitively, it seems obvious that the collision time is the same for both cars, but it is only true for objects traveling at ordinary speeds. This assumption must be modified for objects travelling near the speed of light, without affecting the result that momentum is conserved.

Similarly, the change in momentum of car 2 is

$$\Delta p_2 = F_2 \Delta t, \tag{2.2}$$

where F_2 is the force on car 2 due to car 1, and we assume the duration of the collision Δt is the same for both cars. We know from Newton's third law that $F_2 = -F_1$, and so

$$\Delta p_2 = -F_1 \Delta t = -\Delta p_1. \tag{2.3}$$

Thus, the changes in momentum are equal and opposite, and

$$\Delta p_1 + \Delta p_2 = 0. \tag{2.4}$$

Because the changes in momentum add to zero, the total momentum of the two-car system is constant. That is,

$$p_1 + p_2 = \text{constant}, \tag{2.5}$$

$$p_1 + p_2 = p'_1 + p'_2, \tag{2.6}$$

where p'_1 and p'_2 are the momenta of cars 1 and 2 after the collision. (We often use primes to denote the final state.)

This result—that momentum is conserved—has validity far beyond the preceding one-dimensional case. It can be similarly shown that total momentum is conserved for any isolated system, with any number of objects in it. In equation form, the **conservation of momentum principle** for an isolated system is written

$$\mathbf{p}_{\text{tot}} = \text{constant}, \tag{2.7}$$

or

$$\mathbf{p}_{\text{tot}} = \mathbf{p}'_{\text{tot}}, \tag{2.8}$$

where $\mathbf{p}_{\text{tot}}$ is the total momentum (the sum of the momenta of the individual objects in the system) and $\mathbf{p}'_{\text{tot}}$ is the total momentum some time later. (The total momentum can be shown to be the momentum of the center of mass of the system.) An **isolated system** is defined to be one for which the net external force is zero $(\mathbf{F}_{\text{net}} = 0)$.

Conservation of Momentum Principle

$$\begin{aligned} \mathbf{p}_{\text{tot}} &= \text{constant} \\ \mathbf{p}_{\text{tot}} &= \mathbf{p}'_{\text{tot}} \quad \text{(isolated system)} \end{aligned} \tag{2.9}$$

Isolated System

An isolated system is defined to be one for which the net external force is zero $(\mathbf{F}_{\text{net}} = 0)$.

Perhaps an easier way to see that momentum is conserved for an isolated system is to consider Newton's second law in terms of momentum, $\mathbf{F}_{\text{net}} = \dfrac{\Delta \mathbf{p}_{\text{tot}}}{\Delta t}$. For an isolated system, $(\mathbf{F}_{\text{net}} = 0)$; thus, $\Delta \mathbf{p}_{\text{tot}} = 0$, and $\mathbf{p}_{\text{tot}}$ is constant.

We have noted that the three length dimensions in nature—x, y, and z—are independent, and it is interesting to note that momentum can be conserved in different ways along each dimension. For example, during projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero and momentum is unchanged. But along the vertical direction, the net vertical force is not zero and the momentum of the projectile is not conserved. (See **Figure 2.2**.) However, if the momentum of the projectile-Earth system is considered in the vertical direction, we find that the total momentum is conserved.

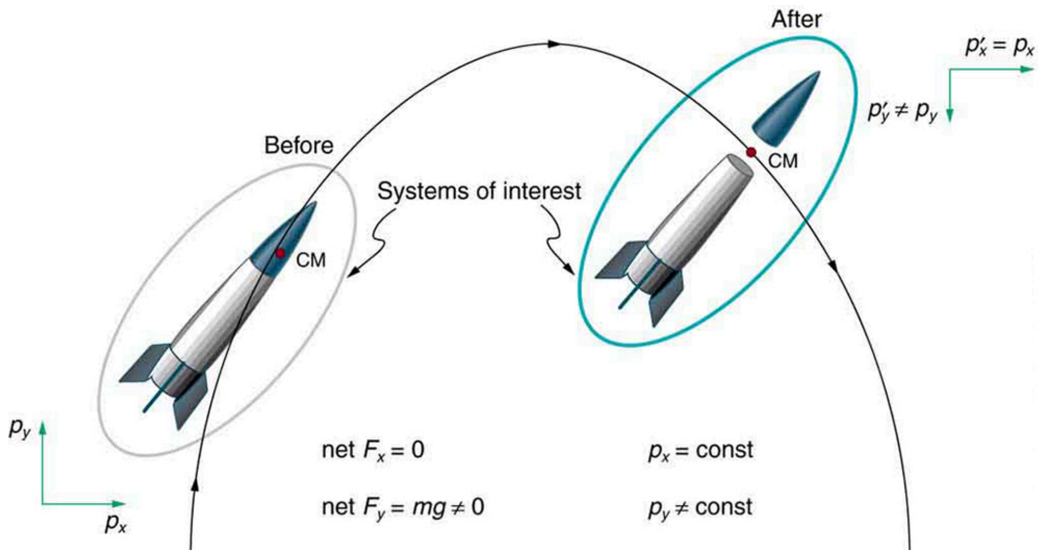


Figure 2.2 The horizontal component of a projectile's momentum is conserved if air resistance is negligible, even in this case where a space probe separates. The forces causing the separation are internal to the system, so that the net external horizontal force $F_{x-\text{net}}$ is still zero. The vertical component of the momentum is not conserved, because the net vertical force $F_{y-\text{net}}$ is not zero. In the vertical direction, the space probe-Earth system needs to be considered and we find that the total momentum is conserved. The center of mass of the space probe takes the same path it would if the separation did not occur.

The conservation of momentum principle can be applied to systems as different as a comet striking Earth and a gas containing huge numbers of atoms and molecules. Conservation of momentum is violated only when the net external force is not zero. But another larger system can always be considered in which momentum is conserved by simply including the source of the external force. For example, in the collision of two cars considered above, the two-car system conserves momentum while each one-car system does not.

Making Connections: Take-Home Investigation—Drop of Tennis Ball and a Basketball

Hold a tennis ball side by side and in contact with a basketball. Drop the balls together. (Be careful!) What happens? Explain your observations. Now hold the tennis ball above and in contact with the basketball. What happened? Explain your observations. What do you think will happen if the basketball ball is held above and in contact with the tennis ball?

Making Connections: Take-Home Investigation—Two Tennis Balls in a Ballistic Trajectory

Tie two tennis balls together with a string about a foot long. Hold one ball and let the other hang down and throw it in a ballistic trajectory. Explain your observations. Now mark the center of the string with bright ink or attach a brightly colored sticker to it and throw again. What happened? Explain your observations.

Some aquatic animals such as jellyfish move around based on the principles of conservation of momentum. A jellyfish fills its umbrella section with water and then pushes the water out resulting in motion in the opposite direction to that of the jet of water. Squids propel themselves in a similar manner but, in contrast with jellyfish, are able to control the direction in which they move by aiming their nozzle forward or backward. Typical squids can move at speeds of 8 to 12 km/h.

The ballistocardiograph (BCG) was a diagnostic tool used in the second half of the 20th century to study the strength of the heart. About once a second, your heart beats, forcing blood into the aorta. A force in the opposite direction is exerted on the rest of your body (recall Newton's third law). A ballistocardiograph is a device that can measure this reaction force. This measurement is done by using a sensor (resting on the person) or by using a moving table suspended from the ceiling. This technique can gather information on the strength of the heart beat and the volume of blood passing from the heart. However, the electrocardiogram (ECG or EKG) and the echocardiogram (cardiac ECHO or ECHO; a technique that uses ultrasound to see an image of the heart) are more widely used in the practice of cardiology.

Making Connections: Conservation of Momentum and Collision

Conservation of momentum is quite useful in describing collisions. Momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

Subatomic Collisions and Momentum

The conservation of momentum principle not only applies to the macroscopic objects, it is also essential to our explorations of atomic and subatomic particles. Giant machines hurl subatomic particles at one another, and researchers evaluate the results by assuming conservation of momentum (among other things).

On the small scale, we find that particles and their properties are invisible to the naked eye but can be measured with our instruments, and models of these subatomic particles can be constructed to describe the results. Momentum is found to be a property of all subatomic particles including massless particles such as photons that compose light. Momentum being a property of particles hints that momentum may have an identity beyond the description of an object's mass multiplied by the object's velocity. Indeed, momentum relates to wave properties and plays a fundamental role in what measurements are taken and how we take these measurements. Furthermore, we find that the conservation of momentum principle is valid when considering systems of particles. We use this principle to analyze the masses and other properties of previously undetected particles, such as the nucleus of an atom and the existence of quarks that make up particles of nuclei. Figure 2.3 below illustrates how a particle scattering backward from another implies that its target is massive and dense. Experiments seeking evidence that **quarks** make up protons (one type of particle that makes up nuclei) scattered high-energy electrons off of protons (nuclei of hydrogen atoms). Electrons occasionally scattered straight backward in a manner that implied a very small and very dense particle makes up the proton—this observation is considered nearly direct evidence of quarks. The analysis was based partly on the same conservation of momentum principle that works so well on the large scale.

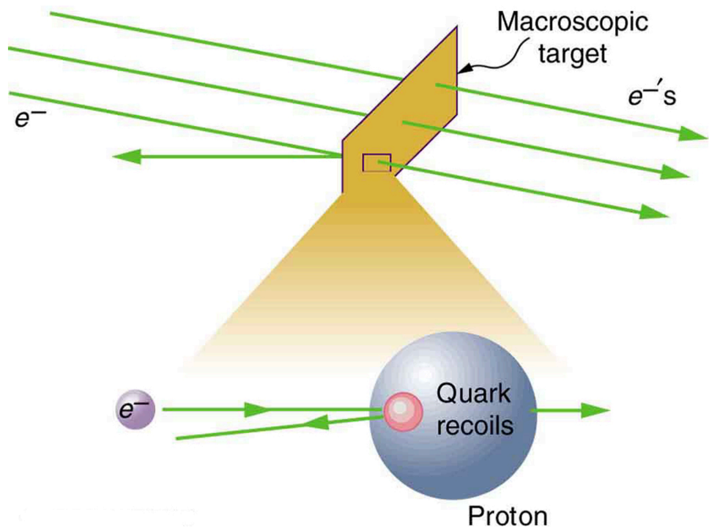


Figure 2.3 A subatomic particle scatters straight backward from a target particle. In experiments seeking evidence for quarks, electrons were observed to occasionally scatter straight backward from a proton.

Section Summary

- The conservation of momentum principle is written

$$\mathbf{p}_{\text{tot}} = \text{constant} \tag{2.10}$$

or

$$\mathbf{p}_{\text{tot}} = \mathbf{p}'_{\text{tot}} \quad \text{(isolated system)}, \tag{2.11}$$

$\mathbf{p}_{\text{tot}}$ is the initial total momentum and $\mathbf{p}'_{\text{tot}}$ is the total momentum some time later.
- An isolated system is defined to be one for which the net external force is zero $(\mathbf{F}_{\text{net}} = 0)$.
- During projectile motion and where air resistance is negligible, momentum is conserved in the horizontal direction because horizontal forces are zero.
- Conservation of momentum applies only when the net external force is zero.
- The conservation of momentum principle is valid when considering systems of particles.

Conceptual Questions

Exercise 2.1

Professional Application

If you dive into water, you reach greater depths than if you do a belly flop. Explain this difference in depth using the concept of conservation of energy. Explain this difference in depth using what you have learned in this chapter.

Exercise 2.2

Under what circumstances is momentum conserved?

Exercise 2.3

Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?

Exercise 2.4

Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.

Exercise 2.5

Professional Application

Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.

Exercise 2.6

Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.

Exercise 2.7

Must the total energy of a system be conserved whenever its momentum is conserved? Explain why or why not.

Problems & Exercises

Exercise 2.8

Professional Application

Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of 150,000 kg and a velocity of 0.300 m/s, and the second having a mass of 110,000 kg and a velocity of $-0.120\ \text{m/s}$. (The minus indicates direction of motion.) What is their final velocity?

Solution
0.122 m/s

Exercise 2.9

Suppose a clay model of a koala bear has a mass of 0.200 kg and slides on ice at a speed of 0.750 m/s. It runs into another clay model, which is initially motionless and has a mass of 0.350 kg. Both being soft clay, they naturally stick together. What is their final velocity?

Exercise 2.10

Professional Application

Consider the following question: *A car moving at 10 m/s crashes into a tree and stops in 0.26 s. Calculate the force the seatbelt exerts on a passenger in the car to bring him to a halt. The mass of the passenger is 70 kg.* Would the answer to this question be different if the car with the 70-kg passenger had collided with a car that has a mass equal to and is traveling in the opposite direction and at the same speed? Explain your answer.

Solution
In a collision with an identical car, momentum is conserved. Afterwards $v_{\text{f}} = 0$ for both cars. The change in momentum will be the same as in the crash with the tree. However, the force on the body is not determined since the time is not known. A padded stop will reduce injurious force on body.

Exercise 2.11

What is the velocity of a 900-kg car initially moving at 30.0 m/s, just after it hits a 150-kg deer initially running at 12.0 m/s in the same direction? Assume the deer remains on the car.

Exercise 2.12

A 1.80-kg falcon catches a 0.650-kg dove from behind in midair. What is their velocity after impact if the falcon's velocity is initially 28.0 m/s and the dove's velocity is 7.00 m/s in the same direction?

Solution
22.4 m/s in the same direction as the original motion

2.3 Elastic Collisions in One Dimension

Let us consider various types of two-object collisions. These collisions are the easiest to analyze, and they illustrate many of the physical principles involved in collisions. The conservation of momentum principle is very useful here, and it can be used whenever the net external force on a system is zero.

We start with the elastic collision of two objects moving along the same line—a one-dimensional problem. An **elastic collision** is one that also conserves internal kinetic energy. **Internal kinetic energy** is the sum of the kinetic energies of the objects in the system. **Figure 2.4** illustrates an elastic collision in which internal kinetic energy and momentum are conserved.

Truly elastic collisions can only be achieved with subatomic particles, such as electrons striking nuclei. Macroscopic collisions can be very nearly, but not quite, elastic—some kinetic energy is always converted into other forms of energy such as heat transfer due to friction and sound. One macroscopic collision that is nearly elastic is that of two steel blocks on ice. Another nearly elastic collision is that between two carts with spring bumpers on an air track. Icy surfaces and air tracks are nearly frictionless, more readily allowing nearly elastic collisions on them.

Elastic Collision

An **elastic collision** is one that conserves internal kinetic energy.

Internal Kinetic Energy

Internal kinetic energy is the sum of the kinetic energies of the objects in the system.

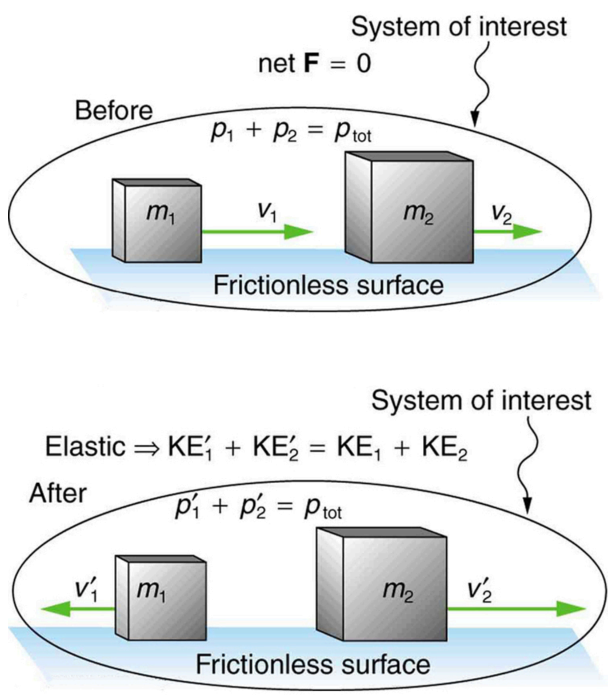


Figure 2.4 An elastic one-dimensional two-object collision. Momentum and internal kinetic energy are conserved.

Now, to solve problems involving one-dimensional elastic collisions between two objects we can use the equations for conservation of momentum and conservation of internal kinetic energy. First, the equation for conservation of momentum for two objects in a one-dimensional collision is

$$p_1 + p_2 = p'_1 + p'_2 \quad (F_{\text{net}} = 0) \tag{2.12}$$

or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 \quad (F_{\text{net}} = 0), \tag{2.13}$$

where the primes (') indicate values after the collision. By definition, an elastic collision conserves internal kinetic energy, and so the sum of kinetic energies before the collision equals the sum after the collision. Thus,

$$\frac{1}{2}m_1 v_1{}^2 + \frac{1}{2}m_2 v_2{}^2 = \frac{1}{2}m_1 v'_1{}^2 + \frac{1}{2}m_2 v'_2{}^2 \quad \text{(two-object elastic collision)} \tag{2.14}$$

expresses the equation for conservation of internal kinetic energy in a one-dimensional collision.

Example 2.1 Calculating Velocities Following an Elastic Collision

Calculate the velocities of two objects following an elastic collision, given that

$$m_1 = 0.500 \text{ kg}, \ m_2 = 3.50 \text{ kg}, \ v_1 = 4.00 \text{ m/s, and } v_2 = 0. \tag{2.15}$$

Strategy and Concept

First, visualize what the initial conditions mean—a small object strikes a larger object that is initially at rest. This situation is slightly simpler than the situation shown in **Figure 2.4** where both objects are initially moving. We are asked to find two unknowns (the final velocities v'_1 and v'_2). To find two unknowns, we must use two independent equations. Because this collision is elastic, we can use the above two equations. Both can be simplified by the fact that object 2 is initially at rest, and thus $v_2 = 0$. Once we simplify these equations, we combine them algebraically to solve for the unknowns.

Solution

For this problem, note that $v_2 = 0$ and use conservation of momentum. Thus,

$$p_1 = p'_1 + p'_2 \tag{2.16}$$

or

$$m_1 v_1 = m_1 v'_1 + m_2 v'_2. \tag{2.17}$$

Using conservation of internal kinetic energy and that $v_2 = 0$,

$$\frac{1}{2}m_1 v_1{}^2 = \frac{1}{2}m_1 v'_1{}^2 + \frac{1}{2}m_2 v'_2{}^2. \tag{2.18}$$

Solving the first equation (momentum equation) for v'_2, we obtain

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1). \tag{2.19}$$

Substituting this expression into the second equation (internal kinetic energy equation) eliminates the variable v'_2, leaving only v'_1 as an unknown (the algebra is left as an exercise for the reader). There are two solutions to any quadratic equation; in this example, they are

$$v'_1 = 4.00 \text{ m/s} \tag{2.20}$$

and

$$v'_1 = -3.00 \text{ m/s}. \tag{2.21}$$

As noted when quadratic equations were encountered in earlier chapters, both solutions may or may not be meaningful. In this case, the first solution is the same as the initial condition. The first solution thus represents the situation before the collision and is discarded. The second solution $(v'_1 = -3.00 \text{ m/s})$ is negative, meaning that the first object bounces backward. When this negative value of v'_1 is used to find the velocity of the second object after the collision, we get

$$v'_2 = \frac{m_1}{m_2}(v_1 - v'_1) = \frac{0.500 \text{ kg}}{3.50 \text{ kg}}[4.00 - (-3.00)] \text{ m/s} \tag{2.22}$$

or

$$v'_2 = 1.00 \text{ m/s}. \tag{2.23}$$

Discussion

The result of this example is intuitively reasonable. A small object strikes a larger one at rest and bounces backward. The larger one is knocked forward, but with a low speed. (This is like a compact car bouncing backward off a full-size SUV that is initially at rest.) As a check, try calculating the internal kinetic energy before and after the collision. You will see that the internal kinetic energy is unchanged at 4.00 J. Also check the total momentum before and after the collision; you will find it, too, is unchanged.

The equations for conservation of momentum and internal kinetic energy as written above can be used to describe any one-dimensional elastic collision of two objects. These equations can be extended to more objects if needed.

Making Connections: Take-Home Investigation—Ice Cubes and Elastic Collision

Find a few ice cubes which are about the same size and a smooth kitchen tabletop or a table with a glass top. Place the ice cubes on the surface several centimeters away from each other. Flick one ice cube toward a stationary ice cube and observe the path and velocities of the ice cubes after the collision. Try to avoid edge-on collisions and collisions with rotating ice cubes. Have you created approximately elastic collisions? Explain the speeds and directions of the ice cubes using momentum.

PhET Explorations: Collision Lab

Investigate collisions on an air hockey table. Set up your own experiments: vary the number of discs, masses and initial conditions. Is momentum conserved? Is kinetic energy conserved? Vary the elasticity and see what happens.

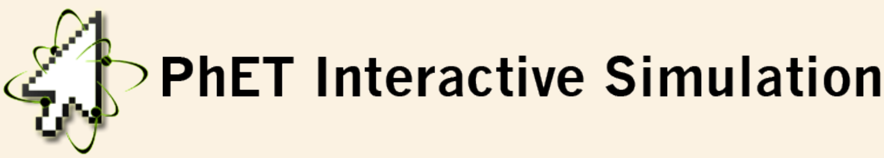


Figure 2.5 Collision Lab (http://cnx.org/content/m42163/1.3/collision-lab_en.jar)

Section Summary

- An elastic collision is one that conserves internal kinetic energy.
- Conservation of kinetic energy and momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one dimensional two-body collisions.

Conceptual Questions

Exercise 2.13

What is an elastic collision?

Problems & Exercises

Exercise 2.14

Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.

Exercise 2.15

Professional Application

Two manned satellites approach one another at a relative speed of 0.250 m/s, intending to dock. The first has a mass of $4.00\times10^3 \text{ kg}$, and the second a mass of $7.50\times10^3 \text{ kg}$. If the two satellites collide elastically rather than dock, what is their final relative velocity?

Solution

0.250 m/s

Exercise 2.16

A 70.0-kg ice hockey goalie, originally at rest, catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. Suppose the goalie and the ice puck have an elastic collision and the puck is reflected back in the direction from which it came. What would their final velocities be in this case?

2.4 Inelastic Collisions in One Dimension

We have seen that in an elastic collision, internal kinetic energy is conserved. An **inelastic collision** is one in which the internal kinetic energy changes (it is not conserved). This lack of conservation means that the forces between colliding objects may remove or add internal kinetic energy. Work done by internal forces may change the forms of energy within a system. For inelastic collisions, such as when colliding objects stick together, this internal work may transform some internal kinetic energy into heat transfer. Or it may convert stored energy into internal kinetic energy, such as when exploding bolts separate a satellite from its launch vehicle.

Inelastic Collision

An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).

Figure 2.6 shows an example of an inelastic collision. Two objects that have equal masses head toward one another at equal speeds and then stick together. Their total internal kinetic energy is initially $\frac{1}{2}mv^2 + \frac{1}{2}mv^2 = mv^2$. The two objects come to rest after sticking together, conserving momentum. But the internal kinetic energy is zero after the collision. A collision in which the objects stick together is sometimes called a **perfectly inelastic collision** because it reduces internal kinetic energy more than does any other type of inelastic collision. In fact, such a collision reduces internal kinetic energy to the minimum it can have while still conserving momentum.

Perfectly Inelastic Collision

A collision in which the objects stick together is sometimes called "perfectly inelastic."

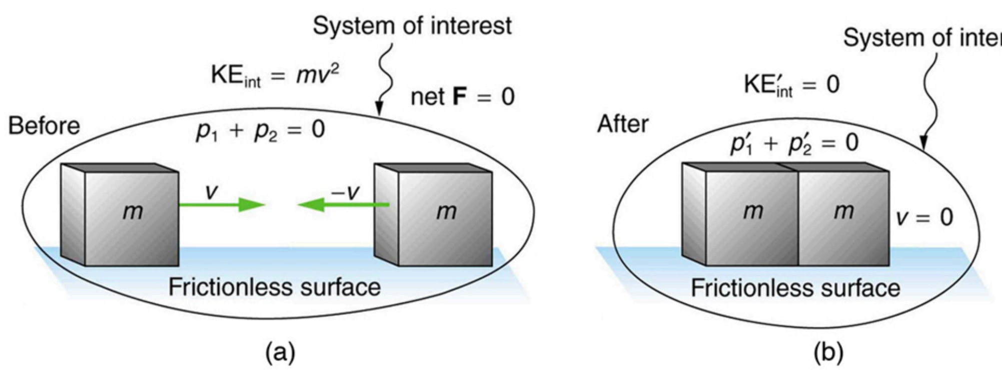


Figure 2.6 An inelastic one-dimensional two-object collision. Momentum is conserved, but internal kinetic energy is not conserved. (a) Two objects of equal mass initially head directly toward one another at the same speed. (b) The objects stick together (a perfectly inelastic collision), and so their final velocity is zero. The internal kinetic energy of the system changes in any inelastic collision and is reduced to zero in this example.

Example 2.2 Calculating Velocity and Change in Kinetic Energy: Inelastic Collision of a Puck and a Goalie

(a) Find the recoil velocity of a 70.0-kg ice hockey goalie, originally at rest, who catches a 0.150-kg hockey puck slapped at him at a velocity of 35.0 m/s. (b) How much kinetic energy is lost during the collision? Assume friction between the ice and the puck-goalie system is negligible. (See Figure 2.7)

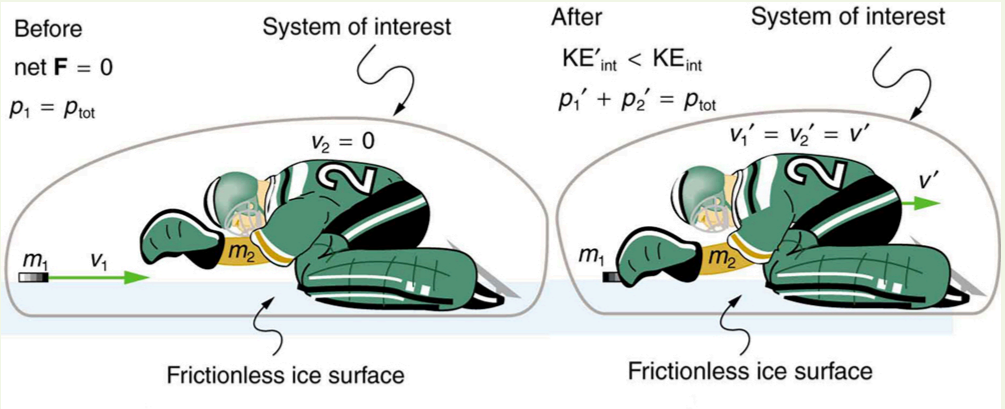


Figure 2.7 An ice hockey goalie catches a hockey puck and recoils backward. The initial kinetic energy of the puck is almost entirely converted to thermal energy and sound in this inelastic collision.

Strategy

Momentum is conserved because the net external force on the puck-goalie system is zero. We can thus use conservation of momentum to find the final velocity of the puck and goalie system. Note that the initial velocity of the goalie is zero and that the final velocity of the puck and goalie are the same. Once the final velocity is found, the kinetic energies can be calculated before and after the collision and compared as requested.

Solution for (a)

Momentum is conserved because the net external force on the puck-goalie system is zero.

Conservation of momentum is

$$p_1 + p_2 = p'_1 + p'_2 \tag{2.24}$$

or

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2. \tag{2.25}$$

Because the goalie is initially at rest, we know $v_2 = 0$. Because the goalie catches the puck, the final velocities are equal, or $v'_1 = v'_2 = v'$. Thus, the conservation of momentum equation simplifies to

$$m_1 v_1 = (m_1 + m_2)v'. \tag{2.26}$$

Solving for v' yields

$$v' = \frac{m_1}{m_1 + m_2} v_1. \tag{2.27}$$

Entering known values in this equation, we get

$$v' = \left(\frac{0.150 \text{ kg}}{70.0 \text{ kg} + 0.150 \text{ kg}}\right)(35.0 \text{ m/s}) = 7.48 \times 10^{-2} \text{ m/s}. \tag{2.28}$$

Discussion for (a)

This recoil velocity is small and in the same direction as the puck's original velocity, as we might expect.

Solution for (b)

Before the collision, the internal kinetic energy KE_{int} of the system is that of the hockey puck, because the goalie is initially at rest. Therefore, KE_{int} is initially

$$\begin{aligned} \text{KE}_{\text{int}} &= \frac{1}{2}mv^2 = \frac{1}{2}(0.150 \text{ kg})(35.0 \text{ m/s})^2 \\ &= 91.9 \text{ J}. \end{aligned} \tag{2.29}$$

After the collision, the internal kinetic energy is

$$\begin{aligned} \text{KE}'_{\text{int}} &= \frac{1}{2}(m + M)v^2 = \frac{1}{2}(70.15 \text{ kg})\left(7.48 \times 10^{-2} \text{ m/s}\right)^2 \\ &= 0.196 \text{ J}. \end{aligned} \tag{2.30}$$

The change in internal kinetic energy is thus

$$\begin{aligned}\mathrm{KE'}_{\mathrm{int}} - \mathrm{KE}_{\mathrm{int}} &= 0.196\ \mathrm{J} - 91.9\ \mathrm{J} \\ &= -91.7\ \mathrm{J}\end{aligned} \tag{2.31}$$

where the minus sign indicates that the energy was lost.

Discussion for (b)

Nearly all of the initial internal kinetic energy is lost in this perfectly inelastic collision. $\mathrm{KE}_{\mathrm{int}}$ is mostly converted to thermal energy and sound.

During some collisions, the objects do not stick together and less of the internal kinetic energy is removed—such as happens in most automobile accidents. Alternatively, stored energy may be converted into internal kinetic energy during a collision. **Figure 2.8** shows a one-dimensional example in which two carts on an air track collide, releasing potential energy from a compressed spring. **Example 2.3** deals with data from such a collision.

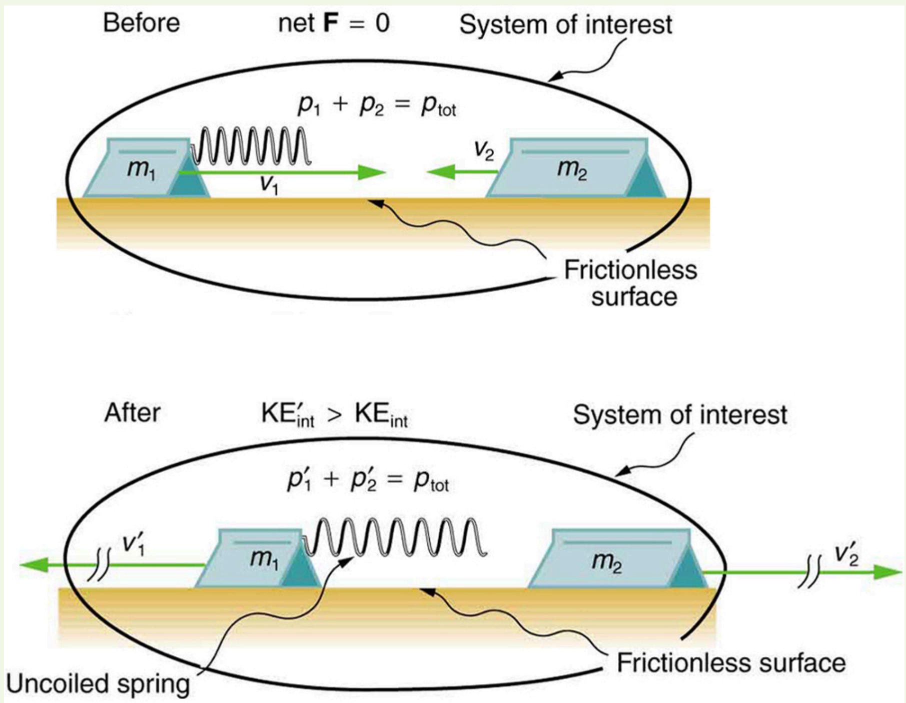


Figure 2.8 An air track is nearly frictionless, so that momentum is conserved. Motion is one-dimensional. In this collision, examined in **Example 2.3**, the potential energy of a compressed spring is released during the collision and is converted to internal kinetic energy.

Collisions are particularly important in sports and the sporting and leisure industry utilizes elastic and inelastic collisions. Let us look briefly at tennis. Recall that in a collision, it is momentum and not force that is important. So, a heavier tennis racquet will have the advantage over a lighter one. This conclusion also holds true for other sports—a lightweight bat (such as a softball bat) cannot hit a hardball very far.

The location of the impact of the tennis ball on the racquet is also important, as is the part of the stroke during which the impact occurs. A smooth motion results in the maximizing of the velocity of the ball after impact and reduces sports injuries such as tennis elbow. A tennis player tries to hit the ball on the “sweet spot” on the racquet, where the vibration and impact are minimized and the ball is able to be given more velocity. Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Take-Home Experiment—Bouncing of Tennis Ball

1. Find a racquet (a tennis, badminton, or other racquet will do). Place the racquet on the floor and stand on the handle. Drop a tennis ball on the strings from a measured height. Measure how high the ball bounces. Now ask a friend to hold the racquet firmly by the handle and drop a tennis ball from the same measured height above the racquet. Measure how high the ball bounces and observe what happens to your friend’s hand during the collision. Explain your observations and measurements.
2. The coefficient of restitution (c) is a measure of the elasticity of a collision between a ball and an object, and is defined as the ratio of the speeds after and before the collision. A perfectly elastic collision has a c of 1. For a ball bouncing off the floor (or a racquet on the floor), c can be shown to be $c = (h/H)^{1/2}$ where h is the height to which the ball bounces and H is the height from which the ball is dropped. Determine c for the cases in Part 1 and

for the case of a tennis ball bouncing off a concrete or wooden floor ($c = 0.85$ for new tennis balls used on a tennis court).

Example 2.3 Calculating Final Velocity and Energy Release: Two Carts Collide

In the collision pictured in **Figure 2.8**, two carts collide inelastically. Cart 1 (denoted m_1 carries a spring which is initially compressed. During the collision, the spring releases its potential energy and converts it to internal kinetic energy. The mass of cart 1 and the spring is 0.350 kg, and the cart and the spring together have an initial velocity of 2.00 m/s . Cart 2 (denoted m_2 in **Figure 2.8**) has a mass of 0.500 kg and an initial velocity of -0.500 m/s . After the collision, cart 1 is observed to recoil with a velocity of -4.00 m/s . (a) What is the final velocity of cart 2? (b) How much energy was released by the spring (assuming all of it was converted into internal kinetic energy)?

Strategy

We can use conservation of momentum to find the final velocity of cart 2, because $F_{\text{net}} = 0$ (the track is frictionless and the force of the spring is internal). Once this velocity is determined, we can compare the internal kinetic energy before and after the collision to see how much energy was released by the spring.

Solution for (a)

As before, the equation for conservation of momentum in a two-object system is

$$m_1 v_1 + m_2 v_2 = m_1 v'_1 + m_2 v'_2 . \tag{2.32}$$

The only unknown in this equation is v'_2 . Solving for v'_2 and substituting known values into the previous equation yields

$$\begin{aligned} v'_2 &= \frac{m_1 v_1 + m_2 v_2 - m_1 v'_1}{m_2} \\ &= \frac{(0.350 \text{ kg})(2.00 \text{ m/s}) + (0.500 \text{ kg})(-0.500 \text{ m/s})}{0.500 \text{ kg}} - \frac{(0.350 \text{ kg})(-4.00 \text{ m/s})}{0.500 \text{ kg}} \\ &= 3.70 \text{ m/s}. \end{aligned} \tag{2.33}$$

Solution for (b)

The internal kinetic energy before the collision is

$$\begin{aligned} \text{KE}_{\text{int}} &= \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 \\ &= \frac{1}{2}(0.350 \text{ kg})(2.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(-0.500 \text{ m/s})^2 \\ &= 0.763 \text{ J}. \end{aligned} \tag{2.34}$$

After the collision, the internal kinetic energy is

$$\begin{aligned} \text{KE}'_{\text{int}} &= \frac{1}{2} m_1 v'^2_1 + \frac{1}{2} m_2 v'^2_2 \\ &= \frac{1}{2}(0.350 \text{ kg})(\text{-}4.00 \text{ m/s})^2 + \frac{1}{2}(0.500 \text{ kg})(3.70 \text{ m/s})^2 \\ &= 6.22 \text{ J}. \end{aligned} \tag{2.35}$$

The change in internal kinetic energy is thus

$$\begin{aligned} \text{KE}'_{\text{int}} - \text{KE}_{\text{int}} &= 6.22 \text{ J} - 0.763 \text{ J} \\ &= 5.46 \text{ J}. \end{aligned} \tag{2.36}$$

Discussion

The final velocity of cart 2 is large and positive, meaning that it is moving to the right after the collision. The internal kinetic energy in this collision increases by 5.46 J. That energy was released by the spring.

Section Summary

- An inelastic collision is one in which the internal kinetic energy changes (it is not conserved).
- A collision in which the objects stick together is sometimes called perfectly inelastic because it reduces internal kinetic energy more than does any other type of inelastic collision.
- Sports science and technologies also use physics concepts such as momentum and rotational motion and vibrations.

Conceptual Questions

Exercise 2.17

What is an inelastic collision? What is a perfectly inelastic collision?

Exercise 2.18

Mixed-pair ice skaters performing in a show are standing motionless at arms length just before starting a routine. They reach out, clasp hands, and pull themselves together by only using their arms. Assuming there is no friction between the blades of their skates and the ice, what is their velocity after their bodies meet?

Exercise 2.19

A small pickup truck that has a camper shell slowly coasts toward a red light with negligible friction. Two dogs in the back of the truck are moving and making various inelastic collisions with each other and the walls. What is the effect of the dogs on the motion of the center of mass of the system (truck plus entire load)? What is their effect on the motion of the truck?

Problems & Exercises

Exercise 2.20

A 0.240-kg billiard ball that is moving at 3.00 m/s strikes the bumper of a pool table and bounces straight back at 2.40 m/s (80% of its original speed). The collision lasts 0.0150 s. (a) Calculate the average force exerted on the ball by the bumper. (b) How much kinetic energy in joules is lost during the collision? (c) What percent of the original energy is left?

Solution

(a) 86.4 N perpendicularly away from the bumper

(b) 0.389 J

(c) 64.0%

Exercise 2.21

During an ice show, a 60.0-kg skater leaps into the air and is caught by an initially stationary 75.0-kg skater. (a) What is their final velocity assuming negligible friction and that the 60.0-kg skater's original horizontal velocity is 4.00 m/s? (b) How much kinetic energy is lost?

Exercise 2.22

Professional Application

Using mass and speed data from **m42156 (https://legacy.cnx.org/content/m42156/latest/#fs-id1356444)** and assuming that the football player catches the ball with his feet off the ground with both of them moving horizontally, calculate: (a) the final velocity if the ball and player are going in the same direction and (b) the loss of kinetic energy in this case. (c) Repeat parts (a) and (b) for the situation in which the ball and the player are going in opposite directions. Might the loss of kinetic energy be related to how much it hurts to catch the pass?

Solution

(a) 8.06 m/s

(b) -56.0 J

(c)(i) 7.88 m/s; (ii) -223 J

Exercise 2.23

A battleship that is $6.00\times10^{7}\ \text{kg}$ and is originally at rest fires a 1100-kg artillery shell horizontally with a velocity of 575 m/s. (a) If the shell is fired straight aft (toward the rear of the ship), there will be negligible friction opposing the ship's recoil. Calculate its recoil velocity. (b) Calculate the increase in internal kinetic energy (that is, for the ship and the shell). This energy is less than the energy released by the gun powder—significant heat transfer occurs.

Exercise 2.24

Professional Application

Two manned satellites approaching one another, at a relative speed of 0.250 m/s, intending to dock. The first has a mass of $4.00\times10^3\ \text{kg}$, and the second a mass of $7.50\times10^3\ \text{kg}$. (a) Calculate the final velocity (after docking) by using the frame of reference in which the first satellite was originally at rest. (b) What is the loss of kinetic energy in this inelastic collision? (c) Repeat both parts by using the frame of reference in which the second satellite was originally at rest. Explain why the change in velocity is different in the two frames, whereas the change in kinetic energy is the same in both.

Solution

(a) 0.163 m/s in the direction of motion of the more massive satellite

(b) 81.6 J

(c) $8.70\times10^{-2}\ \text{m/s}$ in the direction of motion of the less massive satellite, 81.5 J. Because there are no external forces, the velocity of the center of mass of the two-satellite system is unchanged by the collision. The two velocities calculated above are the velocity of the center of mass in each of the two different individual reference frames. The loss in KE is the same in both reference frames because the KE lost to internal forces (heat, friction, etc.) is the same regardless of the coordinate system chosen.

Exercise 2.25

Professional Application

A 30,000-kg freight car is coasting at 0.850 m/s with negligible friction under a hopper that dumps 110,000 kg of scrap metal into it. (a) What is the final velocity of the loaded freight car? (b) How much kinetic energy is lost?

Exercise 2.26

Professional Application

Space probes may be separated from their launchers by exploding bolts. (They bolt away from one another.) Suppose a 4800-kg satellite uses this method to separate from the 1500-kg remains of its launcher, and that 5000 J of kinetic energy is supplied to the two parts. What are their subsequent velocities using the frame of reference in which they were at rest before separation?

Solution

0.704 m/s

–2.25 m/s

Exercise 2.27

A 0.0250-kg bullet is accelerated from rest to a speed of 550 m/s in a 3.00-kg rifle. The pain of the rifle’s kick is much worse if you hold the gun loosely a few centimeters from your shoulder rather than holding it tightly against your shoulder. (a) Calculate the recoil velocity of the rifle if it is held loosely away from the shoulder. (b) How much kinetic energy does the rifle gain? (c) What is the recoil velocity if the rifle is held tightly against the shoulder, making the effective mass 28.0 kg? (d) How much kinetic energy is transferred to the rifle-shoulder combination? The pain is related to the amount of kinetic energy, which is significantly less in this latter situation. (e) Calculate the momentum of a 110-kg football player running at 8.00 m/s. Compare the player’s momentum with the momentum of a hard-thrown 0.410-kg football that has a speed of 25.0 m/s. Discuss its relationship to this problem.

Solution

(a) 4.58 m/s away from the bullet

(b) 31.5 J

(c) –0.491 m/s

(d) 3.38 J

Exercise 2.28

Professional Application

One of the waste products of a nuclear reactor is plutonium-239 $\left({}^{239}\text{Pu}\right)$. This nucleus is radioactive and decays by splitting into a helium-4 nucleus and a uranium-235 nucleus $\left({}^{4}\text{He} + {}^{235}\text{U}\right)$, the latter of which is also radioactive and will itself decay some time later. The energy emitted in the plutonium decay is $8.40 \times 10^{-13}\ \text{J}$ and is entirely converted to kinetic energy of the helium and uranium nuclei. The mass of the helium nucleus is $6.68 \times 10^{-27}\ \text{kg}$, while that of the uranium is $3.92 \times 10^{-25}\ \text{kg}$ (note that the ratio of the masses is 4 to 235). (a) Calculate the velocities of the two nuclei, assuming the plutonium nucleus is originally at rest. (b) How much kinetic energy does each nucleus carry away? Note that the data given here are accurate to three digits only.

Exercise 2.29

Professional Application

The Moon's craters are remnants of meteorite collisions. Suppose a fairly large asteroid that has a mass of $5.00 \times 10^{12}\ \text{kg}$ (about a kilometer across) strikes the Moon at a speed of 15.0 km/s. (a) At what speed does the Moon recoil after the perfectly inelastic collision (the mass of the Moon is $7.36 \times 10^{22}\ \text{kg}$) ? (b) How much kinetic energy is lost in the collision? Such an event may have been observed by medieval English monks who reported observing a red glow and subsequent haze about the Moon. (c) In October 2009, NASA crashed a rocket into the Moon, and analyzed the plume produced by the impact. (Significant amounts of water were detected.) Answer part (a) and (b) for this real-life experiment. The mass of the rocket was 2000 kg and its speed upon impact was 9000 km/h. How does the plume produced alter these results?

Solution

(a) $1.02 \times 10^{-6}\ \text{m/s}$

(b) $\mathbf{5.63 \times 10^{20}\ J}$ (almost all KE lost)

(c) Recoil speed is $6.79 \times 10^{-17}\ \text{m/s}$, energy lost is $6.25 \times 10^{9}\ \text{J}$. The plume will not affect the momentum result because the plume is still part of the Moon system. The plume may affect the kinetic energy result because a significant part of the initial kinetic energy may be transferred to the kinetic energy of the plume particles.

Exercise 2.30

Professional Application

Two football players collide head-on in midair while trying to catch a thrown football. The first player is 95.0 kg and has an initial velocity of 6.00 m/s, while the second player is 115 kg and has an initial velocity of –3.50 m/s. What is their velocity just after impact if they cling together?

Exercise 2.31

What is the speed of a garbage truck that is $1.20 \times 10^{4}\ \text{kg}$ and is initially moving at 25.0 m/s just after it hits and adheres to a trash can that is 80.0 kg and is initially at rest?

Solution

24.8 m/s

Exercise 2.32

During a circus act, an elderly performer thrills the crowd by catching a cannon ball shot at him. The cannon ball has a mass of 10.0 kg and the horizontal component of its velocity is 8.00 m/s when the 65.0-kg performer catches it. If the performer is on nearly frictionless roller skates, what is his recoil velocity?

Exercise 2.33

(a) During an ice skating performance, an initially motionless 80.0-kg clown throws a fake barbell away. The clown's ice skates allow her to recoil frictionlessly. If the clown recoils with a velocity of 0.500 m/s and the barbell is thrown with a velocity of 10.0 m/s, what is the mass of the barbell? (b) How much kinetic energy is gained by this maneuver? (c) Where does the kinetic energy come from?

Solution

(a) 4.00 kg

(b) 210 J

(c) The clown does work to throw the barbell, so the kinetic energy comes from the muscles of the clown. The muscles convert the chemical potential energy of ATP into kinetic energy.

2.5 Collisions of Point Masses in Two Dimensions

In the previous two sections, we considered only one-dimensional collisions; during such collisions, the incoming and outgoing velocities are all along the same line. But what about collisions, such as those between billiard balls, in which objects scatter to the side? These are two-dimensional collisions, and we shall see that their study is an extension of the one-dimensional analysis already presented. The approach taken (similar to the approach in discussing two-dimensional kinematics and dynamics) is to choose a convenient coordinate system and resolve the motion into components along perpendicular axes. Resolving the motion yields a pair of one-dimensional problems to be solved simultaneously.

One complication arising in two-dimensional collisions is that the objects might rotate before or after their collision. For example, if two ice skaters hook arms as they pass by one another, they will spin in circles. We will not consider such rotation until later, and so for now we arrange things so that no rotation is possible. To avoid rotation, we consider only the scattering of **point masses**—that is, structureless particles that cannot rotate or spin.

We start by assuming that $\mathbf{F}_{\text{net}} = 0$, so that momentum $\mathbf{p}$ is conserved. The simplest collision is one in which one of the particles is initially at rest. (See **Figure 2.9**.) The best choice for a coordinate system is one with an axis parallel to the velocity of the incoming particle, as shown in **Figure 2.9**. Because momentum is conserved, the components of momentum along the x-and y-axes $(p_x \text{ and } p_y)$ will also be conserved, but with the chosen coordinate system, p_y is initially zero and p_x is the momentum of the incoming particle. Both facts simplify the analysis. (Even with the simplifying assumptions of point masses, one particle initially at rest, and a convenient coordinate system, we still gain new insights into nature from the analysis of two-dimensional collisions.)

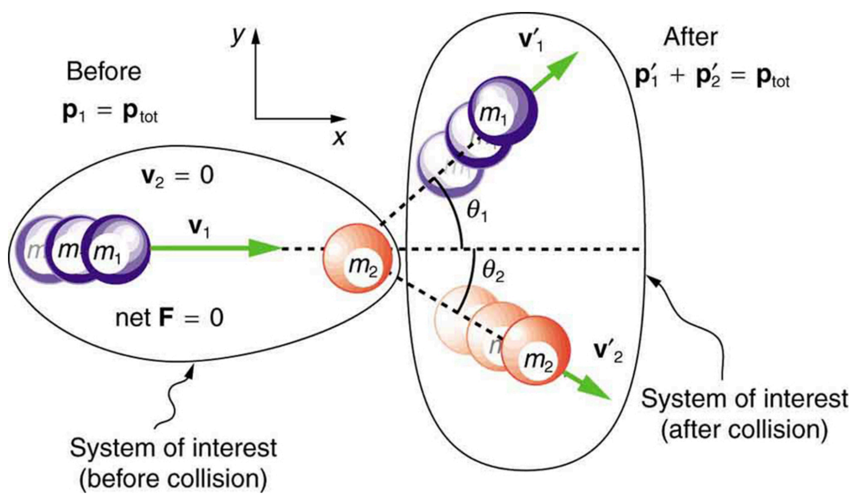


Figure 2.9 A two-dimensional collision with the coordinate system chosen so that m_2 is initially at rest and v_1 is parallel to the x-axis. This coordinate system is sometimes called the laboratory coordinate system, because many scattering experiments have a target that is stationary in the laboratory, while particles are scattered from it to determine the particles that make-up the target and how they are bound together. The particles may not be observed directly, but their initial and final velocities are.

Along the x-axis, the equation for conservation of momentum is

$$p_{1x} + p_{2x} = p'_{1x} + p'_{2x}. \tag{2.37}$$

Where the subscripts denote the particles and axes and the primes denote the situation after the collision. In terms of masses and velocities, this equation is

$$m_1 v_{1x} + m_2 v_{2x} = m_1 v'_{1x} + m_2 v'_{2x}. \tag{2.38}$$

But because particle 2 is initially at rest, this equation becomes

$$m_1 v_{1x} = m_1 v'_{1x} + m_2 v'_{2x}. \tag{2.39}$$

The components of the velocities along the x -axis have the form $v \cos \theta$. Because particle 1 initially moves along the x -axis, we find $v_{1x} = v_1$.

Conservation of momentum along the x -axis gives the following equation:

$$m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2, \tag{2.40}$$

where θ_1 and θ_2 are as shown in **Figure 2.9**.

Conservation of Momentum along the x -axis

$$m_1 v_1 = m_1 v'_1 \cos \theta_1 + m_2 v'_2 \cos \theta_2 \tag{2.41}$$

Along the y -axis, the equation for conservation of momentum is

$$p_{1y} + p_{2y} = p'_{1y} + p'_{2y} \tag{2.42}$$

or

$$m_1 v_{1y} + m_2 v_{2y} = m_1 v'_{1y} + m_2 v'_{2y}. \tag{2.43}$$

But v_{1y} is zero, because particle 1 initially moves along the x -axis. Because particle 2 is initially at rest, v_{2y} is also zero. The equation for conservation of momentum along the y -axis becomes

$$0 = m_1 v'_{1y} + m_2 v'_{2y}. \tag{2.44}$$

The components of the velocities along the y -axis have the form $v \sin \theta$.

Thus, conservation of momentum along the y -axis gives the following equation:

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2. \tag{2.45}$$

Conservation of Momentum along the y -axis

$$0 = m_1 v'_1 \sin \theta_1 + m_2 v'_2 \sin \theta_2 \tag{2.46}$$

The equations of conservation of momentum along the x -axis and y -axis are very useful in analyzing two-dimensional collisions of particles, where one is originally stationary (a common laboratory situation). But two equations can only be used to find two unknowns, and so other data may be necessary when collision experiments are used to explore nature at the subatomic level.

Example 2.4 Determining the Final Velocity of an Unseen Object from the Scattering of Another Object

Suppose the following experiment is performed. A 0.250-kg object (m_1) is slid on a frictionless surface into a dark room, where it strikes an initially stationary object with mass of 0.400 kg (m_2) . The 0.250-kg object emerges from the room at an angle of 45.0° with its incoming direction.

The speed of the 0.250-kg object is originally 2.00 m/s and is 1.50 m/s after the collision. Calculate the magnitude and direction of the velocity $(v'_2$ and $\theta_2)$ of the 0.400-kg object after the collision.

Strategy

Momentum is conserved because the surface is frictionless. The coordinate system shown in **Figure 2.10** is one in which m_2 is originally at rest and the initial velocity is parallel to the x -axis, so that conservation of momentum along the x - and y -axes is applicable.

Everything is known in these equations except v'_2 and θ_2 , which are precisely the quantities we wish to find. We can find two unknowns because we have two independent equations: the equations describing the conservation of momentum in the x - and y -directions.

Solution

Solving $m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2$ for $v'_2 \cos\theta_2$ and $0 = m_1 v'_1 \sin\theta_1 + m_2 v'_2 \sin\theta_2$ for $v'_2 \sin\theta_2$ and taking the ratio yields an equation (in which θ2 is the only unknown quantity. Applying the identity $\left(\tan\theta = \frac{\sin\theta}{\cos\theta}\right)$, we obtain:

$$\tan\theta_2 = \frac{v'_1 \sin\theta_1}{v'_1 \cos\theta_1 - v_1}. \tag{2.47}$$

Entering known values into the previous equation gives

$$\tan\theta_2 = \frac{(1.50\text{ m/s})(0.7071)}{(1.50\text{ m/s})(0.7071) - 2.00\text{ m/s}} = -1.129. \tag{2.48}$$

Thus,

$$\theta_2 = \tan^{-1}(-1.129) = 311.5^\circ \approx 312^\circ. \tag{2.49}$$

Angles are defined as positive in the counter clockwise direction, so this angle indicates that m_2 is scattered to the right in **Figure 2.10**, as expected (this angle is in the fourth quadrant). Either equation for the x- or y-axis can now be used to solve for v'_2, but the latter equation is easiest because it has fewer terms.

$$v'_2 = -\frac{m_1}{m_2} v'_1 \frac{\sin\theta_1}{\sin\theta_2} \tag{2.50}$$

Entering known values into this equation gives

$$v'_2 = -\left(\frac{0.250\text{ kg}}{0.400\text{ kg}}\right)(1.50\text{ m/s})\left(\frac{0.7071}{-0.7485}\right). \tag{2.51}$$

Thus,

$$v'_2 = 0.886\text{ m/s}. \tag{2.52}$$

Discussion

It is instructive to calculate the internal kinetic energy of this two-object system before and after the collision. (This calculation is left as an end-of-chapter problem.) If you do this calculation, you will find that the internal kinetic energy is less after the collision, and so the collision is inelastic. This type of result makes a physicist want to explore the system further.

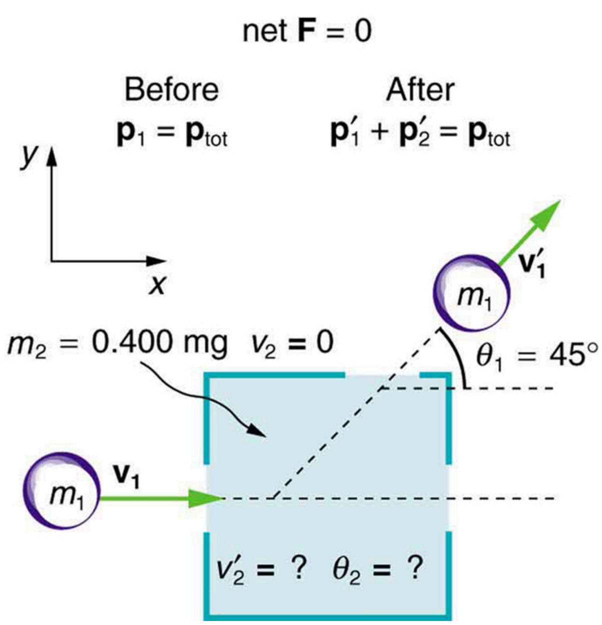


Figure 2.10 A collision taking place in a dark room is explored in **Example 2.4**. The incoming object m_1 is scattered by an initially stationary object. Only the stationary object's mass m_2 is known. By measuring the angle and speed at which m_1 emerges from the room, it is possible to calculate the magnitude and direction of the initially stationary object's velocity after the collision.

Elastic Collisions of Two Objects with Equal Mass

Some interesting situations arise when the two colliding objects have equal mass and the collision is elastic. This situation is nearly the case with colliding billiard balls, and precisely the case with some subatomic particle collisions. We can thus get a mental image of a collision of subatomic particles by thinking about billiards (or pool). (Refer to **Figure 2.9** for masses and angles.) First, an elastic collision conserves internal kinetic energy. Again, let us assume object 2 (m_2) is initially at rest. Then, the internal kinetic energy before and after the collision of two objects that have equal masses is

$$\frac{1}{2}mv_1{}^2 = \frac{1}{2}mv'_1{}^2 + \frac{1}{2}mv'_2{}^2. \tag{2.53}$$

Because the masses are equal, $m_1 = m_2 = m$. Algebraic manipulation (left to the reader) of conservation of momentum in the x- and y-directions can show that

$$\frac{1}{2}mv_1{}^2 = \frac{1}{2}mv'_1{}^2 + \frac{1}{2}mv'_2{}^2 + mv'_1v'_2\cos(\theta_1 - \theta_2). \tag{2.54}$$

(Remember that θ_2 is negative here.) The two preceding equations can both be true only if

$$mv'_1v'_2\cos(\theta_1 - \theta_2) = 0. \tag{2.55}$$

There are three ways that this term can be zero. They are

- $v'_1 = 0$: head-on collision; incoming ball stops
- $v'_2 = 0$: no collision; incoming ball continues unaffected
- $\cos(\theta_1 - \theta_2) = 0$: angle of separation $(\theta_1 - \theta_2)$ is 90° after the collision

All three of these ways are familiar occurrences in billiards and pool, although most of us try to avoid the second. If you play enough pool, you will notice that the angle between the balls is very close to 90° after the collision, although it will vary from this value if a great deal of spin is placed on the ball. (Large spin carries in extra energy and a quantity called *angular momentum*, which must also be conserved.) The assumption that the scattering of billiard balls is elastic is reasonable based on the correctness of the three results it produces. This assumption also implies that, to a good approximation, momentum is conserved for the two-ball system in billiards and pool. The problems below explore these and other characteristics of two-dimensional collisions.

Connections to Nuclear and Particle Physics

Two-dimensional collision experiments have revealed much of what we know about subatomic particles, as we shall see in **Medical Applications of Nuclear Physics (https://legacy.cnx.org/content/m42646/latest/)** and **Particle Physics (https://legacy.cnx.org/content/m42667/latest/)**. Ernest Rutherford, for example, discovered the nature of the atomic nucleus from such experiments.

Section Summary

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes. Choose a coordinate system with the x-axis parallel to the velocity of the incoming particle.
- Two-dimensional collisions of point masses where mass 2 is initially at rest conserve momentum along the initial direction of mass 1 (the x-axis), stated by $m_1v_1 = m_1v'_1\cos\theta_1 + m_2v'_2\cos\theta_2$ and along the direction perpendicular to the initial direction (the y-axis) stated by $0 = m_1v'_{1y} + m_2v'_{2y}$.
- The internal kinetic before and after the collision of two objects that have equal masses is

$$\frac{1}{2}mv_1{}^2 = \frac{1}{2}mv'_1{}^2 + \frac{1}{2}mv'_2{}^2 + mv'_1v'_2\cos(\theta_1 - \theta_2). \tag{2.56}$$

- Point masses are structureless particles that cannot spin.

Conceptual Questions

Exercise 2.34

Figure 2.11 shows a cube at rest and a small object heading toward it. (a) Describe the directions (angle θ_1) at which the small object can emerge after colliding elastically with the cube. How does θ_1 depend on b, the so-called impact parameter? Ignore any effects that might be due to rotation after the collision, and assume that the cube is much more massive than the small object. (b) Answer the same questions if the small object instead collides with a massive sphere.

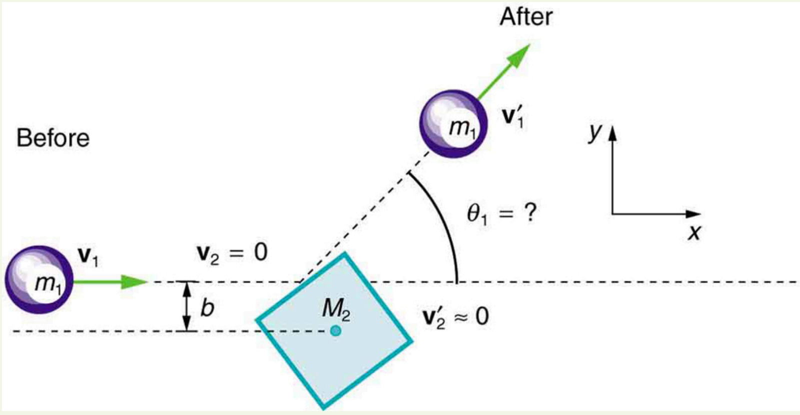


Figure 2.11 A small object approaches a collision with a much more massive cube, after which its velocity has the direction θ_1 . The angles at which the small object can be scattered are determined by the shape of the object it strikes and the impact parameter b .

Problems & Exercises

Exercise 2.35

Two identical pucks collide on an air hockey table. One puck was originally at rest. (a) If the incoming puck has a speed of 6.00 m/s and scatters to an angle of 30.0° ,what is the velocity (magnitude and direction) of the second puck? (You may use the result that $\theta_1 - \theta_2 = 90^\circ$ for elastic collisions of objects that have identical masses.) (b) Confirm that the collision is elastic.

Solution

(a) 3.00 m/s, 60° below x -axis

(b) Find speed of first puck after collision: $0 = mv'_1 \sin 30^\circ - mv'_2 \sin 60^\circ \Rightarrow v'_1 = v'_2 \frac{\sin 60^\circ}{\sin 30^\circ} = 5.196 \text{ m/s}$

Verify that ratio of initial to final KE equals one: $$\left.\begin{array}{l}\text{KE} = \frac{1}{2}mv_1{}^2 = 18m \text{ J} \\ \text{KE} = \frac{1}{2}mv'_1{}^2 + \frac{1}{2}mv'_2{}^2 = 18m \text{ J}\end{array}\right\} \frac{\text{KE}}{\text{KE}'} = 1.00$$

Exercise 2.36

Confirm that the results of the example **Example 2.4** do conserve momentum in both the x - and y -directions.

Exercise 2.37

A 3000-kg cannon is mounted so that it can recoil only in the horizontal direction. (a) Calculate its recoil velocity when it fires a 15.0-kg shell at 480 m/s at an angle of 20.0° above the horizontal. (b) What is the kinetic energy of the cannon? This energy is dissipated as heat transfer in shock absorbers that stop its recoil. (c) What happens to the vertical component of momentum that is imparted to the cannon when it is fired?

Solution

(a) -2.26 m/s

(b) $7.63 \times 10^3 \text{ J}$

(c) The ground will exert a normal force to oppose recoil of the cannon in the vertical direction. The momentum in the vertical direction is transferred to the earth. The energy is transferred into the ground, making a dent where the cannon is. After long barrages, cannon have erratic aim because the ground is full of divots.

Exercise 2.38

Professional Application

A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of 85.0° to the initial direction of the bowling ball and with a speed of 15.0 m/s. (a) Calculate the final velocity (magnitude and direction) of the bowling ball. (b) Is the collision elastic? (c) Linear kinetic energy is greater after the collision. Discuss how spin on the ball might be converted to linear kinetic energy in the collision.

Exercise 2.39

Professional Application

Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei $\left({}^{4}\text{He}\right)$ from gold-197 nuclei $\left({}^{197}\text{Au}\right)$. The energy of the incoming helium nucleus was $8.00\times10^{-13}\text{ J}$, and the masses of the helium and gold nuclei were $6.68\times10^{-27}\text{ kg}$ and $3.29\times10^{-25}\text{ kg}$, respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of 120° during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

Solution

(a) $5.36\times10^{5}\text{ m/s}$ at -29.5°

(b) $7.52\times10^{-13}\text{ J}$

Exercise 2.40

Professional Application

Two cars collide at an icy intersection and stick together afterward. The first car has a mass of 1200 kg and is approaching at 8.00 m/s due south. The second car has a mass of 850 kg and is approaching at 17.0 m/s due west. (a) Calculate the final velocity (magnitude and direction) of the cars. (b) How much kinetic energy is lost in the collision? (This energy goes into deformation of the cars.) Note that because both cars have an initial velocity, you cannot use the equations for conservation of momentum along the x-axis and y-axis; instead, you must look for other simplifying aspects.

Exercise 2.41

Starting with equations $m_1 v_1 = m_1 v'_1 \cos\theta_1 + m_2 v'_2 \cos\theta_2$ and $0 = m_1 v'_1 \sin\theta_1 + m_2 v'_2 \sin\theta_2$ for conservation of momentum in the x- and y-directions and assuming that one object is originally stationary, prove that for an elastic collision of two objects of equal masses,

$$\frac{1}{2}mv_1{}^2 = \frac{1}{2}mv'_1{}^2 + \frac{1}{2}mv'_2{}^2 + mv'_1 v'_2 \cos\left(\theta_1 - \theta_2\right) \tag{2.57}$$

as discussed in the text.

Solution

We are given that $m_1 = m_2 \equiv m$. The given equations then become:

$$v_1 = v_1 \cos\theta_1 + v_2 \cos\theta_2 \tag{2.58}$$

and

$$0 = v'_1 \sin\theta_1 + v'_2 \sin\theta_2. \tag{2.59}$$

Square each equation to get

$$\begin{aligned} v_1{}^2 &= v'_1{}^2 \cos^2\theta_1 + v'_2{}^2 \cos^2\theta_2 + 2v'_1 v'_2 \cos\theta_1 \cos\theta_2 \\ 0 &= v'_1{}^2 \sin^2\theta_1 + v'_2{}^2 \sin^2\theta_2 + 2v'_1 v'_2 \sin\theta_1 \sin\theta_2 . \end{aligned} \tag{2.60}$$

Add these two equations and simplify:

$$\begin{aligned} v_1{}^2 &= v'_1{}^2 + v'_2{}^2 + 2v'_1 v'_2(\cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2) \\ &= v'_1{}^2 + v'_2{}^2 + 2v'_1 v'_2\left[\frac{1}{2}\cos(\theta_1-\theta_2) + \frac{1}{2}\cos(\theta_1+\theta_2) + \frac{1}{2}\cos(\theta_1-\theta_2) - \frac{1}{2}\cos(\theta_1+\theta_2)\right] \\ &= v'_1{}^2 + v'_2{}^2 + 2v'_1 v'_2 \cos(\theta_1-\theta_2). \end{aligned} \tag{2.61}$$

Multiply the entire equation by $\frac{1}{2}m$ to recover the kinetic energy:

$$\frac{1}{2}mv_1{}^2 = \frac{1}{2}mv'_1{}^2 + \frac{1}{2}mv'_2{}^2 + mv'_1 v'_2 \cos(\theta_1-\theta_2) \tag{2.62}$$

Exercise 2.42

Integrated Concepts

A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?

2.6 Introduction to Rocket Propulsion

Rockets range in size from fireworks so small that ordinary people use them to immense Saturn Vs that once propelled massive payloads toward the Moon. The propulsion of all rockets, jet engines, deflating balloons, and even squids and octopuses is explained by the same physical principle—Newton's third law of motion. Matter is forcefully ejected from a system, producing an equal and opposite reaction on what remains. Another common example is the recoil of a gun. The gun exerts a force on a bullet to accelerate it and consequently experiences an equal and opposite force, causing the gun's recoil or kick.

Making Connections: Take-Home Experiment—Propulsion of a Balloon

Hold a balloon and fill it with air. Then, let the balloon go. In which direction does the air come out of the balloon and in which direction does the balloon get propelled? If you fill the balloon with water and then let the balloon go, does the balloon's direction change? Explain your answer.

Figure 2.12 shows a rocket accelerating straight up. In part (a), the rocket has a mass m and a velocity v relative to Earth, and hence a momentum mv. In part (b), a time Δt has elapsed in which the rocket has ejected a mass Δm of hot gas at a velocity v_e relative to the rocket. The remainder of the mass $(m - \Delta m)$ now has a greater velocity $(v + \Delta v)$. The momentum of the entire system (rocket plus expelled gas) has actually decreased because the force of gravity has acted for a time Δt, producing a negative impulse $\Delta p = -mg\Delta t$. (Remember that impulse is the net external force on a system multiplied by the time it acts, and it equals the change in momentum of the system.) So, the center of mass of the system is in free fall but, by rapidly expelling mass, part of the system can accelerate upward. It is a commonly held misconception that the rocket exhaust pushes on the ground. If we consider thrust; that is, the force exerted on the rocket by the exhaust gases, then a rocket's thrust is greater in outer space than in the atmosphere or on the launch pad. In fact, gases are easier to expel into a vacuum.

By calculating the change in momentum for the entire system over Δt, and equating this change to the impulse, the following expression can be shown to be a good approximation for the acceleration of the rocket.

$$a = \frac{v_\mathrm{e}}{m}\frac{\Delta m}{\Delta t} - g \tag{2.63}$$

"The rocket" is that part of the system remaining after the gas is ejected, and g is the acceleration due to gravity.

Acceleration of a Rocket

Acceleration of a rocket is

$$a = \frac{v_\mathrm{e}}{m}\frac{\Delta m}{\Delta t} - g, \tag{2.64}$$

where a is the acceleration of the rocket, v_e is the escape velocity, m is the mass of the rocket, Δm is the mass of the ejected gas, and Δt is the time in which the gas is ejected.

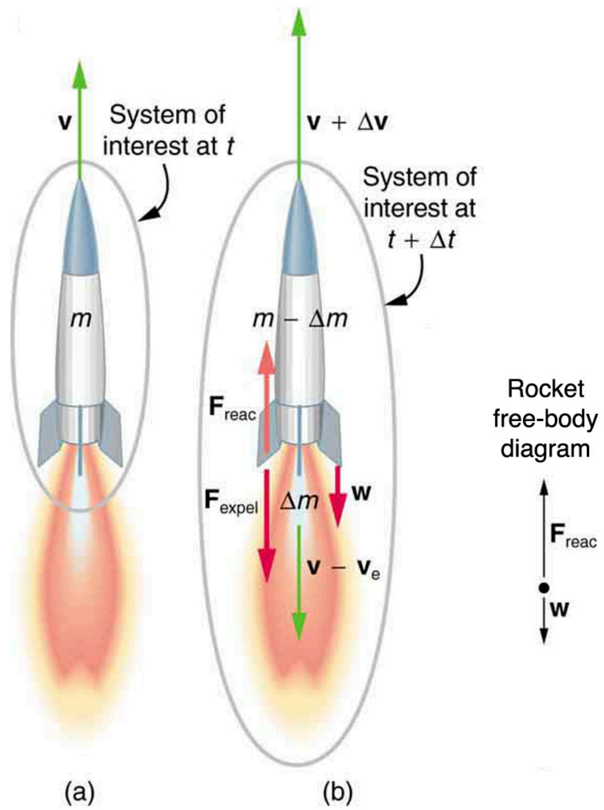


Figure 2.12 (a) This rocket has a mass m and an upward velocity v. The net external force on the system is $-mg$, if air resistance is neglected. (b) A time Δt later the system has two main parts, the ejected gas and the remainder of the rocket. The reaction force on the rocket is what overcomes the gravitational force and accelerates it upward.

A rocket's acceleration depends on three major factors, consistent with the equation for acceleration of a rocket **.** First, the greater the exhaust velocity of the gases relative to the rocket, v_e, the greater the acceleration is. The practical limit for v_e is about $2.5 \times 10^3 \text{ m/s}$ for conventional (non-nuclear) hot-gas propulsion systems. The second factor is the rate at which mass is ejected from the rocket. This is the factor $\Delta m / \Delta t$ in the equation. The quantity $(\Delta m / \Delta t)v_e$, with units of newtons, is called "thrust." The faster the rocket burns its fuel, the greater its thrust, and the greater its acceleration. The third factor is the mass m of the rocket. The smaller the mass is (all other factors being the same), the greater the acceleration. The rocket mass m decreases dramatically during flight because most of the rocket is fuel to begin with, so that acceleration increases continuously, reaching a maximum just before the fuel is exhausted.

Factors Affecting a Rocket's Acceleration

- The greater the exhaust velocity v_e of the gases relative to the rocket, the greater the acceleration.
- The faster the rocket burns its fuel, the greater its acceleration.
- The smaller the rocket's mass (all other factors being the same), the greater the acceleration.

Example 2.5 Calculating Acceleration: Initial Acceleration of a Moon Launch

A Saturn V's mass at liftoff was $2.80 \times 10^6 \text{ kg}$, its fuel-burn rate was $1.40 \times 10^4 \text{ kg/s}$, and the exhaust velocity was $2.40 \times 10^3 \text{ m/s}$. Calculate its initial acceleration.

Strategy

This problem is a straightforward application of the expression for acceleration because a is the unknown and all of the terms on the right side of the equation are given.

Solution

Substituting the given values into the equation for acceleration yields

$$\begin{aligned} a &= \frac{v_e}{m}\frac{\Delta m}{\Delta t} - g \\ &= \frac{2.40\times10^3\ \text{m/s}}{2.80\times10^6\ \text{kg}}\left(1.40\times10^4\ \text{kg/s}\right) - 9.80\ \text{m/s}^2 \\ &= 2.20\ \text{m/s}^2. \end{aligned} \tag{2.65}$$

Discussion

This value is fairly small, even for an initial acceleration. The acceleration does increase steadily as the rocket burns fuel, because m decreases while v_e and $\frac{\Delta m}{\Delta t}$ remain constant. Knowing this acceleration and the mass of the rocket, you can show that the thrust of the engines was $3.36\times10^7\ \text{N}$.

To achieve the high speeds needed to hop continents, obtain orbit, or escape Earth's gravity altogether, the mass of the rocket other than fuel must be as small as possible. It can be shown that, in the absence of air resistance and neglecting gravity, the final velocity of a one-stage rocket initially at rest is

$$v = v_e \ln \frac{m_0}{m_r}, \tag{2.66}$$

where $\ln(m_0 / m_r)$ is the natural logarithm of the ratio of the initial mass of the rocket (m_0) to what is left (m_r) after all of the fuel is exhausted. (Note that v is actually the change in velocity, so the equation can be used for any segment of the flight. If we start from rest, the change in velocity equals the final velocity.) For example, let us calculate the mass ratio needed to escape Earth's gravity starting from rest, given that the escape velocity from Earth is about $11.2\times10^3\ \text{m/s}$, and assuming an exhaust velocity $v_e = 2.5\times10^3\ \text{m/s}$.

$$\ln \frac{m_0}{m_r} = \frac{v}{v_e} = \frac{11.2\times10^3\ \text{m/s}}{2.5\times10^3\ \text{m/s}} = 4.48 \tag{2.67}$$

Solving for m_0 / m_r gives

$$\frac{m_0}{m_r} = e^{4.48} = 88. \tag{2.68}$$

Thus, the mass of the rocket is

$$m_r = \frac{m_0}{88}. \tag{2.69}$$

This result means that only $1 / 88$ of the mass is left when the fuel is burnt, and $87 / 88$ of the initial mass was fuel. Expressed as percentages, 98.9% of the rocket is fuel, while payload, engines, fuel tanks, and other components make up only 1.10%. Taking air resistance and gravitational force into account, the mass m_r remaining can only be about $m_0 / 180$. It is difficult to build a rocket in which the fuel has a mass 180 times everything else. The solution is multistage rockets. Each stage only needs to achieve part of the final velocity and is discarded after it burns its fuel. The result is that each successive stage can have smaller engines and more payload relative to its fuel. Once out of the atmosphere, the ratio of payload to fuel becomes more favorable, too.

The space shuttle was an attempt at an economical vehicle with some reusable parts, such as the solid fuel boosters and the craft itself. (See **Figure 2.13**) The shuttle's need to be operated by humans, however, made it at least as costly for launching satellites as expendable, unmanned rockets. Ideally, the shuttle would only have been used when human activities were required for the success of a mission, such as the repair of the Hubble space telescope. Rockets with satellites can also be launched from airplanes. Using airplanes has the double advantage that the initial velocity is significantly above zero and a rocket can avoid most of the atmosphere's resistance.

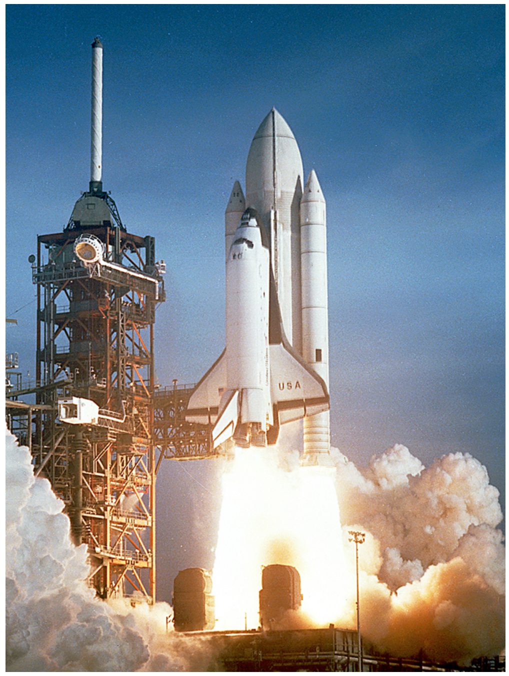


Figure 2.13 The space shuttle had a number of reusable parts. Solid fuel boosters on either side were recovered and refueled after each flight, and the entire orbiter returned to Earth for use in subsequent flights. The large liquid fuel tank was expended. The space shuttle was a complex assemblage of technologies, employing both solid and liquid fuel and pioneering ceramic tiles as reentry heat shields. As a result, it permitted multiple launches as opposed to single-use rockets. (credit: NASA)

PhET Explorations: Lunar Lander

Can you avoid the boulder field and land safely, just before your fuel runs out, as Neil Armstrong did in 1969? Our version of this classic video game accurately simulates the real motion of the lunar lander with the correct mass, thrust, fuel consumption rate, and lunar gravity. The real lunar lander is very hard to control.

PhET Interactive Simulation

Figure 2.14 Lunar Lander (http://legacy.cnx.org/content/m42166/1.6/lunar-lander_en.jar)

Section Summary

- Newton's third law of motion states that to every action, there is an equal and opposite reaction.
- Acceleration of a rocket is $a = \frac{v_e}{m}\frac{\Delta m}{\Delta t} - g$.
- A rocket's acceleration depends on three main factors. They are
  1. The greater the exhaust velocity of the gases, the greater the acceleration.
  2. The faster the rocket burns its fuel, the greater its acceleration.
  3. The smaller the rocket's mass, the greater the acceleration.

Conceptual Questions

Exercise 2.43

Professional Application

Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How is the motion of the center of mass affected by the explosion? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?

Exercise 2.44

Professional Application

During a visit to the International Space Station, an astronaut was positioned motionless in the center of the station, out of reach of any solid object on which he could exert a force. Suggest a method by which he could move himself away from this position, and explain the physics involved.

Exercise 2.45

Professional Application

It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?

Problems & Exercises

Exercise 2.46

Professional Application

Antiballistic missiles (ABMs) are designed to have very large accelerations so that they may intercept fast-moving incoming missiles in the short time available. What is the takeoff acceleration of a 10,000-kg ABM that expels 196 kg of gas per second at an exhaust velocity of $2.50\times10^{3}\ \text{m/s}$?

Solution

$39.2\ \text{m/s}^2$

Exercise 2.47

Professional Application

What is the acceleration of a 5000-kg rocket taking off from the Moon, where the acceleration due to gravity is only $1.6\ \text{m/s}^2$, if the rocket expels 8.00 kg of gas per second at an exhaust velocity of $2.20\times10^{3}\ \text{m/s}$?

Exercise 2.48

Professional Application

Calculate the increase in velocity of a 4000-kg space probe that expels 3500 kg of its mass at an exhaust velocity of $2.00\times10^{3}\ \text{m/s}$. You may assume the gravitational force is negligible at the probe's location.

Solution

$4.16\times10^{3}\ \text{m/s}$

Exercise 2.49

Professional Application

Ion-propulsion rockets have been proposed for use in space. They employ atomic ionization techniques and nuclear energy sources to produce extremely high exhaust velocities, perhaps as great as $8.00\times10^{6}\ \text{m/s}$. These techniques allow a much more favorable payload-to-fuel ratio. To illustrate this fact: (a) Calculate the increase in velocity of a 20,000-kg space probe that expels only 40.0-kg of its mass at the given exhaust velocity. (b) These engines are usually designed to produce a very small thrust for a very long time—the type of engine that might be useful on a trip to the outer planets, for example. Calculate the acceleration of such an engine if it expels $4.50\times10^{-6}\ \text{kg/s}$ at the given velocity, assuming the acceleration due to gravity is negligible.

Exercise 2.50

Derive the equation for the vertical acceleration of a rocket.

Solution

The force needed to give a small mass Δm an acceleration $a_{\Delta m}$ is $F = \Delta m a_{\Delta m}$. To accelerate this mass in the small time interval Δt at a speed v_e requires $v_e = a_{\Delta m}\Delta t$, so $F = v_e \frac{\Delta m}{\Delta t}$. By Newton's third law, this force is equal in magnitude to the thrust force acting on the rocket, so $F_{\text{thrust}} = v_e \frac{\Delta m}{\Delta t}$, where all quantities are positive. Applying Newton's second law to the rocket gives $F_{\text{thrust}} - mg = ma \Rightarrow a = \frac{v_e}{m}\frac{\Delta m}{\Delta t} - g$, where m is the mass of the rocket and unburnt fuel.

Exercise 2.51

Professional Application

(a) Calculate the maximum rate at which a rocket can expel gases if its acceleration cannot exceed seven times that of gravity. The mass of the rocket just as it runs out of fuel is 75,000-kg, and its exhaust velocity is 2.40×10^3 m/s. Assume that the acceleration of gravity is the same as on Earth's surface $\left(9.80\text{ m/s}^2\right)$. (b) Why might it be necessary to limit the acceleration of a rocket?

Exercise 2.52

Given the following data for a fire extinguisher-toy wagon rocket experiment, calculate the average exhaust velocity of the gases expelled from the extinguisher. Starting from rest, the final velocity is 10.0 m/s. The total mass is initially 75.0 kg and is 70.0 kg after the extinguisher is fired.

Exercise 2.53

How much of a single-stage rocket that is 100,000 kg can be anything but fuel if the rocket is to have a final speed of 8.00 km/s, given that it expels gases at an exhaust velocity of 2.20×10^3 m/s?

Solution

2.63×10^3 kg

Exercise 2.54

Professional Application

(a) A 5.00-kg squid initially at rest ejects 0.250-kg of fluid with a velocity of 10.0 m/s. What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-N frictional force opposing the squid's movement. (b) How much energy is lost to work done against friction?

Solution

(a) 0.421 m/s away from the ejected fluid.

(b) 0.237 J.

Exercise 2.55

Unreasonable Results

Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water. (a) Calculate the initial speed of the squid if it leaves the water at an angle of 20.0°, assuming negligible lift from the air and negligible air resistance. (b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

Exercise 2.56

Construct Your Own Problem

Consider an astronaut in deep space cut free from her space ship and needing to get back to it. The astronaut has a few packages that she can throw away to move herself toward the ship. Construct a problem in which you calculate the time it takes her to get back by throwing all the packages at one time compared to throwing them one at a time. Among the things to be considered are the masses involved, the force she can exert on the packages through some distance, and the distance to the ship.

Exercise 2.57

Construct Your Own Problem

Consider an artillery projectile striking armor plating. Construct a problem in which you find the force exerted by the projectile on the plate. Among the things to be considered are the mass and speed of the projectile and the distance over which its speed is reduced. Your instructor may also wish for you to consider the relative merits of depleted uranium versus lead projectiles based on the greater density of uranium.

Glossary

conservation of momentum principle: when the net external force is zero, the total momentum of the system is conserved or constant

elastic collision: a collision that also conserves internal kinetic energy

inelastic collision: a collision in which internal kinetic energy is not conserved

internal kinetic energy: the sum of the kinetic energies of the objects in a system

isolated system: a system in which the net external force is zero

perfectly inelastic collision: a collision in which the colliding objects stick together

point masses: structureless particles with no rotation or spin

quark: fundamental constituent of matter and an elementary particle

3 INTRODUCTION TO TORQUE AND CENTER OF MASS

3.1 Torque and Center of Mass Introduction

In our study of physics so far, we have modeled everything as a dot for free-body diagrams. Where we applied our force to objects and how the object was shaped didn't matter. And, in a lot of cases, this is fine. When you push a box across the floor, pushing the box from upper part of one side is roughly the same as pushing the box from the lower part, which is also roughly the same as just pushing from the center of the side. However, there are also a lot of cases where the place that you push matters. In fact, there's one case that we encounter daily: doors.

If you can, try getting up and opening or closing a door by pushing it near the hinge. You might feel a bit silly, but you'll notice that you need considerably more force to move the door than if you pushed on it near end with the doorknob. Another exercise to try would be to try opening a door by pulling or pushing at an angle from the doorknob. You'll find that it's also harder to do that than just pushing or pulling straight onto the doorknob. In this chapter, we will be exploring this idea of where you push matters, which plays into two physics ideas: torque and center of mass.

3.2 Center of Mass

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Given a series of masses, be able to calculate the position of the center of mass
- Construct a definition of center of mass that contains all of its attributes

What is Center of Mass?

Center of mass is an idea with multiple facets that are all related. The center of mass is the mass weighted average location of an object. Furthermore, the center of mass is the point at which gravity acts. When we get into our discussion of torque, we're going to be interested in the location at which each force is applied, which means we're going to need to figure out where the force of gravity acts. The force of gravity acts at the center of mass. So, the first part of the definition of center of mass is a mass weighted average location of an object. Another term for center of mass is center of gravity; there are some differences between the two, but for this course, these two terms will be synonymous.

Now, this is a lot going on, so, let's think about what a weighted average is first in a context with which you are probably more familiar with the idea. Think about your grades; not each assignment in a course counts the same. For example, in this class, the weights for the different assignments are provided in the table on the left.

Individual components (61% total)		Team components (39% of total)	
Online Homework	10%	Laboratory and other in-class activities	18%
iRATs	9% (50% of total RAT score)	tRATs	9% (50% of total RAT score)
Individual Exam I	14% (78% of total exam score)	Team Exam I	4% (22% of total exam score)
Individual Exam II	14%	Team Exam II	4%
Individual Exam III	14%	Team Exam III	4%

Individual components (61% total)		Team components (39% of total)	
Online Homework	95%	Laboratory and other in-class activities	90%
iRATs	75% (50% of total RAT score)	tRATs	95% (50% of total RAT score)
Individual Exam I	70%	Team Exam I	85%
Individual Exam II	70%	Team Exam II	85%
Individual Exam III	70%	Team Exam III	85%

Now, if you got the grades in the table on the right, what would your final grade be in this course? The weighted average is equal to the sum of all the values times its worth over the total, or

$$\sum \frac{Value * Worth}{Total}$$

Let's work this through for this example. So, in this example this hypothetical student achieved in 95, so that would be the value on the online homework, and the online homework is worth 10 out of a total of 100.

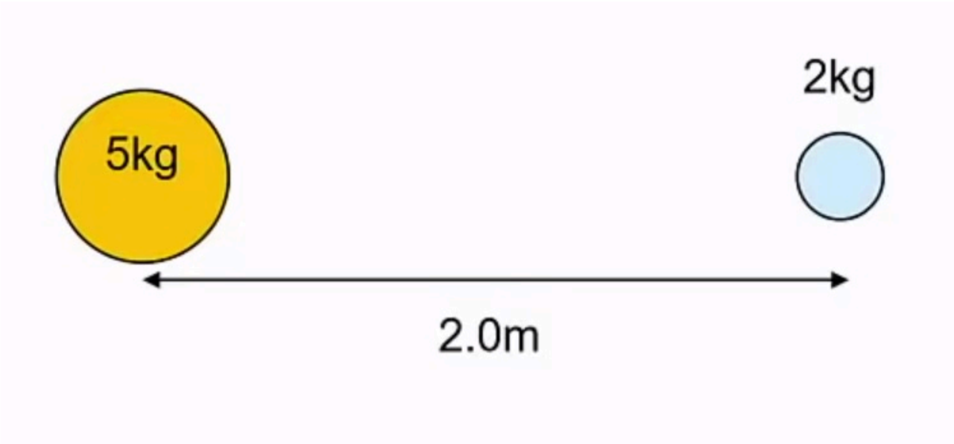
$$(95) * (\frac{10}{100})$$

Now I just repeat this for all the different categories, and then add them all up.

$$(95)\left(\frac{10}{100}\right)+(75)\left(\frac{9}{100}\right)+(70)\left(\frac{14}{100}\right)+(70)\left(\frac{14}{100}\right)+(70)\left(\frac{14}{100}\right)+(70)\left(\frac{14}{100}\right)+(90)\left(\frac{18}{100}\right)+(95)\left(\frac{9}{100}\right)+(85)\left(\frac{4}{100}\right)+(85)\left(\frac{4}{100}\right)+(85)\left(\frac{4}{100}\right)$$

If you do out this calculation, you get an 81.05, which if you go and look at the syllabus is a B+. This is a weighted average of scores for the course.

Center of mass is similarly a weighted average, only we're using mass to weight our average, and we're averaging position. So, let's look at this example of a 5-kilogram object and a 2-kilogram object, and let's calculate the center of mass for these two objects.



The first step is to establish a coordinate system. We're talking about positions I need a coordinate system, so I'm going to establish the positive x direction to be towards the right. Now we can apply the same idea with the grades here with the weight. We can set the position of the 5-kilogram to 0m, and the 2kg to 2m, and the weighted totals will be the position of each weight times its mass over the total mass, or:

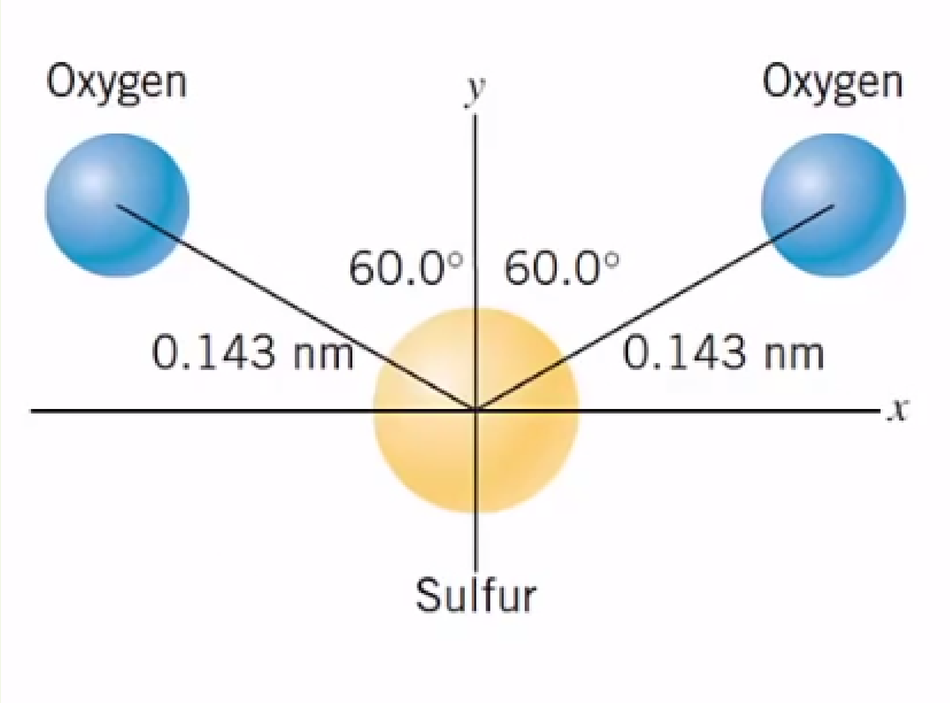
$$(0m)\left(\frac{5kg}{7kg}\right)+(2m)\left(\frac{2kg}{7kg}\right)$$

Notice that the kilograms units will cancel out, leaving it in meters. Since meters is what we're looking for, since we're looking for the position of the center of mass, this makes sense. If I plug this into a calculator, we can see that I get a numerical value of 0.57 meters, which means that the center of mass is somewhere here, much closer to the 5-kilogram object than to the 2-kilogram.

Now let's consider a 2-dimensional example.

Example 3.1

The drawing shows a sulfur dioxide molecule. The sulfur atom is twice as massive as the oxygen atom. Using this information and the data provided in the drawing, find the x and y coordinates of the center of mass of the sulfur dioxide molecule.



As usual, when working in two dimensions, we separate the x and y directions. There's no mass given in this problem, but we can just give the mass of the oxygen the variable M, which would make the mass of the sulfur atom 2M, since it's twice as massive. We also need to set up our coordinate system; let's have the positive x direction be towards the right, the positive y direction to be up, and we can set the sulfur atom to be at 0m in the x and 0m in the y.

Let's look at the sulfur atom to start. The set up for the center of mass contribution of the sulfur atom in the x-direction is:

$$x_{cm} = (x)\left(\frac{2M}{M_{tot}}\right)$$

M_{tot} will be the sum of the masses, which will be the mass of both oxygen atoms and the sulfur atom, or 4M. x is 0m, since we set the position of the sulfur atom to be at 0m. This 0m will make the whole value go to zero, so x_{cm} is 0 for the sulfur. Similarly, since the y position is also 0m, y_{cm} is also 0m.

Now let's look at the oxygen atoms. Again, we use the same setup, and we can substitute M_{tot} with 4M:

$$x_{cm} = (x)\left(\frac{M}{4M}\right)$$

The Ms cancel out here, giving us:

$$x_{cm} = (x)\left(\frac{1}{4}\right)$$

Doing a little trigonometry will give us our x-distance; making a triangle with the line going from the sulfur atom to the oxygen atom and the oxygen atom to the y-axis gives us an x distance sin(60)*0.153nm, or 0.124nm. For the oxygen atom on the left, this value is negative, since it's going left from the origin, and the one on the right is positive, since it's going right of the origin. Plugging these into the center of mass calculation gives us:

$$(0.124nm)\left(\frac{1}{4}\right) + (-0.124nm)\left(\frac{1}{4}\right)$$

Doing the calculation out gives us 0nm.

Alternatively, you can consider the symmetry of the problem. Since the masses are symmetrical both to the left and the right, we can conclude that the center of mass is going to fall along the y-axis at x=0nm. Noticing these symmetries can save you some time, and give you a way to check if your answer makes sense.

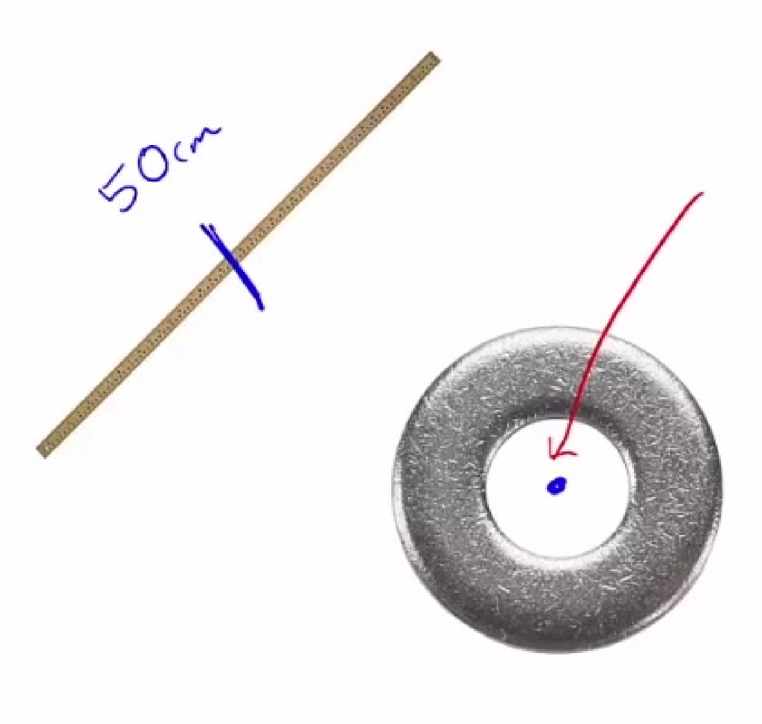
Moving on to the y-direction, we can do the same set up as the x-direction. Doing the trigonometry for the y position gives us cos(60)0.143nm, or 0.072nm. Since both oxygen atoms are above the origin, they are both positive distances away, so our calculation would be:

$$(0.072nm)\left(\frac{1}{4}\right) + (0.072nm)\left(\frac{1}{4}\right)$$

Note that unlike the x-components, the y-components do not cancel out. Doing the calculation gives us 0.036nm. We can write our final answer in coordinates as (0nm, 0.036nm).

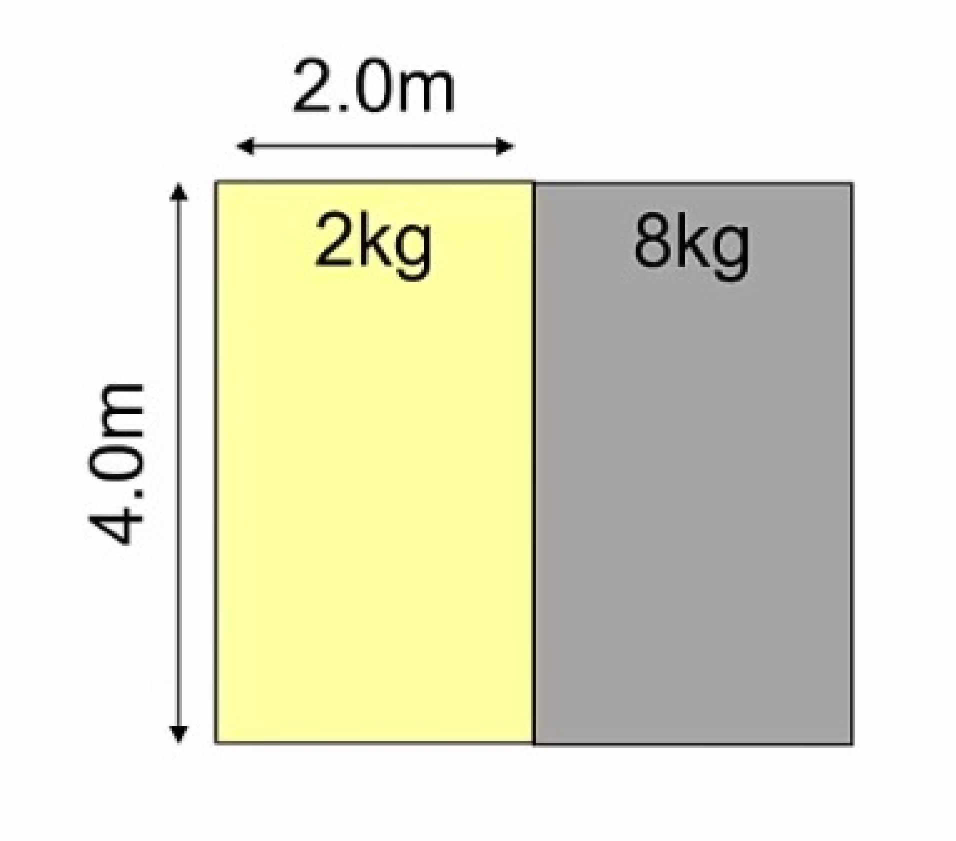
Non-point particles

But what about objects that aren't collections of point particles? Well, if the object is uniform, and this is a very important caveat, then the center of mass will be at the geometric center of the object. For example, if we have say a meter stick, then the center of mass will be at the 50-centimeter mark as that's the middle of the meter stick. If I have a metal washer, then the center of mass will be at the geometric center, right in the middle. You can notice that the center of mass does not need to be inside the body of the object. It can in fact be some random point in space. The center of mass of the washer is not within the metal, it's within the center of the hole.



But what about objects that are not uniform? Well, the answer in this case is divide the object up into uniform chunks, find the center of mass of each chunk, treat each chunk is a point particle located at the center of mass, and then calculate as usual.

So, here I have a nice example problem of a slab made of a light half and a heavy half, and let's think about trying to find the center of mass of this object. Thinking ahead, I expect it to be on this side of the middle, because this is the heavier side of the object. Furthermore, I can already say from the symmetry of the problem that the center of mass is going to lie along this line, right in the middle, vertically, of the object.



Now we follow the same procedure with point masses. First, we establish a coordinate system; we'll set the origin to be at the center of mass of the left chunk. This will simplify our calculation for reasons we'll explore in a bit. We'll also have to break down the problem into x and y coordinates. However, like earlier in this section, we can look at the symmetry of the problem to simplify it. You'll notice that the object is symmetrical in the y direction, and therefore we can assume that the center of mass will be at y=0m. Next, we set up our center of mass calculations for each chunk. For the left chunk, we have:

$$(0m) * (\frac{2kg}{2kg + 8kg})$$

Again, we take the position of the point mass and weight it with its mass over the total mass. Since the position is at 0m, however, we know it'll go to 0; this is why we set the origin to be at the center of mass of one of the objects. Moving on to the right chunk, we'll need to find the position of its center of mass. The center of mass of the object will be at its geometric center, and we can see that there's 1 meter on each side of the center of mass, and 2 meters separating both center of masses. So, we have:

$$(2m) * (\frac{8kg}{2kg + 8kg})$$

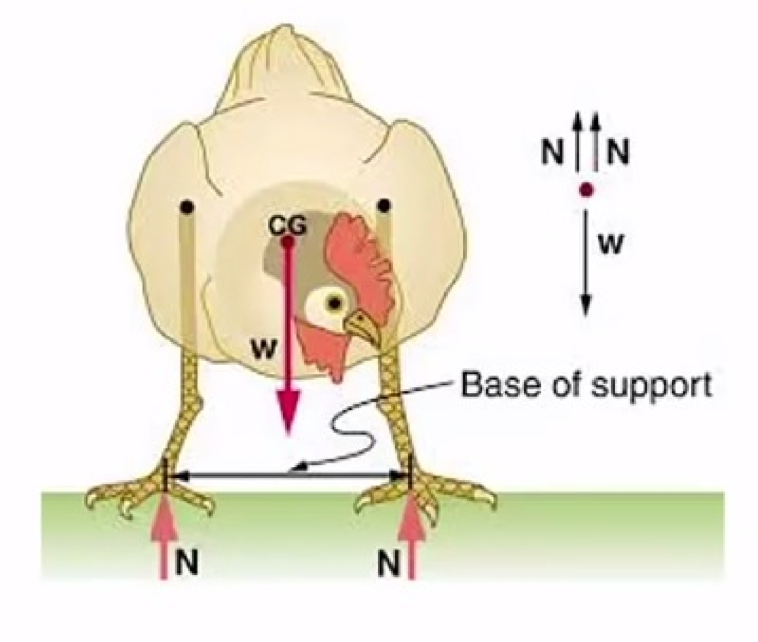
And the total weighted average of the center of mass will be:

$$x_{com} = 0 + (2m) * (\frac{8kg}{2kg + 8kg})$$

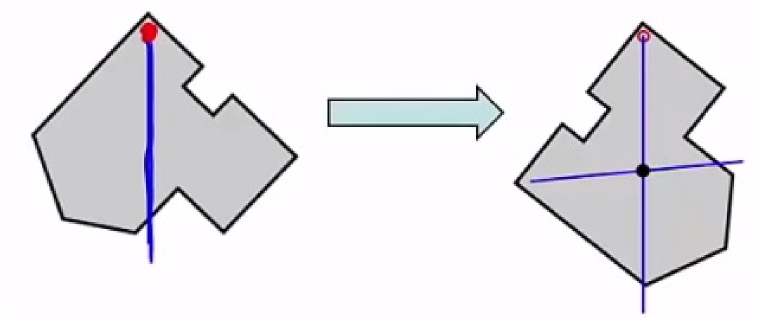
Doing the calculation, we get 1.6m, so the center of mass will be just to the left of the center of mass of the right chunk.

So, that's one part of the definition for center of mass, the mass weighted average position of the object. The center of mass is also the location where gravity can be said to act. As I said at the beginning of this video, when we discuss torque, we'll be interested in where each force acts. For example, when I open a door, I tend to apply the force at the knob in the door. Gravity acts at the center of mass.

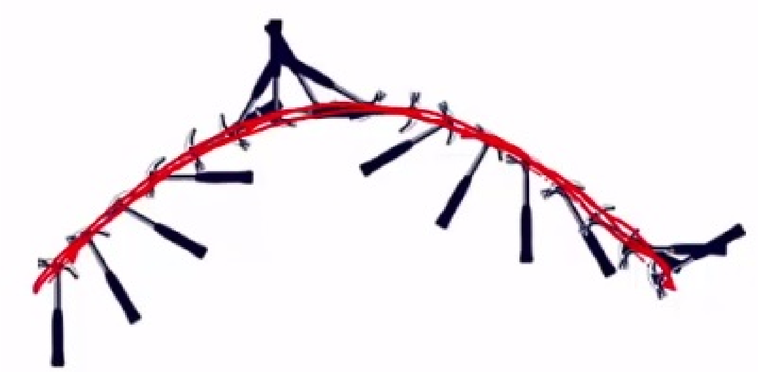
So, if we look at this chicken and we were to make a free body diagram, we would say there are two normal forces, one for each foot, and the weight force, but now we're going to start thinking about where each force is being applied. The two normal forces are being applied one on each foot, so they're being applied there, and the weight force is applied at the center of mass, or the center of gravity, remember that these are synonymous terms as far as this class is concerned, which is roughly in the middle of the chicken.



So, the normal forces get applied where the feet meet the ground and the weight force gets applied at the center of gravity, or the center of mass. So, this is the second part of the definition of center of mass. The center of mass is the point where gravity can be set to act. This part of the definition of center of mass has some consequences. The first piece is that if an object is suspended from a point, then the center of mass will be below the point from which it is hung.



So, let's say we have some oddly shaped object, and I suspend it from a point like here, and I draw a nice line hanging straight down. The center of mass is somewhere on that line. Now, if I take this same object and hang it from a different point, I know that the center of mass is somewhere on this line. Where the two lines cross will be the center of mass. This technique is useful for finding the center of mass of irregular objects. Another consequence of the definition of center of mass as being the point where gravity is said to act means that the center of mass is what follows the parabolic path that we know and love for objects and projectile motion. So, this hammer follows a rather complicated path, but the center of mass, which is closer to the head of the hammer, follows a nice parabolic path just like a ball would.



A final consequence of the center of mass being the point which gravity said to act deals with the balance of an object. Now, if you hold an object under its center of mass, it will balance. So, for the hammer in the previous example, if I put my finger close to the head of the hammer, I can balance the hammer at that point. However, more complicated objects we're usually interested in not balancing on a single point, so we need to define a quantity known as base of support, and the base of support is the region where the object contacts the ground, plus the space in between.

So, for the example of our chicken the base of support is this area between the feet of the chicken, or if you were to look at a person in some rather fancy shoes, the base of support is everywhere the foot contacts the ground, so, this line here connected by a line, plus all the region in between. All of this is the base of support. if the center of mass of an object is above the base of support, then an object will balance. We will explore this in a laboratory activity in class in more tail and become more comfortable with this idea. Right now, what I want you to take away from it is the definition of base of support, and the fact that center of mass sort of has an aspect of its definition, which is that if the center of mass is over the base of support than an object or balance.

Summary

The center of mass in the center of gravity are, as far as this course is concerned, the same idea, so we might alternate between these two different terms, but as far as we're concerned, they're the same. And the definition of center of mass has many different aspects. The first aspect is that the center of mass is the mass weighted average position of an object, and that this point does not need to be inside the object itself, as we saw in the example of the washer. Second aspect of the center of mass is the center of mass is the point at which the force of gravity can be thought to act, and this aspect of the center of mass is definition has a couple of important consequences. The first is that, if an object is suspended, then the center of mass will be below this suspension point on a straight line. We can also say that the center of mass is what follows the parabolic path in projectile motion. A perhaps subtler consequence of this aspect of the center of mass' definition, as being the point at which gravity can be thought to act, is that an object will balance if its center of mass is over its base of support which again, base of support is defined as all of the points where an object meets the ground connected by straight lines, and all of the area inside.

3.3 Torque

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Calculate the net torque exerted by any force on any object including sign
- Describe how to get the maximum torque on an object for a given force
- Contrast force and torque
- Identify the net torque and net force acting on an object at rest

This section is also available as a video lecture, available here:

- **Introduction to Torque (https://www.youtube.com/watch?v=gWHELBPNNKU)**
- **Torque Examples (https://www.youtube.com/watch?v=HjEojBTKveo)**

Before we go into torque, here's a little exercise to try out (yes, it involves doors). With a friend and a door, have your friend push the door open normally, by the doorknob. While they are pushing the door open, try and push back door close to the hinge (be careful not to apply too much force and slam the door into each other). Notice how much force you need to apply to resist the door opening. Now switch positions; notice how much force you need to apply to open the door.

This raises an interesting question; why does the person pushing near the hinge need a lot more force to overcome the force being applied near the doorknob? To understand what's happening here, we need to add another tool to our physics toolbox: torque.

Torque is the turning effect of a force. If a force would cause an object to rotate, like pushing on a door, that force is applying a torque. Mathematically, we represent torque as

$$\tau = rF_{\perp}$$

The r is the distance from the axis of rotation in which the force is being applied, and the F is the force perpendicular to the r. Notice that torque is a vector, and therefore has a direction. However, for torque, we don't apply the traditional up, down, left, right, and so on. Instead, we consider torque to be acting in two directions, clockwise and counterclockwise. A net torque of zero means that all the torque going counterclockwise is equal to the force going clockwise. The units of torque or Nm.

Let's go back to our door example, and try to figure out why the person pushing the door by the doorknob will generally beat out the person pushing near the hinge. If both pushers are pushing at relatively the same amount of force, we can say the force part of the torque is the same for both pushers. However, torque is both dependent on force and the distance from the axis of rotation, in this case the hinge. The person pushing closer to the hinge has a smaller r than that of the person pushing by the doorknob. Notice that in the equation for torque, if r increases, torque increases as well. Thus, we can see why the doorknob pusher has the advantage; their r is larger, so their torque will generally be larger as well.

The r in our equation above references an axis of rotation; what if the object is not rotating at all? This situation gives a freedom when working with problems. If the net torque is zero, then we can put the axis of rotation wherever we want along the axis of (something something). For example, in the case with the door, if it's not rotating, then we can put the axis of rotation anywhere along the axis of the door; we could put it on the doorknob, the hinge, the middle of the door, or even miles away from the door, as long as the door is not rotating. A general rule of thumb is to put the axis of rotation on a point where a force is acting that we do not know and we're not looking for.

Example 3.2

The pedals of a bicycle rotate in a circle with a diameter of 40cm. What is the maximum force a 55-kg rider can apply by putting all her weight on one pedal?

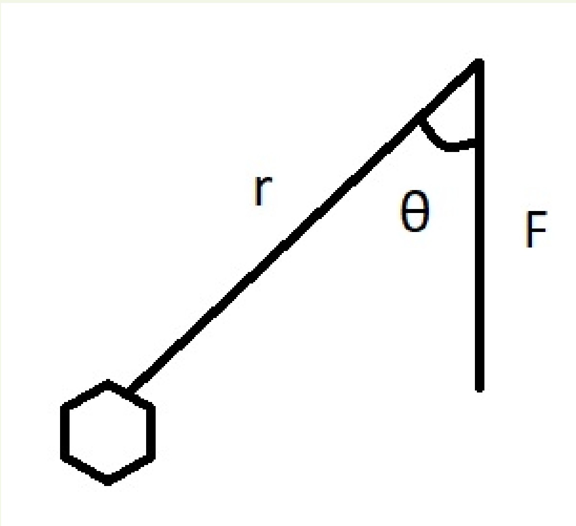
Let's start by considering the situation. The rider applies their force at the end of the pedal, and the pedal rotates at the center, so the distance away from the axis of rotation the force is being applied to should be the distance of the pedal from the center, or the radius of the circle the pedal makes. The diameter is 40 cm, so the radius is 20 cm, or 0.2 m. Next, if the maximum force the rider can apply is all their weight, then the maximum force is equal to the force of weight on the rider, mg. The mass of the rider is 55 kg, so the force is (55 kg)(9.81 kg/N), or 539.55 N. Now we can take our definition of torque and solve for these values, so: $\tau = (0.2m)(539.55N)$

Evaluating this gives us 107.91 Nm, which is the maximum torque.

Example 3.3

A bolt is screwed into a machine you are trying to disassemble, and it needs a torque of 20 Nm to unscrew. How much force do you need to apply to the end of a 30 cm wrench, at an angle of 30 degrees from the wrench, to unscrew the bolt?

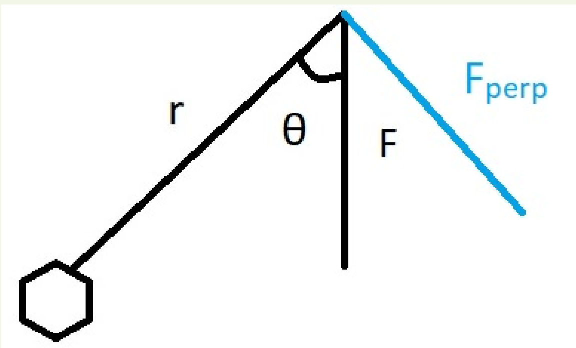
A good starting place with any problem is a simple drawing of the problem:



Here, r is the length of the wrench, F is the force applied, and θ is the angle the force makes with the wrench. Going back to our definition of torque,

$$\boldsymbol{\tau} = rF_{\perp}$$

we have the torque, which is the minimum torque needed to unscrew the bolt, and we have the distance from the axis of rotation to the force, which is the length of the screw. Since the force we're looking for is the force perpendicular to the lever arm, in this case the wrench, and so we need to find the perpendicular part of the force:



Since the perpendicular part of the force is perpendicular to the wrench, we know the angle it makes is 90 degrees, and so we know that the angle between the force and its perpendicular part is (90-30) degrees, or 60 degrees. $F_{\perp}$ is the adjacent to the force, which tells us that $F_{\perp}=F\cos(60)$. Now we have all the parts to solve for F. Substituting our values into the definition of torque:

$$20Nm = (.3m)(Fcos(60))$$

Solving for F brings us to:

$$F = \frac{20Nm}{.3m(\cos(60))}$$

Which gives us a force of 133.33 N, which is the force needed to unscrew the bolt.

4 WORK

4.1 Work: The Scientific Definition

What It Means to Do Work

The scientific definition of work differs in some ways from its everyday meaning. Certain things we think of as hard work, such as writing an exam or carrying a heavy load on level ground, are not work as defined by a scientist. The scientific definition of work reveals its relationship to energy—whenever work is done, energy is transferred.

For work, in the scientific sense, to be done, a force must be exerted and there must be displacement in the direction of the force.

Formally, the **work** done on a system by a constant force is defined to be *the product of the component of the force in the direction of motion times the distance through which the force acts*. For one-way motion in one dimension, this is expressed in equation form as

$$W = |\mathbf{F}|(\cos\theta)|\mathbf{d}|, \tag{4.1}$$

where W is work, $\mathbf{d}$ is the displacement of the system, and θ is the angle between the force vector $\mathbf{F}$ and the displacement vector $\mathbf{d}$, as in **Figure 4.1**. We can also write this as

$$W = Fd\cos\theta. \tag{4.2}$$

To find the work done on a system that undergoes motion that is not one-way or that is in two or three dimensions, we divide the motion into one-way one-dimensional segments and add up the work done over each segment.

What is Work?

The work done on a system by a constant force is *the product of the component of the force in the direction of motion times the distance through which the force acts*. For one-way motion in one dimension, this is expressed in equation form as

$$W = Fd\cos\theta, \tag{4.3}$$

where W is work, F is the magnitude of the force on the system, d is the magnitude of the displacement of the system, and θ is the angle between the force vector $\mathbf{F}$ and the displacement vector $\mathbf{d}$.

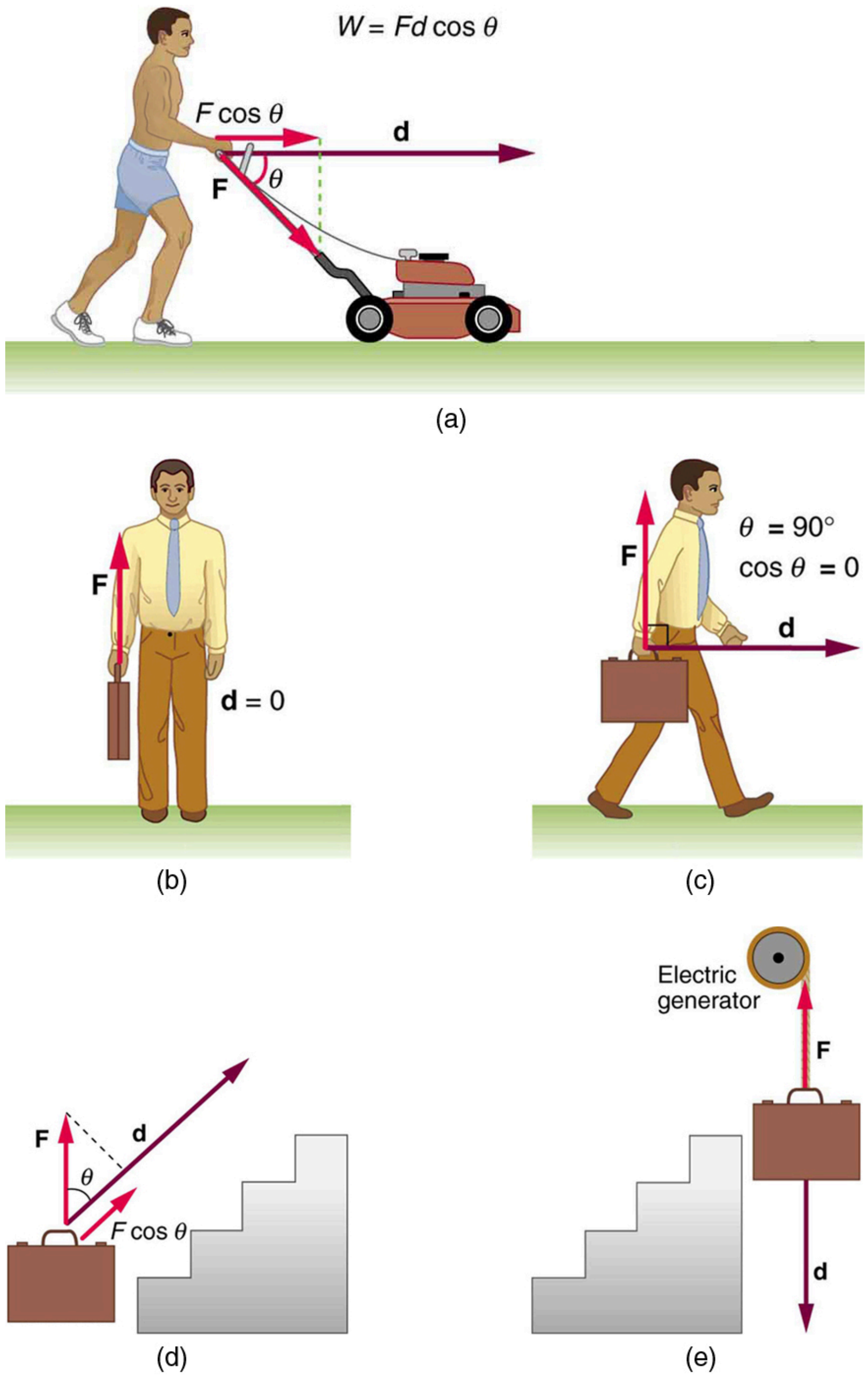


Figure 4.1 Examples of work. (a) The work done by the force $\mathbf{F}$ on this lawn mower is $Fd\cos\theta$. Note that $F\cos\theta$ is the component of the force in the direction of motion. (b) A person holding a briefcase does no work on it, because there is no displacement. No energy is transferred to or from the briefcase. (c) The person moving the briefcase horizontally at a constant speed does no work on it, and transfers no energy to it. (d) Work *is* done on the briefcase by carrying it up stairs at constant speed, because there is necessarily a component of force $\mathbf{F}$ in the direction of the motion. Energy is transferred to the briefcase and could in turn be used to do work. (e) When the briefcase is lowered, energy is transferred out of the briefcase and into an electric generator. Here the work done on the briefcase by the generator is negative, removing energy from the briefcase, because $\mathbf{F}$ and $\mathbf{d}$ are in opposite directions.

To examine what the definition of work means, let us consider the other situations shown in **Figure 4.1**. The person holding the briefcase in **Figure 4.1**(b) does no work, for example. Here $d = 0$, so $W = 0$. Why is it you get tired just holding a load? The answer is that your muscles are doing work against one another, *but they are doing no work on the system of interest* (the "briefcase-Earth system"—see **Gravitational Potential Energy (https://legacy.cnx.org/content/m42148/latest/)** for more details). There must be displacement for work to be done, and there must be a component of the force in the direction of the motion. For example, the person carrying the briefcase on level ground in **Figure 4.1**(c) does no work on it, because the force is perpendicular to the motion. That is, $\cos 90^\circ = 0$, and so $W = 0$.

In contrast, when a force exerted on the system has a component in the direction of motion, such as in **Figure 4.1**(d), work *is* done—energy is transferred to the briefcase. Finally, in **Figure 4.1**(e), energy is transferred from the briefcase to a generator. There are two good ways to interpret this energy transfer. One interpretation is that the briefcase's weight does work on the generator, giving it energy. The other interpretation is that the generator does negative work on the briefcase, thus removing energy from it. The drawing shows the latter, with the force from the generator upward on the briefcase, and the displacement downward. This makes $\theta = 180^\circ$, and $\cos 180^\circ = -1$; therefore, W is negative.

Calculating Work

Work and energy have the same units. From the definition of work, we see that those units are force times distance. Thus, in SI units, work and energy are measured in **newton-meters**. A newton-meter is given the special name **joule** (J), and $1\text{ J} = 1\text{ N}\cdot\text{m} = 1\text{ kg}\cdot\text{m}^2/\text{s}^2$. One joule is not a large amount of energy; it would lift a small 100-gram apple a distance of about 1 meter.

Example 4.1 Calculating the Work You Do to Push a Lawn Mower Across a Large Lawn

How much work is done on the lawn mower by the person in **Figure 4.1**(a) if he exerts a constant force of 75.0 N at an angle 35° below the horizontal and pushes the mower 25.0 m on level ground? Convert the amount of work from joules to kilocalories and compare it with this person's average daily intake of $10,000\text{ kJ}$ (about 2400 kcal) of food energy. One *calorie* (1 cal) of heat is the amount required to warm 1 g of water by 1°C, and is equivalent to 4.184 J, while one *food calorie* (1 kcal) is equivalent to 4184 J.

Strategy

We can solve this problem by substituting the given values into the definition of work done on a system, stated in the equation $W = Fd\cos\theta$. The force, angle, and displacement are given, so that only the work W is unknown.

Solution

The equation for the work is

$$W = Fd\cos\theta. \tag{4.4}$$

Substituting the known values gives

$$\begin{aligned} W &= (75.0\text{ N})(25.0\text{ m})\cos(35.0^\circ) \\ &= 1536\text{ J} = 1.54\times10^3\text{ J}. \end{aligned} \tag{4.5}$$

Converting the work in joules to kilocalories yields $W = (1536\text{ J})(1\text{ kcal}/4184\text{ J}) = 0.367\text{ kcal}$. The ratio of the work done to the daily consumption is

$$\frac{W}{2400\text{ kcal}} = 1.53\times10^{-4}. \tag{4.6}$$

Discussion

This ratio is a tiny fraction of what the person consumes, but it is typical. Very little of the energy released in the consumption of food is used to do work. Even when we "work" all day long, less than 10% of our food energy intake is used to do work and more than 90% is converted to thermal energy or stored as chemical energy in fat.

Section Summary

- Work is the transfer of energy by a force acting on an object as it is displaced.
- The work W that a force $\mathbf{F}$ does on an object is the product of the magnitude F of the force, times the magnitude d of the displacement, times the cosine of the angle θ between them. In symbols,

$$W = Fd\cos\theta. \tag{4.7}$$

- The SI unit for work and energy is the joule (J), where $1\text{ J} = 1\text{ N}\cdot\text{m} = 1\text{ kg}\cdot\text{m}^2/\text{s}^2$.
- The work done by a force is zero if the displacement is either zero or perpendicular to the force.
- The work done is positive if the force and displacement have the same direction, and negative if they have opposite direction.

Conceptual Questions

Exercise 4.1

Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.

Exercise 4.2

Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.

Exercise 4.3

Describe a situation in which a force is exerted for a long time but does no work. Explain.

Problems & Exercises

Exercise 4.4

How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N? Express your answer in joules and kilocalories.

Solution

$$3.00\text{ J} = 7.17\times10^{-4}\text{ kcal} \tag{4.8}$$

Exercise 4.5

A 75.0-kg person climbs stairs, gaining 2.50 meters in height. Find the work done to accomplish this task.

Exercise 4.6

(a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the lift by the gravitational force in this process? (c) What is the total work done on the lift?

Solution

(a) $5.92\times10^{5}\text{ J}$

(b) $-5.88\times10^{5}\text{ J}$

(c) The net force is zero.

Exercise 4.7

Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (See **m42151 (https://legacy.cnx.org/content/m42151/latest/#import-auto-id2866785)** for the energy content of gasoline.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed? (b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?

Exercise 4.8

Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of 20.0° with the horizontal. (See **Figure 4.2**.) He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate *and* on his body to get up the ramp.

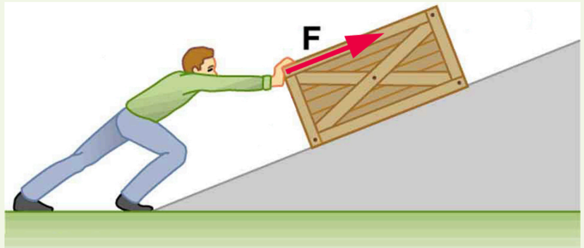


Figure 4.2 A man pushes a crate up a ramp.

Solution

$$3.14\times10^{3}\ \mathrm{J} \tag{4.9}$$

Exercise 4.9

How much work is done by the boy pulling his sister 30.0 m in a wagon as shown in **Figure 4.3**? Assume no friction acts on the wagon.

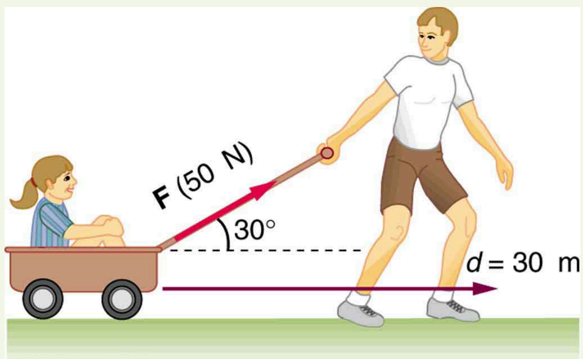


Figure 4.3 The boy does work on the system of the wagon and the child when he pulls them as shown.

Exercise 4.10

A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force. He pushes in a direction 25.0^{o} below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?

Solution

(a) $-700\ \mathrm{J}$

(b) 0

(c) 700 J

(d) 38.6 N

(e) 0

Exercise 4.11

Suppose the ski patrol lowers a rescue sled and victim, having a total mass of 90.0 kg, down a 60.0^{o} slope at constant speed, as shown in **Figure 4.4**. The coefficient of friction between the sled and the snow is 0.100. (a) How much work is done by friction as the sled moves 30.0 m along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?

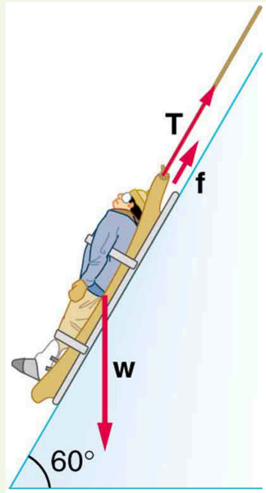


Figure 4.4 A rescue sled and victim are lowered down a steep slope.

4.2 Representing Work Graphically

Net Work and the Work-Energy Theorem

We know from the study of Newton's laws in **Dynamics: Force and Newton's Laws of Motion (https://legacy.cnx.org/content/m42129/latest/)** that net force causes acceleration. We will see in this section that work done by the net force gives a system energy of motion, and in the process we will also find an expression for the energy of motion.

Let us start by considering the total, or net, work done on a system. Net work is defined to be the sum of work done by all external forces—that is, **net work** is the work done by the net external force $\mathbf{F}_{\text{net}}$. In equation form, this is $W_{\text{net}} = F_{\text{net}} d \cos \theta$ where θ is the angle between the force vector and the displacement vector.

Figure 4.5(a) shows a graph of force versus displacement for the component of the force in the direction of the displacement—that is, an $F \cos \theta$ vs. d graph. In this case, $F \cos \theta$ is constant. You can see that the area under the graph is $Fd \cos \theta$, or the work done. **Figure 4.5**(b) shows a more general process where the force varies. The area under the curve is divided into strips, each having an average force $(F \cos \theta)_{i(\text{ave})}$. The work done is $(F \cos \theta)_{i(\text{ave})} d_i$ for each strip, and the total work done is the sum of the W_i. Thus the total work done is the total area under the curve, a useful property to which we shall refer later.

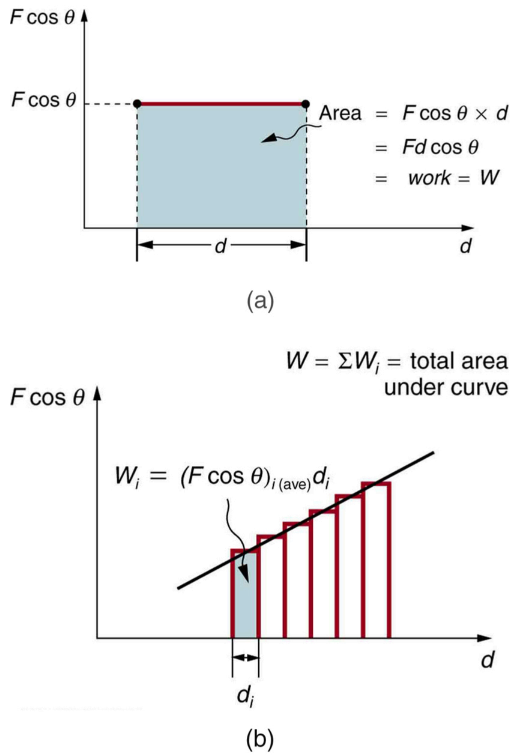


Figure 4.5 (a) A graph of $F \cos\theta$ vs. d, when $F \cos\theta$ is constant. The area under the curve represents the work done by the force. (b) A graph of $F \cos\theta$ vs. d in which the force varies. The work done for each interval is the area of each strip; thus, the total area under the curve equals the total work done.

4.3 Work in Terms of Pressure

UMASS AMHERST Instructor's Notes

This sections is also available as a video: **https://www.youtube.com/watch?v=7F4kS0tXkCU (https://www.youtube.com/watch?v=7F4kS0tXkCU)**

Let's explore another way to think about work. So, why do we need another way to think about work, why do we need another definition for this idea of work? Well the definition of work that we have, Fdcos(θ), works really well for objects like people and cars and blocks. However, in the life sciences and beginning in the next unit as well, we are often interested in the behavior of things that physicists call fluids, which means gases and liquids. For gases and liquids, you tend to not be interested in the force you apply to them but instead the pressure. To remind you, pressure is the force divided by the area, which means that it would have units of N/m^2. Some other common units of pressure that you might have seen are Pascals or atmospheres. Think about the air pressure in a tire. You might see that quoted as N/cm^2 or in atmospheres. Or, you might also think about the osmotic pressure of a fluid inside of a cell. It is therefore useful to be able to think about the work done on the fluid in terms of pressure.

How will we figure out the expression for work in terms of pressure? Well, we will do this simple example of a gas inside of a container with a piston.

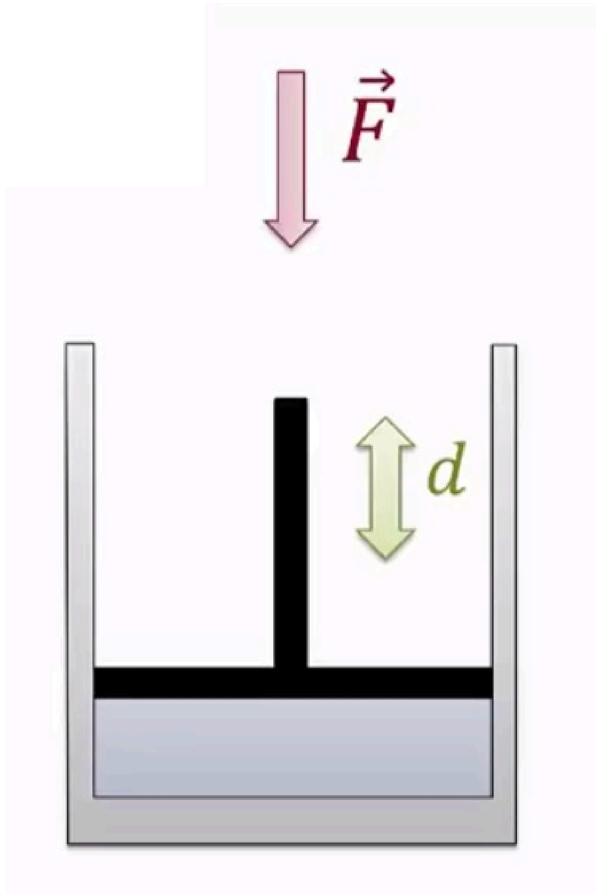


Figure 4.6 A gas being compressed by a piston. The piston does work on the gas equal to the force exerted on it times the distance it moves.

As we compress the gas, we apply a force on the piston for some distance. We are doing work on the gas, and since the force and the displacement are in the same direction, we know that that work is positive. Most the time in the life sciences in particular, we are interested in gases at what we call constant pressure, because organisms and chemical reactions are open to the air and therefore everything happens at a constant pressure of one at. In our example up to this point, the gas is going to increase in pressure as we compress. So, to solve that problem, let's puncture a little hole in the bottom, and let some gas escape to maintain constant pressure of the gas as we compress it.

How much work is done on the gas by our force in this admittedly very contrived case? Well, let's think about this. The gas has some amount of volume before we compress it. We can think about the area of the bottom of the container A, and the height h, and if you multiply these two quantities together, you'll get the volume, Ah. As we apply our force to the piston, compressing it and having some of the gas leak out to maintain constant pressure, the area stays the same, but the height shrinks, and it in shrinks by the exact amount of the displacement d that we compress the piston. So, we can say the change in height is equal to -d, where we have this negative sign because the height is getting smaller as the distance is getting larger.

The change in volume of the gas is then A, the area, which doesn't change, times the change in height or, -Ad. If we multiply this change in volume by the pressure of the gas inside, $P\Delta V$, and then we replace the pressure with this definition of force over area and volume $A\Delta h$, and then we use our relationship we've just discovered that the change in height is equal to negative the change in distance, then we see that the area's cancel out, and we are left with just a force times the distance. We're left with the amount of work that we did, Fd is the amount of work we did on this piston. The only difference is we get a negative sign.

Therefore, we can conclude that the work done on the gas is minus the pressure of the gas times the change in volume. We're getting this negative sign because if the force and the displacement are in the same direction, that means the volume of the gas is going to get smaller. So, a positive work on the gas will result in a smaller volume, a $-\Delta V$.

In summary, for a fluid, i.e. a gas or liquid, at constant pressure, the work done by some external force on the fluid can be written in terms of the pressure of the fluid and the change in the volume of the fluid. Mathematically, we say that the work is equal to the $-P\Delta V$. Mathematically, we say that the $W=-P\Delta V$, and this negative sign results because positive work done on the gas will result in the gas compressing, and therefore shrinking volume. This concludes this video.

Glossary

energy: the ability to do work

joule: SI unit of work and energy, equal to one newton-meter

work: the transfer of energy by a force that causes an object to be displaced; the product of the component of the force in the direction of the displacement and the magnitude of the displacement

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Unit IV
Energy

Photo credit: Phelan, John. *English: South College, College of Humanities and Fine Arts, University of Massachusetts Amherst, Amherst Massachusetts*, March 7, 2014. Own work. https://commons.wikimedia.org/wiki/File:South_College,_College_of_Humanities_and_Fine_Arts,_UMass,_Amherst_MA.jpg.

Table of Contents

UNIT 4 OVERVIEW

This overview is based on
umdberg / Reductionism and emergence (2015). Available at: ***http://umdberg.pbworks.com/w/page/68371403/Reductionism%20and%20emergence%20(2015) (http://umdberg.pbworks.com/w/page/68371403/Reductionism%20and%20emergence%20(2015))*** *. (Accessed: 23rd August 2017)*

Science considers a wide variety of scales. In physics and biology we consider

- the nanoscale of atoms and molecules -- distances of nanometers (atomic size) and timescales of microseconds;
- the microscopic scale used in cellular biology -- distances of microns to millimeters and times of fractions of a second;
- the macroscopic human scale we are used to in everyday life -- distances of meters to miles and times of seconds to years;
- the global/deep time scale of ecology and evolution -- distances spanning thousands of kilometers and times of millions of years.

Physics considers scales beyond this including subatomic and even subnuclear structures and cosmological scales to the size of the entire visible universe and timescales of the lifetime of the universe.

We tend to analyze each system on its own terms using the concepts appropriate to the scale. But some of the most interesting scientific insights come from crossing scales. When we explain the properties of systems at one scale in terms of its component parts at a finer scale it's called ***reductionism***. So when we analyze the conductivity of copper in terms of the bonding properties of copper atoms and how they share electrons from one atom in a crystal to the next, it's reductionism. When a patient experiences palpitations, sweating, dizziness, and headache, a caregiver might interpret this as occurring as a result of an imbalance in a chemical in the bloodstream -- insulin. Both of these explanations of a macro event in terms of microscopic properties are reductionism. We are looking at a system at a large scale and its characteristics are explained by something happening at a smaller scale.

We can consider the phenomenon of scale crossing in the other direction. If we are looking at a system at one scale, we might find that effects that seem very small add coherently to produce a dramatic and important effect when we step back and look at things at a larger scale. When the properties of a system at the scale we are considering have effects on a system of a larger scale, it's called ***emergence***, especially when the phenomenon might be almost un-noticable at the scale we are considering.

A physical example is polarization. If we put an atom in an electric field of typical macroscopic values (a few volts per meter), the electrons and the nucleus are pulled apart -- but only by a very tiny amount, perhaps 1 part in 100,000 of the atom's diameter. We might assume this is so tiny an effect that it can be ignored; but if every atom in a macroscopic object undergoes the same slight separation, the total effect might be that we can pick up the object, lifting it against gravity with the sum of the tiny electric forces we are exerting on each atom. The fact that we have so many atoms multiplies what looked to be a tiny effect. A similar phenomenon occurs in biology when there is a small survivability advantage to a mutation. One might not see any effect for many generations, but given thousands of generations, the gene pool of a population can be completely transformed by natural selection.

Of course, these are the same phenomenon, just looked at from different angles. But whatever scale we are considering, keeping these two perspectives in mind will help us to look for structures both at smaller scales that might provide reductionist explanations and at larger scales that might have emergent properties.

1 INTRODUCTION TO THE MICROSCOPIC WORLD

1.1 Introduction

UMASS AMHERST Instructor's Notes

We expect that all of the material in this chapter is review for you from other courses, and is included here for reference. The topics with which you are least likely to be familiar are:

- The expression of the ideal gas law in terms of molecules instead of moles and k_B vs R in section 12.5
- The limitations of the ideal gas law in section 12.5
- The Einsteinian Solid in section 12.6

Your quiz will cover:

- Converting between atoms, grams, and moles for any compound given a periodic table.
- The limitations of the ideal gas law

This introduction is based from
umdberg / The micro to macro connection (2013). Available at: **http://umdberg.pbworks.com/w/page/68405576/The%20micro%20to%20macro%20connection%20(2013) (http://umdberg.pbworks.com/w/page/68405576/The%20micro%20to%20macro%20connection%20(2013))** . (Accessed: 2nd August 2017)

Much of what we have done so far -- the Newtonian framework, describing the properties of solids and liquids, and the concepts of heat and temperature -- are *macroscopic* concepts: they describe things we see, feel, and experience. They express the regularities and consistencies of the behavior of physical systems. Much of this was well known by the middle of the nineteenth century. But one of the most extraordinary and important pieces of knowledge that humanity has garnered since then is the idea of the *microscopic*. By this, physicists don't mean "what you can see in a microscope", but rather the fact that everything we regularly experience is made up of a small number of different kinds of atoms (91 in the natural world, a few more that have been created by humans). The essential point about this is that we believe that all properties of the macroscopic world are ultimately due to the properties and interactions of those 91 distinct elements. Although some phenomena require a description at a higher level (see the discussion of emergent phenomena), at some level (even if it's not convenient or useful for us to explicate), everything we see is a result of atomic properties.

A major component of modern biology is working at the microscopic -- atomic and molecular -- level and learning what are the critical elements that underlie basic biological mechanisms. Much of the research and development that can be expected to transform both biology and medicine over the next few decades will depend on making sense of the micro to macro connection. In this class, we will develop a few of the basic tools needed for making this connection. One set of tools involves *statistical physics*. Since there is a huge amount of energy distributed in all objects at common temperatures, and since these energies tend to be *randomly distributed* among the atoms and molecules of a substance, the science of figuring out the implications of randomness is critical for understanding many biological phenomena.

We will begin our study of the implications of microscopic properties and randomness with two phenomena: *kinetic theory* and *diffusion*. Kinetic theory is about understanding thermal phenomena in molecular terms, and diffusion is about what happens when materials are not uniformly distributed. Analyzing both of these using the methods of statistical physics will give us insights into the mechanism of a large class of complex and important phenomena.

In this chapter, we will look at some important properties of matter at the molecular scale such as the idea of a mole which you may know from a previous course. We will then develop molecule-based pictures of gases and solids. We will use these models of matter to help us to develop a coherent picture of energy that spans from our everyday world to the world of molecules.

1.2 Atomic Structure and Symbolism

The development of modern atomic theory revealed much about the inner structure of atoms. It was learned that an atom contains a very small nucleus composed of positively charged protons and uncharged neutrons, surrounded by a much larger volume of space containing negatively charged electrons. The nucleus contains the majority of an atom's mass because protons and neutrons are much heavier than electrons, whereas electrons occupy almost all of an atom's volume. The diameter of an atom is on the order of 10^{-10} m, whereas the diameter of the nucleus is roughly 10^{-15} m—about 100,000 times smaller. For a perspective about their relative sizes, consider this: If the nucleus were the size of a blueberry, the atom would be about the size of a football stadium (**Figure 1.1**).

Figure 1.1 If an atom could be expanded to the size of a football stadium, the nucleus would be the size of a single blueberry. (credit middle: modification of work by "babyknight"/Wikimedia Commons; credit right: modification of work by Paxson Woelber)

Atoms—and the protons, neutrons, and electrons that compose them—are extremely small. For example, a carbon atom weighs less than 2×10^{-23} g, and an electron has a charge of less than 2×10^{-19} C (coulomb). When describing the properties of tiny objects such as atoms, we use appropriately small units of measure, such as the **atomic mass unit (amu)** and the **fundamental unit of charge (e)**. The amu was originally defined based on hydrogen, the lightest element, then later in terms of oxygen. Since 1961, it has been defined with regard to the most abundant isotope of carbon, atoms of which are assigned masses of exactly 12 amu. (This isotope is known as "carbon-12" as will be discussed later in this module.) Thus, one amu is exactly $\frac{1}{12}$ of the mass of one carbon-12 atom: 1 amu = 1.6605×10^{-24} g. (The **Dalton (Da)** and the **unified atomic mass unit (u)** are alternative units that are equivalent to the amu.) The fundamental unit of charge (also called the elementary charge) equals the magnitude of the charge of an electron (e) with e = 1.602×10^{-19} C.

A proton has a mass of 1.0073 amu and a charge of 1+. A neutron is a slightly heavier particle with a mass 1.0087 amu and a charge of zero; as its name suggests, it is neutral. The electron has a charge of 1− and is a much lighter particle with a mass of about 0.00055 amu (it would take about 1800 electrons to equal the mass of one proton. The properties of these fundamental particles are summarized in **Table 1.1**. (An observant student might notice that the sum of an atom's subatomic particles does not equal the atom's actual mass: The total mass of six protons, six neutrons, and six electrons is 12.0993 amu, slightly larger than 12.00 amu. This "missing" mass is known as the mass defect, and you will learn about it in the chapter on nuclear chemistry.)

Table 1.1

Properties of Subatomic Particles					
Name	**Location**	**Charge (C)**	**Unit Charge**	**Mass (amu)**	**Mass (g)**
electron	outside nucleus	-1.602×10^{-19}	1−	0.00055	0.00091×10^{-24}
proton	nucleus	1.602×10^{-19}	1+	1.00727	1.67262×10^{-24}
neutron	nucleus	0	0	1.00866	1.67493×10^{-24}

The number of protons in the nucleus of an atom is its **atomic number (Z)**. This is the defining trait of an element: Its value determines the identity of the atom. For example, any atom that contains six protons is the element carbon and has the atomic number 6, regardless of how many neutrons or electrons it may have. A neutral atom must contain the same number of positive and negative charges, so the number of protons equals the number of electrons. Therefore, the atomic number also indicates the number of electrons in an atom. The total number of protons and neutrons in an atom is called its **mass number (A)**. The number of neutrons is therefore the difference between the mass number and the atomic number: A – Z = number of neutrons.

$$\begin{aligned} \text{atomic number (Z)} &= \text{number of protons} \\ \text{mass number (A)} &= \text{number of protons} + \text{number of neutrons} \\ \text{A} - \text{Z} &= \text{number of neutrons} \end{aligned} \tag{1.1}$$

Atoms are electrically neutral if they contain the same number of positively charged protons and negatively charged electrons. When the numbers of these subatomic particles are *not* equal, the atom is electrically charged and is called an **ion**. The charge of an atom is defined as follows:

Atomic charge = number of protons − number of electrons

As will be discussed in more detail later in this chapter, atoms (and molecules) typically acquire charge by gaining or losing electrons. An atom that gains one or more electrons will exhibit a negative charge and is called an **anion**. Positively charged atoms called **cations** are formed when an atom loses one or more electrons. For example, a neutral sodium atom (Z = 11) has 11 electrons. If this atom loses one electron, it will become a cation with a 1+ charge (11 − 10 = 1+). A neutral oxygen atom (Z = 8) has eight electrons, and if it gains two electrons it will become an anion with a 2− charge (8 − 10 = 2−).

Example 1.1

Composition of an Atom

Iodine is an essential trace element in our diet; it is needed to produce thyroid hormone. Insufficient iodine in the diet can lead to the development of a goiter, an enlargement of the thyroid gland (**Figure 1.2**).

Figure 1.2 (a) Insufficient iodine in the diet can cause an enlargement of the thyroid gland called a goiter. (b) The addition of small amounts of iodine to salt, which prevents the formation of goiters, has helped eliminate this concern in the US where salt consumption is high. (credit a: modification of work by “Almazi”/Wikimedia Commons; credit b: modification of work by Mike Mozart)

The addition of small amounts of iodine to table salt (iodized salt) has essentially eliminated this health concern in the United States, but as much as 40% of the world’s population is still at risk of iodine deficiency. The iodine atoms are added as anions, and each has a 1− charge and a mass number of 127. Determine the numbers of protons, neutrons, and electrons in one of these iodine anions.

Solution

The atomic number of iodine (53) tells us that a neutral iodine atom contains 53 protons in its nucleus and 53 electrons outside its nucleus. Because the sum of the numbers of protons and neutrons equals the mass number, 127, the number of neutrons is 74 (127 − 53 = 74). Since the iodine is added as a 1− anion, the number of electrons is 54 [53 – (1–) = 54].

Check Your Learning

An ion of platinum has a mass number of 195 and contains 74 electrons. How many protons and neutrons does it contain, and what is its charge?

Answer:

78 protons; 117 neutrons; charge is 4+

Chemical Symbols

A **chemical symbol** is an abbreviation that we use to indicate an element or an atom of an element. For example, the symbol for mercury is Hg (**Figure 1.3**). We use the same symbol to indicate one atom of mercury (microscopic domain) or to label a container of many atoms of the element mercury (macroscopic domain).

Figure 1.3 The symbol Hg represents the element mercury regardless of the amount; it could represent one atom of mercury or a large amount of mercury.

The symbols for several common elements and their atoms are listed in **Table 1.2**. Some symbols are derived from the common name of the element; others are abbreviations of the name in another language. Most symbols have one or two letters, but three-letter symbols have been used to describe some elements that have atomic numbers greater than 112. To avoid confusion with other notations, only the first letter of a symbol is capitalized. For example, Co is the symbol for the element cobalt, but CO is the notation for the compound carbon monoxide, which contains atoms of the elements carbon (C) and oxygen (O). All known elements and their symbols are in the periodic table in **m51003 (https://legacy.cnx.org/content/m51003/latest/#CNX_Chem_02_05_PerTable1)** (also found in **m51209 (https://legacy.cnx.org/content/m51209/latest/#fs-idp64991424)**).

Table 1.2

Some Common Elements and Their Symbols			
Element	**Symbol**	**Element**	**Symbol**
aluminum	Al	iron	Fe (from *ferrum*)
bromine	Br	lead	Pb (from *plumbum*)
calcium	Ca	magnesium	Mg
carbon	C	mercury	Hg (from *hydrargyrum*)
chlorine	Cl	nitrogen	N
chromium	Cr	oxygen	O
cobalt	Co	potassium	K (from *kalium*)
copper	Cu (from *cuprum*)	silicon	Si
fluorine	F	silver	Ag (from *argentum*)
gold	Au (from *aurum*)	sodium	Na (from *natrium*)
helium	He	sulfur	S
hydrogen	H	tin	Sn (from *stannum*)
iodine	I	zinc	Zn

Traditionally, the discoverer (or discoverers) of a new element names the element. However, until the name is recognized by the International Union of Pure and Applied Chemistry (IUPAC), the recommended name of the new element is based on the Latin word(s) for its atomic number. For example, element 106 was called unnilhexium (Unh), element 107 was called unnilseptium (Uns), and element 108 was called unniloctium (Uno) for several years. These elements are now named after scientists (or occasionally locations); for example, element 106 is now known as *seaborgium* (Sg) in honor of Glenn Seaborg, a Nobel Prize winner who was active in the discovery of several heavy elements.

Visit this **site (http://openstaxcollege.org/l/16IUPAC)** to learn more about IUPAC, the International Union of Pure and Applied Chemistry, and explore its periodic table.

Isotopes

The symbol for a specific isotope of any element is written by placing the mass number as a superscript to the left of the element symbol (**Figure 1.4**). The atomic number is sometimes written as a subscript preceding the symbol, but since this number defines the element's identity, as does its symbol, it is often omitted. For example, magnesium exists as a mixture of three isotopes, each with an atomic number of 12 and with mass numbers of 24, 25, and 26, respectively. These isotopes can be identified as ^{24}Mg, ^{25}Mg, and ^{26}Mg. These isotope symbols are read as "element, mass number" and can be symbolized consistent with this reading. For instance, ^{24}Mg is read as "magnesium 24," and can be written as "magnesium-24" or "Mg-24." ^{25}Mg is read as "magnesium 25," and can be written as "magnesium-25" or "Mg-25." All magnesium atoms have 12 protons in their nucleus. They differ only because a ^{24}Mg atom has 12 neutrons in its nucleus, a ^{25}Mg atom has 13 neutrons, and a ^{26}Mg has 14 neutrons.

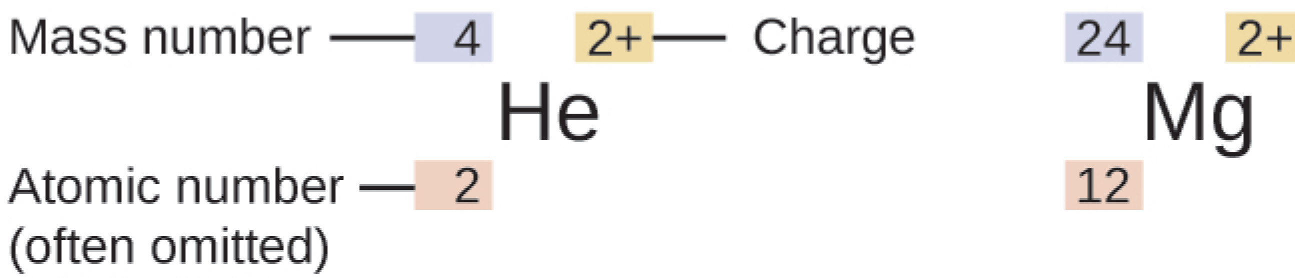


Figure 1.4 The symbol for an atom indicates the element via its usual two-letter symbol, the mass number as a left superscript, the atomic number as a left subscript (sometimes omitted), and the charge as a right superscript.

Information about the naturally occurring isotopes of elements with atomic numbers 1 through 10 is given in **Table 1.3**. Note that in addition to standard names and symbols, the isotopes of hydrogen are often referred to using common names and accompanying symbols. Hydrogen-2, symbolized ^{2}H, is also called deuterium and sometimes symbolized D. Hydrogen-3, symbolized ^{3}H, is also called tritium and sometimes symbolized T.

Table 1.3

Nuclear Compositions of Atoms of the Very Light Elements						
Element	Symbol	Atomic Number	Number of Protons	Number of Neutrons	Mass (amu)	% Natural Abundance
hydrogen	${}_{1}^{1}\text{H}$ (protium)	1	1	0	1.0078	99.989
	${}_{1}^{2}\text{H}$ (deuterium)	1	1	1	2.0141	0.0115
	${}_{1}^{3}\text{H}$ (tritium)	1	1	2	3.01605	— (trace)
helium	${}_{2}^{3}\text{He}$	2	2	1	3.01603	0.00013
	${}_{2}^{4}\text{He}$	2	2	2	4.0026	100
lithium	${}_{3}^{6}\text{Li}$	3	3	3	6.0151	7.59
	${}_{3}^{7}\text{Li}$	3	3	4	7.0160	92.41
beryllium	${}_{4}^{9}\text{Be}$	4	4	5	9.0122	100
boron	${}_{5}^{10}\text{B}$	5	5	5	10.0129	19.9
	${}_{5}^{11}\text{B}$	5	5	6	11.0093	80.1
carbon	${}_{6}^{12}\text{C}$	6	6	6	12.0000	98.89
	${}_{6}^{13}\text{C}$	6	6	7	13.0034	1.11
	${}_{6}^{14}\text{C}$	6	6	8	14.0032	— (trace)
nitrogen	${}_{7}^{14}\text{N}$	7	7	7	14.0031	99.63
	${}_{7}^{15}\text{N}$	7	7	8	15.0001	0.37
oxygen	${}_{8}^{16}\text{O}$	8	8	8	15.9949	99.757
	${}_{8}^{17}\text{O}$	8	8	9	16.9991	0.038
	${}_{8}^{18}\text{O}$	8	8	10	17.9992	0.205
fluorine	${}_{9}^{19}\text{F}$	9	9	10	18.9984	100
neon	${}_{10}^{20}\text{Ne}$	10	10	10	19.9924	90.48
	${}_{10}^{21}\text{Ne}$	10	10	11	20.9938	0.27
	${}_{10}^{22}\text{Ne}$	10	10	12	21.9914	9.25

Use this **Build an Atom simulator (http://openstaxcollege.org/l/16PhetAtomBld)** to build atoms of the first 10 elements, see which isotopes exist, check nuclear stability, and gain experience with isotope symbols.

Atomic Mass

Because each proton and each neutron contribute approximately one amu to the mass of an atom, and each electron contributes far less, the **atomic mass** of a single atom is approximately equal to its mass number (a whole number). However, the average masses of atoms of most elements are not whole numbers because most elements exist naturally as mixtures of two or more isotopes.

The mass of an element shown in a periodic table or listed in a table of atomic masses is a weighted, average mass of all the isotopes present in a naturally occurring sample of that element. This is equal to the sum of each individual isotope's mass multiplied by its fractional abundance.

$$\text{average mass} = \sum_i (\text{fractional abundance} \times \text{isotopic mass})_i \tag{1.2}$$

For example, the element boron is composed of two isotopes: About 19.9% of all boron atoms are ^{10}B with a mass of 10.0129 amu, and the remaining 80.1% are ^{11}B with a mass of 11.0093 amu. The average atomic mass for boron is calculated to be:

$$\begin{aligned} \text{boron average mass} &= (0.199 \times 10.0129\ \text{amu}) + (0.801 \times 11.0093\ \text{amu}) \\ &= 1.99\ \text{amu} + 8.82\ \text{amu} \\ &= 10.81\ \text{amu} \end{aligned} \tag{1.3}$$

It is important to understand that no single boron atom weighs exactly 10.8 amu; 10.8 amu is the average mass of all boron atoms, and individual boron atoms weigh either approximately 10 amu or 11 amu.

Example 1.2

Calculation of Average Atomic Mass

A meteorite found in central Indiana contains traces of the noble gas neon picked up from the solar wind during the meteorite's trip through the solar system. Analysis of a sample of the gas showed that it consisted of 91.84% ^{20}Ne (mass 19.9924 amu), 0.47% ^{21}Ne (mass 20.9940 amu), and 7.69% ^{22}Ne (mass 21.9914 amu). What is the average mass of the neon in the solar wind?

Solution

$$\begin{aligned} \text{average mass} &= (0.9184 \times 19.9924\ \text{amu}) + (0.0047 \times 20.9940\ \text{amu}) + (0.0769 \times 21.9914\ \text{amu}) \\ &= (18.36 + 0.099 + 1.69)\ \text{amu} \\ &= 20.15\ \text{amu} \end{aligned} \tag{1.4}$$

The average mass of a neon atom in the solar wind is 20.15 amu. (The average mass of a terrestrial neon atom is 20.1796 amu. This result demonstrates that we may find slight differences in the natural abundance of isotopes, depending on their origin.)

Check Your Learning

A sample of magnesium is found to contain 78.70% of ^{24}Mg atoms (mass 23.98 amu), 10.13% of ^{25}Mg atoms (mass 24.99 amu), and 11.17% of ^{26}Mg atoms (mass 25.98 amu). Calculate the average mass of a Mg atom.

Answer:

24.31 amu

We can also do variations of this type of calculation, as shown in the next example.

Example 1.3

Calculation of Percent Abundance

Naturally occurring chlorine consists of ^{35}Cl (mass 34.96885 amu) and ^{37}Cl (mass 36.96590 amu), with an average mass of 35.453 amu. What is the percent composition of Cl in terms of these two isotopes?

Solution

The average mass of chlorine is the fraction that is ^{35}Cl times the mass of ^{35}Cl plus the fraction that is ^{37}Cl times the mass of ^{37}Cl.

$$\text{average mass} = (\text{fraction of } ^{35}\text{Cl} \times \text{mass of } ^{35}\text{Cl}) + (\text{fraction of } ^{37}\text{Cl} \times \text{mass of } ^{37}\text{Cl}) \tag{1.5}$$

If we let *x* represent the fraction that is ^{35}Cl, then the fraction that is ^{37}Cl is represented by $1.00 - x$.

(The fraction that is ^{35}Cl + the fraction that is ^{37}Cl must add up to 1, so the fraction of ^{37}Cl must equal 1.00 − the fraction of ^{35}Cl.)

Substituting this into the average mass equation, we have:

$$\begin{aligned} 35.453\text{ amu} &= (x \times 34.96885\text{ amu}) + [(1.00 - x) \times 36.96590\text{ amu}] \\ 35.453 &= 34.96885x + 36.96590 - 36.96590x \\ 1.99705x &= 1.513 \\ x &= \frac{1.513}{1.99705} = 0.7576 \end{aligned} \tag{1.6}$$

So solving yields: $x = 0.7576$, which means that $1.00 - 0.7576 = 0.2424$. Therefore, chlorine consists of 75.76% ^{35}Cl and 24.24% ^{37}Cl.

Check Your Learning

Naturally occurring copper consists of ^{63}Cu (mass 62.9296 amu) and ^{65}Cu (mass 64.9278 amu), with an average mass of 63.546 amu. What is the percent composition of Cu in terms of these two isotopes?

Answer:

69.15% Cu-63 and 30.85% Cu-65

Visit this **site (http://openstaxcollege.org/l/16PhetAtomMass)** to make mixtures of the main isotopes of the first 18 elements, gain experience with average atomic mass, and check naturally occurring isotope ratios using the Isotopes and Atomic Mass simulation.

The occurrence and natural abundances of isotopes can be experimentally determined using an instrument called a mass spectrometer. Mass spectrometry (MS) is widely used in chemistry, forensics, medicine, environmental science, and many other fields to analyze and help identify the substances in a sample of material. In a typical mass spectrometer (**Figure 1.5**), the sample is vaporized and exposed to a high-energy electron beam that causes the sample's atoms (or molecules) to become electrically charged, typically by losing one or more electrons. These cations then pass through a (variable) electric or magnetic field that deflects each cation's path to an extent that depends on both its mass and charge (similar to how the path of a large steel ball bearing rolling past a magnet is deflected to a lesser extent that that of a small steel BB). The ions are detected, and a plot of the relative number of ions generated versus their mass-to-charge ratios (a *mass spectrum*) is made. The height of each vertical feature or peak in a mass spectrum is proportional to the fraction of cations with the specified mass-to-charge ratio. Since its initial use during the development of modern atomic theory, MS has evolved to become a powerful tool for chemical analysis in a wide range of applications.

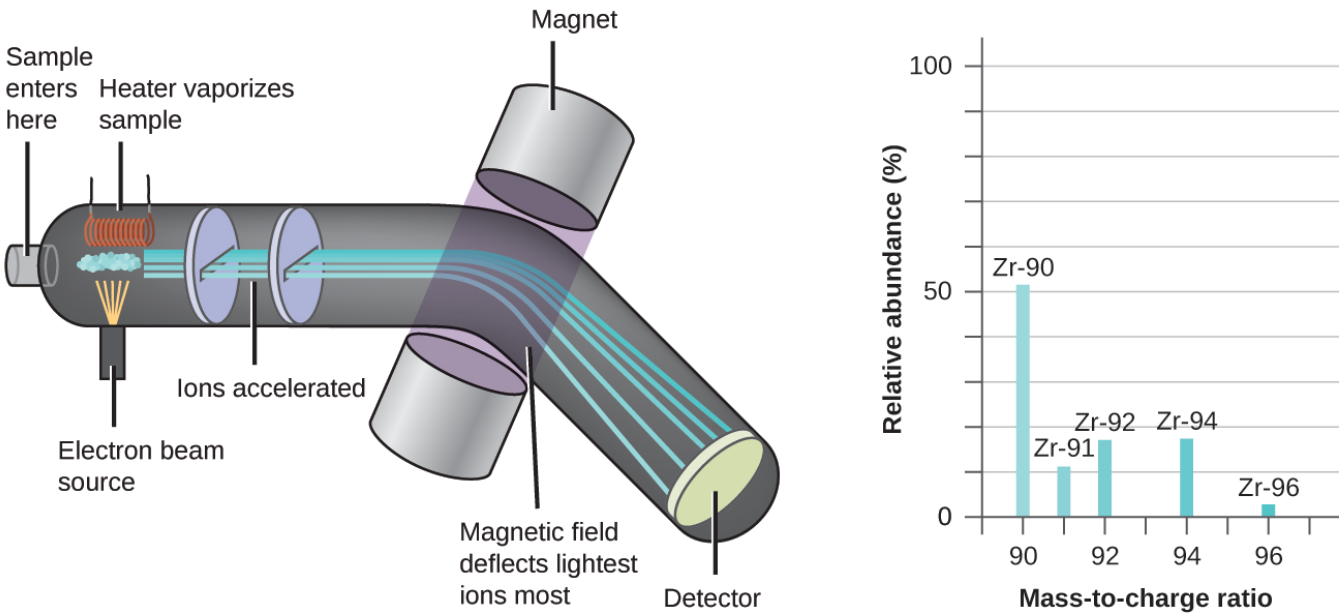


Figure 1.5 Analysis of zirconium in a mass spectrometer produces a mass spectrum with peaks showing the different isotopes of Zr.

See an **animation (http://openstaxcollege.org/l/16MassSpec)** that explains mass spectrometry. Watch this **video (http://openstaxcollege.org/l/16RSChemistry)** from the Royal Society for Chemistry for a brief description of the rudiments of mass spectrometry.

Key Concepts and Summary

An atom consists of a small, positively charged nucleus surrounded by electrons. The nucleus contains protons and neutrons; its diameter is about 100,000 times smaller than that of the atom. The mass of one atom is usually expressed in atomic mass units (amu), which is referred to as the atomic mass. An amu is defined as exactly $\frac{1}{12}$ of the mass of a carbon-12 atom and is equal to 1.6605×10^{-24} g.

Protons are relatively heavy particles with a charge of 1+ and a mass of 1.0073 amu. Neutrons are relatively heavy particles with no charge and a mass of 1.0087 amu. Electrons are light particles with a charge of 1− and a mass of 0.00055 amu. The number of protons in the nucleus is called the atomic number (Z) and is the property that defines an atom’s elemental identity. The sum of the numbers of protons and neutrons in the nucleus is called the mass number and, expressed in amu, is approximately equal to the mass of the atom. An atom is neutral when it contains equal numbers of electrons and protons.

Isotopes of an element are atoms with the same atomic number but different mass numbers; isotopes of an element, therefore, differ from each other only in the number of neutrons within the nucleus. When a naturally occurring element is composed of several isotopes, the atomic mass of the element represents the average of the masses of the isotopes involved. A chemical symbol identifies the atoms in a substance using symbols, which are one-, two-, or three-letter abbreviations for the atoms.

Key Equations

- $\text{average mass} = \sum_i (\text{fractional abundance} \times \text{isotopic mass})_i$

Chemistry End of Chapter Exercises

Exercise 1.1

In what way are isotopes of a given element always different? In what way(s) are they always the same?

Exercise 1.2

Write the symbol for each of the following ions:

(a) the ion with a 1+ charge, atomic number 55, and mass number 133

(b) the ion with 54 electrons, 53 protons, and 74 neutrons

(c) the ion with atomic number 15, mass number 31, and a 3− charge

(d) the ion with 24 electrons, 30 neutrons, and a 3+ charge

Solution

(a) $^{133}Cs^{+}$; (b) $^{127}I^{-}$; (c) $^{31}P^{3-}$; (d) $^{57}Co^{3+}$

Exercise 1.3

Write the symbol for each of the following ions:

(a) the ion with a 3+ charge, 28 electrons, and a mass number of 71

(b) the ion with 36 electrons, 35 protons, and 45 neutrons

(c) the ion with 86 electrons, 142 neutrons, and a 4+ charge

(d) the ion with a 2+ charge, atomic number 38, and mass number 87

Exercise 1.4

Open the **Build an Atom simulation (http://openstaxcollege.org/l/16PhetAtomBld)** and click on the Atom icon.

(a) Pick any one of the first 10 elements that you would like to build and state its symbol.

(b) Drag protons, neutrons, and electrons onto the atom template to make an atom of your element.
State the numbers of protons, neutrons, and electrons in your atom, as well as the net charge and mass number.

(c) Click on “Net Charge” and “Mass Number,” check your answers to (b), and correct, if needed.

(d) Predict whether your atom will be stable or unstable. State your reasoning.

(e) Check the “Stable/Unstable” box. Was your answer to (d) correct? If not, first predict what you can do to make a stable atom of your element, and then do it and see if it works. Explain your reasoning.

Solution

(a) Carbon-12, ^{12}C; (b) This atom contains six protons and six neutrons. There are six electrons in a neutral ^{12}C atom. The net charge of such a neutral atom is zero, and the mass number is 12. (c) The preceding answers are correct. (d) The atom will be stable since C-12 is a stable isotope of carbon. (e) The preceding answer is correct. Other answers for this exercise are possible if a different element of isotope is chosen.

Exercise 1.5

Open the **Build an Atom simulation (http://openstaxcollege.org/l/16PhetAtomBld)**

(a) Drag protons, neutrons, and electrons onto the atom template to make a neutral atom of Oxygen-16 and give the isotope symbol for this atom.

(b) Now add two more electrons to make an ion and give the symbol for the ion you have created.

Exercise 1.6

Open the **Build an Atom simulation (http://openstaxcollege.org/l/16PhetAtomBld)**

(a) Drag protons, neutrons, and electrons onto the atom template to make a neutral atom of Lithium-6 and give the isotope symbol for this atom.

(b) Now remove one electron to make an ion and give the symbol for the ion you have created.

Solution

(a) Lithium-6 contains three protons, three neutrons, and three electrons. The isotope symbol is ^{6}Li or $^{6}_{3}Li$. (b) $^{6}Li^{+}$ or $^{6}_{3}Li^{+}$

Exercise 1.7

Determine the number of protons, neutrons, and electrons in the following isotopes that are used in medical diagnoses:

(a) atomic number 9, mass number 18, charge of 1−

(b) atomic number 43, mass number 99, charge of 7+

(c) atomic number 53, atomic mass number 131, charge of 1−

(d) atomic number 81, atomic mass number 201, charge of 1+

(e) Name the elements in parts (a), (b), (c), and (d).

Exercise 1.8

The following are properties of isotopes of two elements that are essential in our diet. Determine the number of protons, neutrons and electrons in each and name them.

(a) atomic number 26, mass number 58, charge of 2+

(b) atomic number 53, mass number 127, charge of 1−

Solution

(a) Iron, 26 protons, 24 electrons, and 32 neutrons; (b) iodine, 53 protons, 54 electrons, and 74 neutrons

Exercise 1.9

Give the number of protons, electrons, and neutrons in neutral atoms of each of the following isotopes:

(a) $^{10}_{5}B$

(b) $^{199}_{80}Hg$

(c) $^{63}_{29}Cu$

(d) $^{13}_{6}C$

(e) $^{77}_{34}Se$

Exercise 1.10

Give the number of protons, electrons, and neutrons in neutral atoms of each of the following isotopes:

(a) $^{7}_{3}Li$

(b) $^{125}_{52}Te$

(c) $^{109}_{47}Ag$

(d) $^{15}_{7}N$

(e) $^{31}_{15}P$

Solution

(a) 3 protons, 3 electrons, 4 neutrons; (b) 52 protons, 52 electrons, 73 neutrons; (c) 47 protons, 47 electrons, 62 neutrons; (d) 7 protons, 7 electrons, 8 neutrons; (e) 15 protons, 15 electrons, 16 neutrons

Exercise 1.11

Click on the **site (http://openstaxcollege.org/l/16PhetAtomMass)** and select the "Mix Isotopes" tab, hide the "Percent Composition" and "Average Atomic Mass" boxes, and then select the element boron.

(a) Write the symbols of the isotopes of boron that are shown as naturally occurring in significant amounts.

(b) Predict the relative amounts (percentages) of these boron isotopes found in nature. Explain the reasoning behind your choice.

(c) Add isotopes to the black box to make a mixture that matches your prediction in (b). You may drag isotopes from their bins or click on "More" and then move the sliders to the appropriate amounts.

(d) Reveal the "Percent Composition" and "Average Atomic Mass" boxes. How well does your mixture match with your prediction? If necessary, adjust the isotope amounts to match your prediction.

(e) Select "Nature's" mix of isotopes and compare it to your prediction. How well does your prediction compare with the naturally occurring mixture? Explain. If necessary, adjust your amounts to make them match "Nature's" amounts as closely as possible.

Exercise 1.12

Repeat **Exercise 1.11** using an element that has three naturally occurring isotopes.

Solution
Let us use neon as an example. Since there are three isotopes, there is no way to be sure to accurately predict the abundances to make the total of 20.18 amu average atomic mass. Let us guess that the abundances are 9% Ne-22, 91% Ne-20, and only a trace of Ne-21. The average mass would be 20.18 amu. Checking the nature's mix of isotopes shows that the abundances are 90.48% Ne-20, 9.25% Ne-22, and 0.27% Ne-21, so our guessed amounts have to be slightly adjusted.

Exercise 1.13

An element has the following natural abundances and isotopic masses: 90.92% abundance with 19.99 amu, 0.26% abundance with 20.99 amu, and 8.82% abundance with 21.99 amu. Calculate the average atomic mass of this element.

Exercise 1.14

Average atomic masses listed by IUPAC are based on a study of experimental results. Bromine has two isotopes ^{79}Br and ^{81}Br, whose masses (78.9183 and 80.9163 amu) and abundances (50.69% and 49.31%) were determined in earlier experiments. Calculate the average atomic mass of bromine based on these experiments.

Solution
79.904 amu

Exercise 1.15

Variations in average atomic mass may be observed for elements obtained from different sources. Lithium provides an example of this. The isotopic composition of lithium from naturally occurring minerals is 7.5% ^{6}Li and 92.5% ^{7}Li, which have masses of 6.01512 amu and 7.01600 amu, respectively. A commercial source of lithium, recycled from a military source, was 3.75% ^{6}Li (and the rest ^{7}Li). Calculate the average atomic mass values for each of these two sources.

Exercise 1.16

The average atomic masses of some elements may vary, depending upon the sources of their ores. Naturally occurring boron consists of two isotopes with accurately known masses (^{10}B, 10.0129 amu and ^{11}B, 11.0931 amu). The actual atomic mass of boron can vary from 10.807 to 10.819, depending on whether the mineral source is from Turkey or the United States. Calculate the percent abundances leading to the two values of the average atomic masses of boron from these two countries.

Solution
Turkey source: 26.49% (of 10.0129 amu isotope); US source: 25.37% (of 10.0129 amu isotope)

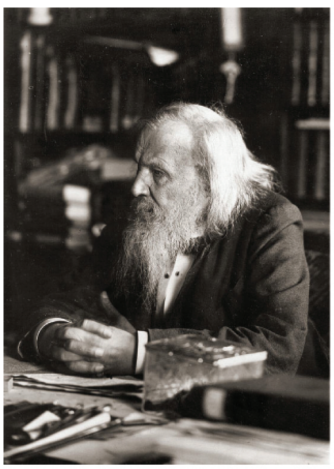
Exercise 1.17

The ^{18}O:^{16}O abundance ratio in some meteorites is greater than that used to calculate the average atomic mass of oxygen on earth. Is the average mass of an oxygen atom in these meteorites greater than, less than, or equal to that of a terrestrial oxygen atom?

1.3 The Periodic Table

As early chemists worked to purify ores and discovered more elements, they realized that various elements could be grouped together by their similar chemical behaviors. One such grouping includes lithium (Li), sodium (Na), and potassium (K): These elements all are shiny, conduct heat and electricity well, and have similar chemical properties. A second grouping includes calcium (Ca), strontium (Sr), and barium (Ba), which also are shiny, good conductors of heat and electricity, and have chemical properties in common. However, the specific properties of these two groupings are notably different from each other. For example: Li, Na, and K are much more reactive than are Ca, Sr, and Ba; Li, Na, and K form compounds with oxygen in a ratio of two of their atoms to one oxygen atom, whereas Ca, Sr, and Ba form compounds with one of their atoms to one oxygen atom. Fluorine (F), chlorine (Cl), bromine (Br), and iodine (I) also exhibit similar properties to each other, but these properties are drastically different from those of any of the elements above.

Dimitri Mendeleev in Russia (1869) and Lothar Meyer in Germany (1870) independently recognized that there was a periodic relationship among the properties of the elements known at that time. Both published tables with the elements arranged according to increasing atomic mass. But Mendeleev went one step further than Meyer: He used his table to predict the existence of elements that would have the properties similar to aluminum and silicon, but were yet unknown. The discoveries of gallium (1875) and germanium (1886) provided great support for Mendeleev's work. Although Mendeleev and Meyer had a long dispute over priority, Mendeleev's contributions to the development of the periodic table are now more widely recognized (**Figure 1.6**).



(a)

Reihen	Gruppo I. — R^2O	Gruppo II. — RO	Gruppo III. — R^2O^3	Gruppe IV. RH^4 RO^2	Gruppe V. RH^3 R^2O^5	Gruppe VI. RH^2 RO^3	Gruppe VII. RH R^2O^7	Gruppo VIII. — RO^4
1	H=1							
2	Li=7	Be=9,4	B=11	C=12	N=14	O=16	F=19	
3	Na=23	Mg=24	Al=27,3	Si=28	P=31	S=32	Cl=35,5	
4	K=39	Ca=40	—=44	Ti=48	V=51	Cr=52	Mn=55	Fe=56, Co=59, Ni=59, Cu=63.
5	(Cu=63)	Zn=65	—=68	—=72	As=75	Se=78	Br=80	
6	Rb=85	Sr=87	?Yt=88	Zr=90	Nb=94	Mo=96	—=100	Ru=104, Rh=104, Pd=106, Ag=108.
7	(Ag=108)	Cd=112	In=113	Sn=118	Sb=122	Te=125	J=127	
8	Cs=133	Ba=137	?Di=138	?Ce=140	—	—	—	— — — —
9	(—)	—	—	—	—	—	—	
10	—	—	?Er=178	?La=180	Ta=182	W=184	—	Os=195, Ir=197, Pt=198, Au=199.
11	(Au=199)	Hg=200	Tl=204	Pb=207	Bi=208	—	—	
12	—	—	—	Th=231	—	U=240	—	— — — —

(b)

Figure 1.6 (a) Dimitri Mendeleev is widely credited with creating (b) the first periodic table of the elements. (credit a: modification of work by Serge Lachinov; credit b: modification of work by "Den fjättrade ankan"/Wikimedia Commons)

By the twentieth century, it became apparent that the periodic relationship involved atomic numbers rather than atomic masses. The modern statement of this relationship, the **periodic law**, is as follows: *the properties of the elements are periodic functions of their atomic numbers*. A modern **periodic table** arranges the elements in increasing order of their atomic numbers and groups atoms with similar properties in the same vertical column (**Figure 1.7**). Each box represents an element and contains its atomic number, symbol, average atomic mass, and (sometimes) name. The elements are arranged in seven horizontal rows, called **periods** or **series**, and 18 vertical columns, called **groups**. Groups are labeled at the top of each column. In the United States, the labels traditionally were numerals with capital letters. However, IUPAC recommends that the numbers 1 through 18 be used, and these labels are more common. For the table to fit on a single page, parts of two of the rows, a total of 14 columns, are usually written below the main body of the table.

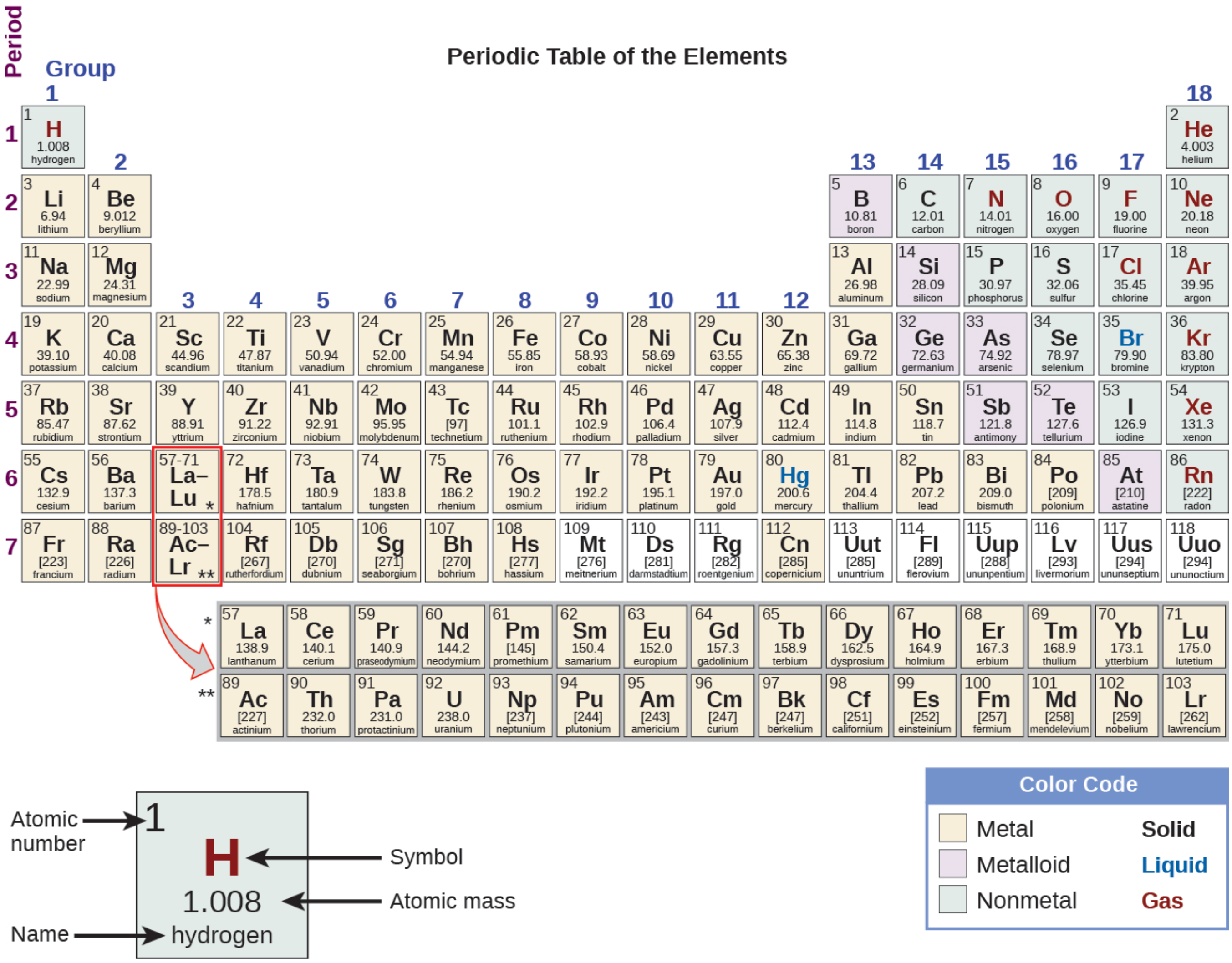


Figure 1.7 Elements in the periodic table are organized according to their properties.

1.4 The Mole

Formula Mass

Formula Mass for Covalent Substances

For covalent substances, the formula represents the numbers and types of atoms composing a single molecule of the substance; therefore, the formula mass may be correctly referred to as a molecular mass. Consider chloroform ($CHCl_3$), a covalent compound once used as a surgical anesthetic and now primarily used in the production of tetrafluoroethylene, the building block for the "anti-stick" polymer, Teflon. The molecular formula of chloroform indicates that a single molecule contains one carbon atom, one hydrogen atom, and three chlorine atoms. The average molecular mass of a chloroform molecule is therefore equal to the sum of the average atomic masses of these atoms. **Figure 1.8** outlines the calculations used to derive the molecular mass of chloroform, which is 119.37 amu.

Element	Quantity		Average atomic mass (amu)		Subtotal (amu)
C	1	×	12.01	=	12.01
H	1	×	1.008	=	1.008
Cl	3	×	35.45	=	106.35
			Molecular mass		119.37

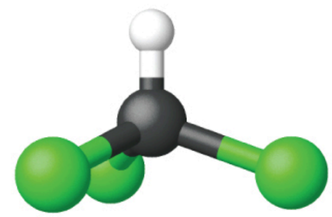


Figure 1.8 The average mass of a chloroform molecule, $CHCl_3$, is 119.37 amu, which is the sum of the average atomic masses of each of its constituent atoms. The model shows the molecular structure of chloroform.

Likewise, the molecular mass of an aspirin molecule, $C_9H_8O_4$, is the sum of the atomic masses of nine carbon atoms, eight hydrogen atoms, and four oxygen atoms, which amounts to 180.15 amu (**Figure 1.9**).

Element	Quantity		Average atomic mass (amu)		Subtotal (amu)
C	9	×	12.01	=	108.09
H	8	×	1.008	=	8.064
O	4	×	16.00	=	64.00
			Molecular mass		180.15

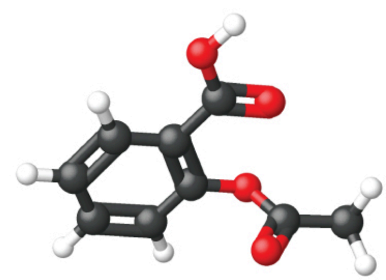


Figure 1.9 The average mass of an aspirin molecule is 180.15 amu. The model shows the molecular structure of aspirin, $C_9H_8O_4$.

Example 1.4

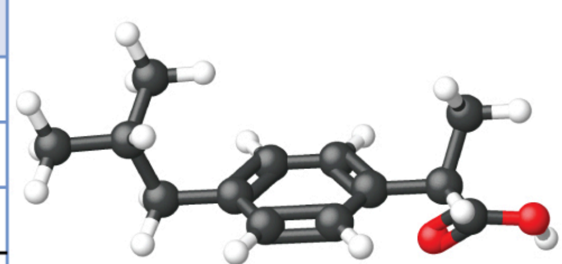
Computing Molecular Mass for a Covalent Compound

Ibuprofen, $C_{13}H_{18}O_2$, is a covalent compound and the active ingredient in several popular nonprescription pain medications, such as Advil and Motrin. What is the molecular mass (amu) for this compound?

Solution

Molecules of this compound are comprised of 13 carbon atoms, 18 hydrogen atoms, and 2 oxygen atoms. Following the approach described above, the average molecular mass for this compound is therefore:

Element	Quantity		Average atomic mass (amu)		Subtotal (amu)
C	13	×	12.01	=	156.13
H	18	×	1.008	=	18.114
O	2	×	16.00	=	32.00
			Molecular mass		206.27



Check Your Learning

Acetaminophen, $C_8H_9NO_2$, is a covalent compound and the active ingredient in several popular nonprescription pain medications, such as Tylenol. What is the molecular mass (amu) for this compound?

Answer:

151.16 amu

Formula Mass for Ionic Compounds

Ionic compounds are composed of discrete cations and anions combined in ratios to yield electrically neutral bulk matter. The formula mass for an ionic compound is calculated in the same way as the formula mass for covalent compounds: by summing the average atomic masses of all the atoms in the compound's formula. Keep in mind, however, that the formula for an ionic compound does not represent the composition of a discrete molecule, so it may not correctly be referred to as the "molecular mass."

As an example, consider sodium chloride, NaCl, the chemical name for common table salt. Sodium chloride is an ionic compound composed of sodium cations, Na^+, and chloride anions, Cl^-, combined in a 1:1 ratio. The formula mass for this compound is computed as 58.44 amu (see **Figure 1.10**).

Element	Quantity		Average atomic mass (amu)		Subtotal
Na	1	×	22.99	=	22.99
Cl	1	×	35.45	=	35.45
				Formula mass	58.44

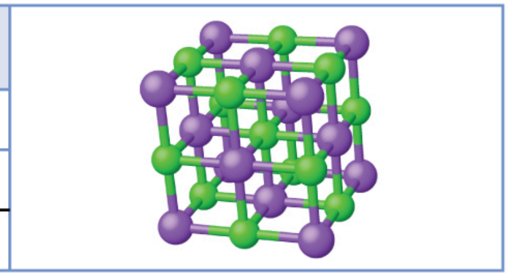


Figure 1.10 Table salt, NaCl, contains an array of sodium and chloride ions combined in a 1:1 ratio. Its formula mass is 58.44 amu.

Note that the average masses of neutral sodium and chlorine atoms were used in this computation, rather than the masses for sodium cations and chlorine anions. This approach is perfectly acceptable when computing the formula mass of an ionic compound. Even though a sodium cation has a slightly smaller mass than a sodium atom (since it is missing an electron), this difference will be offset by the fact that a chloride anion is slightly more massive than a chloride atom (due to the extra electron). Moreover, the mass of an electron is negligibly small with respect to the mass of a typical atom. Even when calculating the mass of an isolated ion, the missing or additional electrons can generally be ignored, since their contribution to the overall mass is negligible, reflected only in the nonsignificant digits that will be lost when the computed mass is properly rounded. The few exceptions to this guideline are very light ions derived from elements with precisely known atomic masses.

Example 1.5

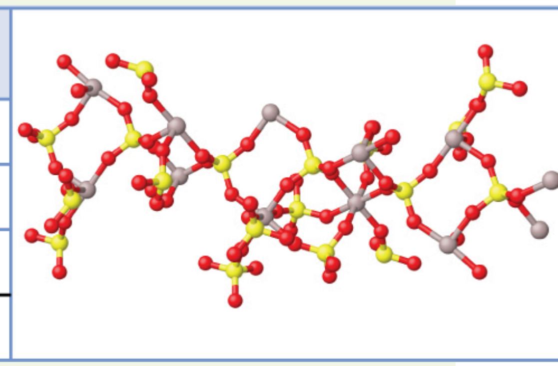
Computing Formula Mass for an Ionic Compound

Aluminum sulfate, $Al_2(SO_4)_3$, is an ionic compound that is used in the manufacture of paper and in various water purification processes. What is the formula mass (amu) of this compound?

Solution

The formula for this compound indicates it contains Al^{3+} and $SO_4{}^{2-}$ ions combined in a 2:3 ratio. For purposes of computing a formula mass, it is helpful to rewrite the formula in the simpler format, $Al_2S_3O_{12}$. Following the approach outlined above, the formula mass for this compound is calculated as follows:

Element	Quantity		Average atomic mass (amu)		Subtotal (amu)
Al	2	×	26.98	=	53.96
S	3	×	32.06	=	96.18
O	12	×	16.00	=	192.00
				Molecular mass	342.14



Check Your Learning

Calcium phosphate, $Ca_3(PO_4)_2$, is an ionic compound and a common anti-caking agent added to food products. What is the formula mass (amu) of calcium phosphate?

Answer:

310.18 amu

The Mole

The identity of a substance is defined not only by the types of atoms or ions it contains, but by the quantity of each type of atom or ion. For example, water, H_2O, and hydrogen peroxide, H_2O_2, are alike in that their respective molecules are composed of hydrogen and oxygen atoms. However, because a hydrogen peroxide molecule contains two oxygen atoms, as opposed to the water molecule, which has only one, the two substances exhibit very different properties. Today, we possess sophisticated instruments that allow the direct measurement of these defining microscopic traits; however, the same traits were originally derived from the measurement of macroscopic properties (the masses and volumes of bulk quantities of matter) using relatively simple tools (balances and volumetric glassware). This experimental approach required the introduction of a new unit for amount of substances, the *mole*, which remains indispensable in modern chemical science.

The mole is an amount unit similar to familiar units like pair, dozen, gross, etc. It provides a specific measure of *the number* of atoms or molecules in a bulk sample of matter. A **mole** is defined as the amount of substance containing the same number of discrete entities (such as atoms, molecules, and ions) as the number of atoms in a sample of pure ^{12}C weighing exactly 12 g. One Latin connotation for the word “mole” is “large mass” or “bulk,” which is consistent with its use as the name for this unit. The mole provides a link between an easily measured macroscopic property, bulk mass, and an extremely important fundamental property, number of atoms, molecules, and so forth.

The number of entities composing a mole has been experimentally determined to be $6.02214179 \times 10^{23}$, a fundamental constant named **Avogadro’s number (N_A)** or the Avogadro constant in honor of Italian scientist Amedeo Avogadro. This constant is properly reported with an explicit unit of “per mole,” a conveniently rounded version being 6.022×10^{23}/mol.

Consistent with its definition as an amount unit, 1 mole of any element contains the same number of atoms as 1 mole of any other element. The masses of 1 mole of different elements, however, are different, since the masses of the individual atoms are drastically different. The **molar mass** of an element (or compound) is the mass in grams of 1 mole of that substance, a property expressed in units of grams per mole (g/mol) (see **Figure 1.11**).

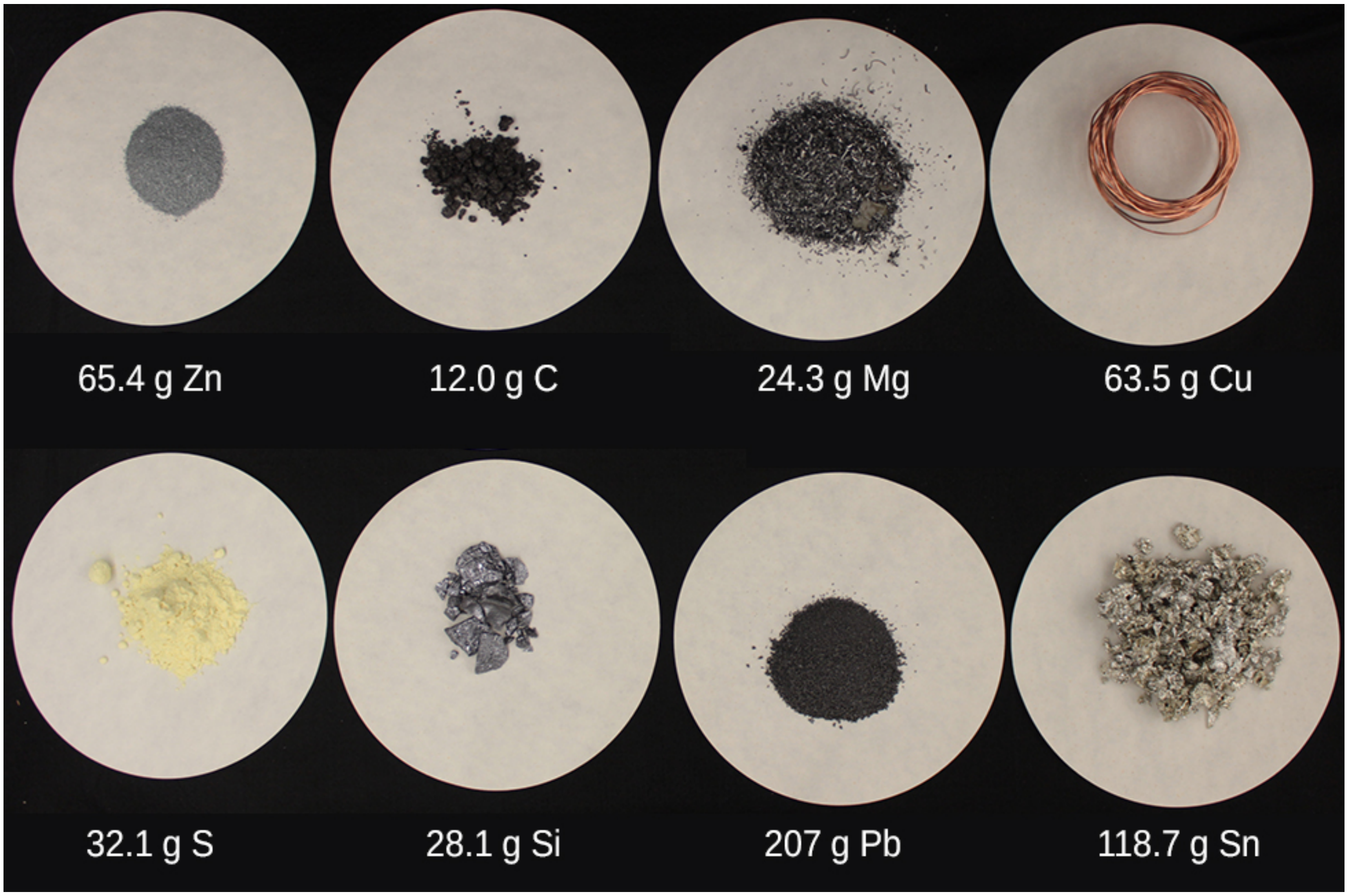


Figure 1.11 Each sample contains 6.022×10^{23} atoms —1.00 mol of atoms. From left to right (top row): 65.4 g zinc, 12.0 g carbon, 24.3 g magnesium, and 63.5 g copper. From left to right (bottom row): 32.1 g sulfur, 28.1 g silicon, 207 g lead, and 118.7 g tin. (credit: modification of work by Mark Ott)

Because the definitions of both the mole and the atomic mass unit are based on the same reference substance, ^{12}C, the molar mass of any substance is numerically equivalent to its atomic or formula weight in amu. Per the amu definition, a single ^{12}C atom weighs 12 amu (its atomic mass is 12 amu). According to the definition of the mole, 12 g of ^{12}C contains 1 mole of ^{12}C atoms (its molar mass is 12 g/mol). This relationship holds for all elements, since their atomic masses are measured relative to that of the amu-reference substance, ^{12}C. Extending this principle, the molar mass of a compound in grams is likewise numerically equivalent to its formula mass in amu (**Figure 1.12**).

Figure 1.12 Each sample contains 6.02×10^{23} molecules or formula units—1.00 mol of the compound or element. Clock-wise from the upper left: 130.2 g of $C_8H_{17}OH$ (1-octanol, formula mass 130.2 amu), 454.4 g of HgI_2 (mercury(II) iodide, formula mass 454.4 amu), 32.0 g of CH_3OH (methanol, formula mass 32.0 amu) and 256.5 g of S_8 (sulfur, formula mass 256.5 amu). (credit: Sahar Atwa)

Table 1.4

Element	Average Atomic Mass (amu)	Molar Mass (g/mol)	Atoms/Mole
C	12.01	12.01	6.022×10^{23}
H	1.008	1.008	6.022×10^{23}
O	16.00	16.00	6.022×10^{23}
Na	22.99	22.99	6.022×10^{23}
Cl	35.45	33.45	6.022×10^{23}

While atomic mass and molar mass are numerically equivalent, keep in mind that they are vastly different in terms of scale, as represented by the vast difference in the magnitudes of their respective units (amu versus g). To appreciate the enormity of the mole, consider a small drop of water weighing about 0.03 g (see **Figure 1.13**). Although this represents just a tiny fraction of 1 mole of water (~18 g), it contains more water molecules than can be clearly imagined. If the molecules were distributed equally among the roughly seven billion people on earth, each person would receive more than 100 billion molecules.

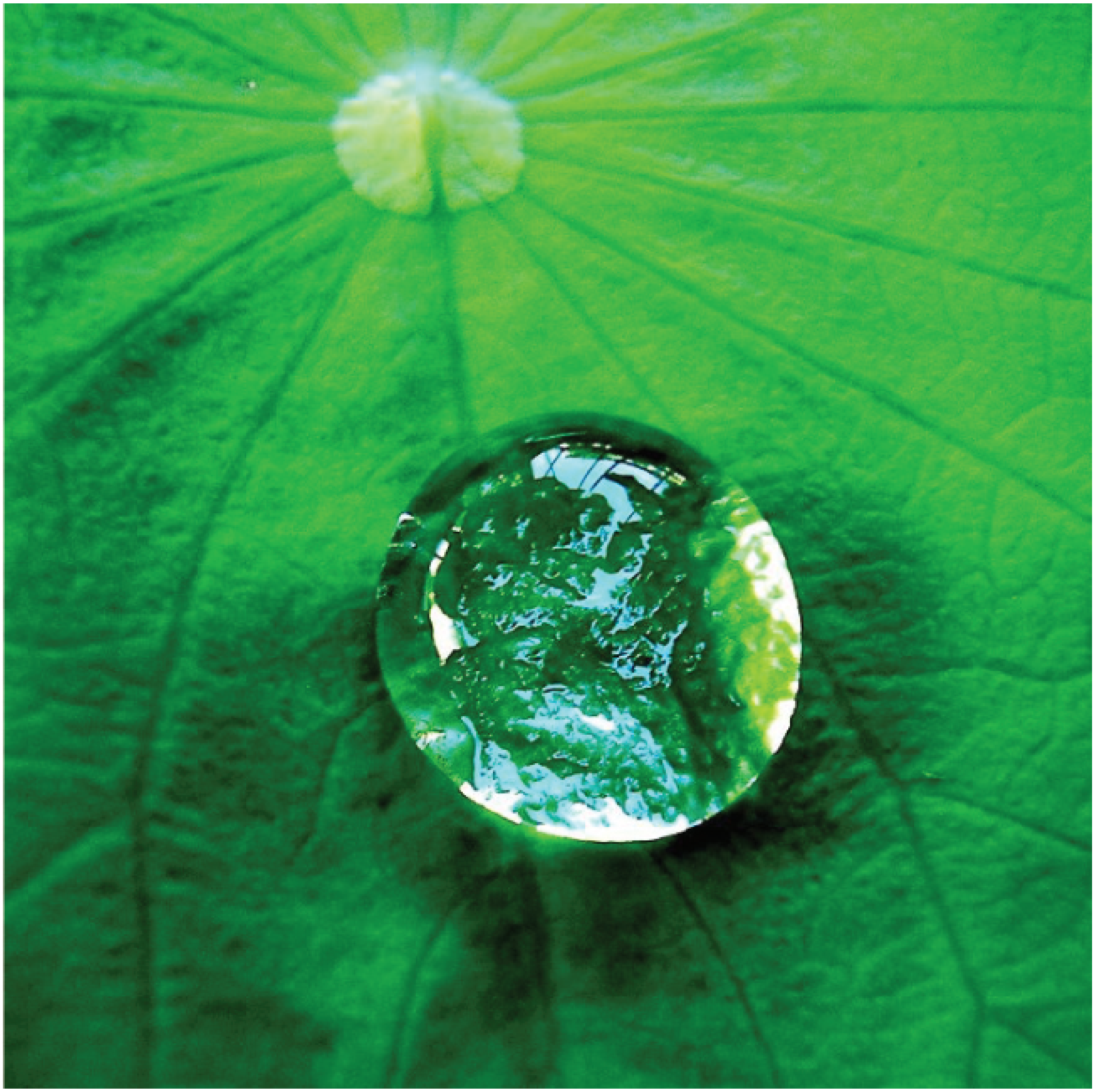


Figure 1.13 The number of molecules in a single droplet of water is roughly 100 billion times greater than the number of people on earth. (credit: "tanakawho"/Wikimedia commons)

The mole is used in chemistry to represent 6.022×10^{23} of something, but it can be difficult to conceptualize such a large number. Watch this **video (http://openstaxcollege.org/l/16molevideo)** and then complete the "Think" questions that follow. Explore more about the mole by reviewing the information under "Dig Deeper."

The relationships between formula mass, the mole, and Avogadro's number can be applied to compute various quantities that describe the composition of substances and compounds. For example, if we know the mass and chemical composition of a substance, we can determine the number of moles and calculate number of atoms or molecules in the sample. Likewise, if we know the number of moles of a substance, we can derive the number of atoms or molecules and calculate the substance's mass.

Example 1.6

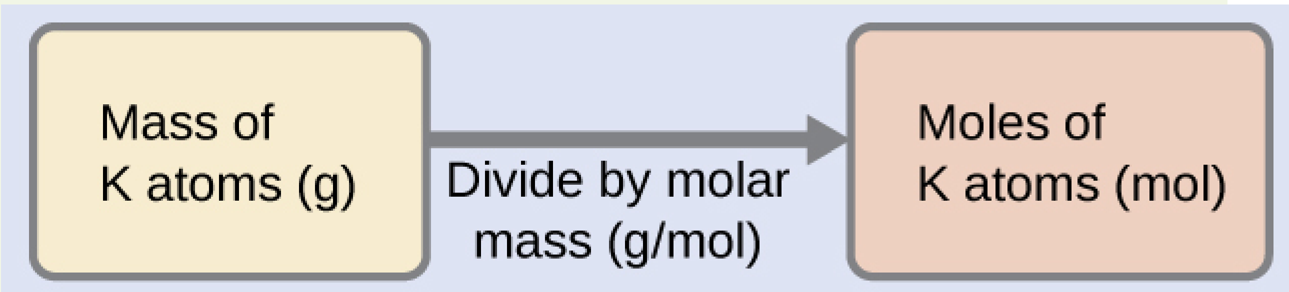
Deriving Moles from Grams for an Element

According to nutritional guidelines from the US Department of Agriculture, the estimated average requirement for dietary potassium is 4.7 g. What is the estimated average requirement of potassium in moles?

Solution

The mass of K is provided, and the corresponding amount of K in moles is requested. Referring to the periodic table, the atomic mass of K is 39.10 amu, and so its molar mass is 39.10 g/mol. The given mass of K (4.7 g) is a bit more than one-tenth the molar mass (39.10 g), so a reasonable "ballpark" estimate of the number of moles would be slightly greater than 0.1 mol.

The molar amount of a substance may be calculated by dividing its mass (g) by its molar mass (g/mol):



The factor-label method supports this mathematical approach since the unit "g" cancels and the answer has units of "mol:"

$$4.7\ \cancel{g}\ K \left(\frac{\text{mol K}}{39.10\ \cancel{g}}\right) = 0.12\ \text{mol K} \tag{1.7}$$

The calculated magnitude (0.12 mol K) is consistent with our ballpark expectation, since it is a bit greater than 0.1 mol.

Check Your Learning

Beryllium is a light metal used to fabricate transparent X-ray windows for medical imaging instruments. How many moles of Be are in a thin-foil window weighing 3.24 g?

Answer:

0.360 mol

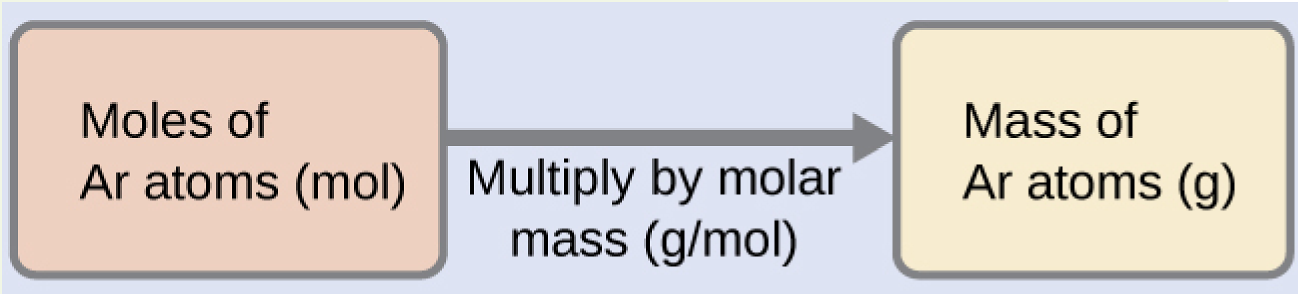
Example 1.7

Deriving Grams from Moles for an Element

A liter of air contains 9.2×10^{-4} mol argon. What is the mass of Ar in a liter of air?

Solution

The molar amount of Ar is provided and must be used to derive the corresponding mass in grams. Since the amount of Ar is less than 1 mole, the mass will be less than the mass of 1 mole of Ar, approximately 40 g. The molar amount in question is approximately one-one thousandth (~10^{-3}) of a mole, and so the corresponding mass should be roughly one-one thousandth of the molar mass (~0.04 g):



In this case, logic dictates (and the factor-label method supports) multiplying the provided amount (mol) by the molar mass (g/mol):

$$9.2 \times 10^{-4}\ \cancel{\text{mol}}\ \text{Ar}\left(\frac{39.95\ \text{g}}{\cancel{\text{mol}}\ \text{Ar}}\right) = 0.037\ \text{g Ar} \tag{1.8}$$

The result is in agreement with our expectations, around 0.04 g Ar.

Check Your Learning

What is the mass of 2.561 mol of gold?

Answer:

504.4 g

Example 1.8

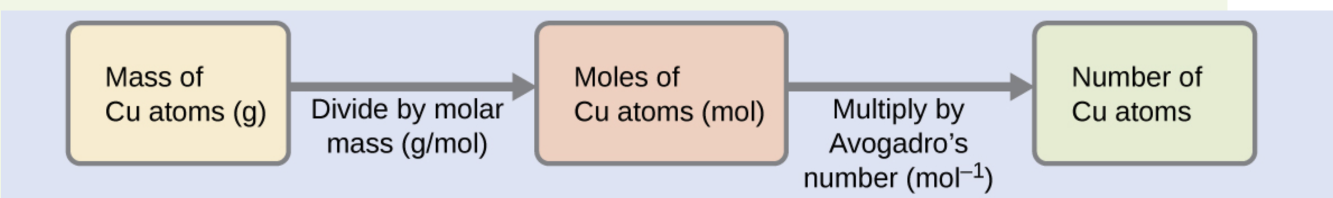
Deriving Number of Atoms from Mass for an Element

Copper is commonly used to fabricate electrical wire (**Figure 1.14**). How many copper atoms are in 5.00 g of copper wire?

Figure 1.14 Copper wire is composed of many, many atoms of Cu. (credit: Emilian Robert Vicol)

Solution

The number of Cu atoms in the wire may be conveniently derived from its mass by a two-step computation: first calculating the molar amount of Cu, and then using Avogadro's number (N_A) to convert this molar amount to number of Cu atoms:



Considering that the provided sample mass (5.00 g) is a little less than one-tenth the mass of 1 mole of Cu (~64 g), a reasonable estimate for the number of atoms in the sample would be on the order of one-tenth N_A, or approximately 10^{22} Cu atoms. Carrying out the two-step computation yields:

$$5.00\ \cancel{\text{g}}\ \text{Cu}\left(\frac{\cancel{\text{mol}}\ \text{Cu}}{63.55\ \cancel{\text{g}}}\right)\left(\frac{6.022\times10^{23}\ \text{atoms}}{\cancel{\text{mol}}}\right) = 4.74\times10^{22}\ \text{atoms of copper} \tag{1.9}$$

The factor-label method yields the desired cancellation of units, and the computed result is on the order of 10^{22} as expected.

Check Your Learning

A prospector panning for gold in a river collects 15.00 g of pure gold. How many Au atoms are in this quantity of gold?

Answer:

4.586×10^{22} Au atoms

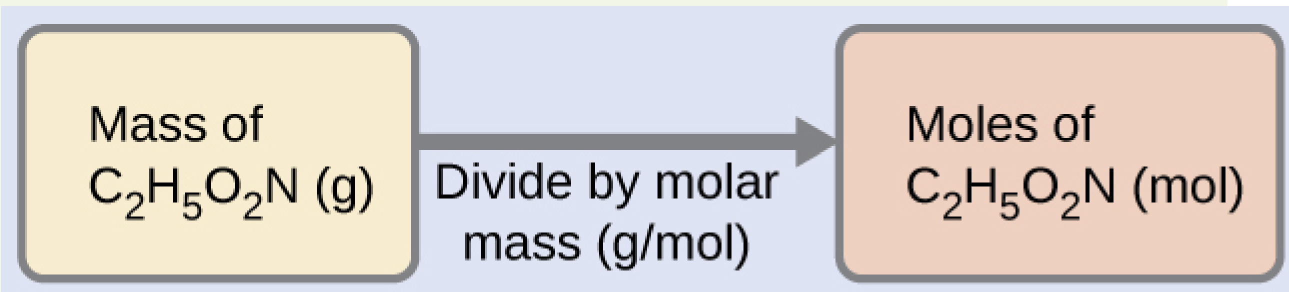
Example 1.9

Deriving Moles from Grams for a Compound

Our bodies synthesize protein from amino acids. One of these amino acids is glycine, which has the molecular formula $C_2H_5O_2N$. How many moles of glycine molecules are contained in 28.35 g of glycine?

Solution

We can derive the number of moles of a compound from its mass following the same procedure we used for an element in **Example 1.6**:



The molar mass of glycine is required for this calculation, and it is computed in the same fashion as its molecular mass. One mole of glycine, $C_2H_5O_2N$, contains 2 moles of carbon, 5 moles of hydrogen, 2 moles of oxygen, and 1 mole of nitrogen:

Element	Quantity (mol element/ mol compound)		Molar mass (g/mol element)		Subtotal (g/mol compound)
C	2	×	12.01	=	24.02
H	5	×	1.008	=	5.040
O	2	×	16.00	=	32.00
N	1	×	14.007	=	14.007
Molecular mass (g/mol compound)					75.07

The provided mass of glycine (~28 g) is a bit more than one-third the molar mass (~75 g/mol), so we would expect the computed result to be a bit greater than one-third of a mole (~0.33 mol). Dividing the compound's mass by its molar mass yields:

$$28.35\ \cancel{\text{g}}\ \text{glycine}\left(\frac{\text{mol glycine}}{75.07\ \cancel{\text{g}}}\right) = 0.378\ \text{mol glycine} \tag{1.10}$$

This result is consistent with our rough estimate.

Check Your Learning

How many moles of sucrose, $C_{12}H_{22}O_{11}$, are in a 25-g sample of sucrose?

Answer:

0.073 mol

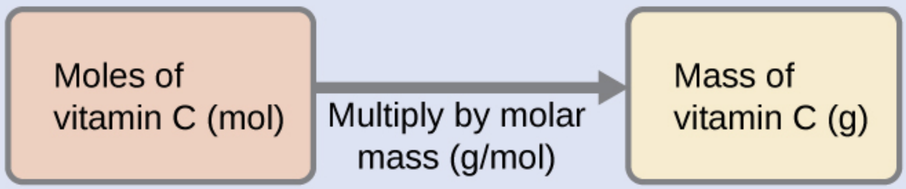
Example 1.10

Deriving Grams from Moles for a Compound

Vitamin C is a covalent compound with the molecular formula $C_6H_8O_6$. The recommended daily dietary allowance of vitamin C for children aged 4–8 years is 1.42×10^{-4} mol. What is the mass of this allowance in grams?

Solution

As for elements, the mass of a compound can be derived from its molar amount as shown:



The molar mass for this compound is computed to be 176.124 g/mol. The given number of moles is a very small fraction of a mole (~10^{-4} or one-ten thousandth); therefore, we would expect the corresponding mass to be about one-ten thousandth of the molar mass (~0.02 g). Performing the calculation, we get:

$$1.42 \times 10^{-4} \cancel{\text{mol}} \text{ vitamin C} \left(\frac{176.124 \text{ g}}{\cancel{\text{mol}} \text{ vitamin C}}\right) = 0.0250 \text{ g vitamin C} \tag{1.11}$$

This is consistent with the anticipated result.

Check Your Learning

What is the mass of 0.443 mol of hydrazine, N_2H_4?

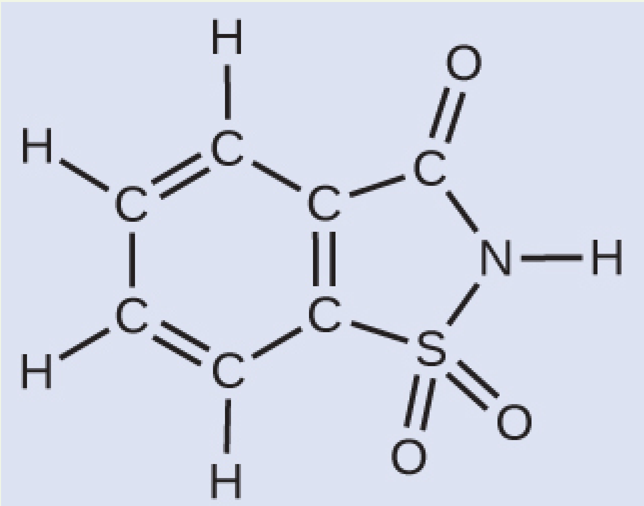
Answer:

14.2 g

Example 1.11

Deriving the Number of Atoms and Molecules from the Mass of a Compound

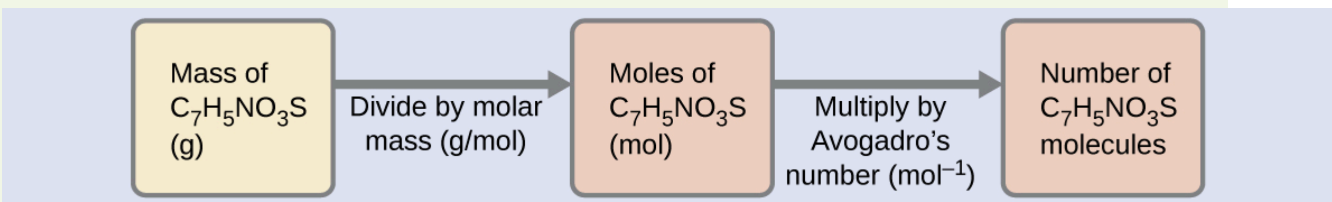
A packet of an artificial sweetener contains 40.0 mg of saccharin ($C_7H_5NO_3S$), which has the structural formula:



Given that saccharin has a molar mass of 183.18 g/mol, how many saccharin molecules are in a 40.0-mg (0.0400-g) sample of saccharin? How many carbon atoms are in the same sample?

Solution

The number of molecules in a given mass of compound is computed by first deriving the number of moles, as demonstrated in **Example 1.9**, and then multiplying by Avogadro's number:



Using the provided mass and molar mass for saccharin yields:

$$0.0400\,\cancel{\text{g}}\,\mathrm{C_7H_5NO_3S}\left(\frac{\cancel{\text{mol}}\,\mathrm{C_7H_5NO_3S}}{183.18\,\cancel{\text{g}}\,\mathrm{C_7H_5NO_3S}}\right)\left(\frac{6.022\times10^{23}\,\mathrm{C_7H_5NO_3S}\text{ molecules}}{1\,\cancel{\text{mol}}\,\mathrm{C_7H_5NO_3S}}\right) \tag{1.12}$$

$$= 1.31\times10^{20}\,\mathrm{C_7H_5NO_3S}\text{ molecules}$$

The compound's formula shows that each molecule contains seven carbon atoms, and so the number of C atoms in the provided sample is:

$$1.31\times10^{20}\,\mathrm{C_7H_5NO_3S}\text{ molecules}\left(\frac{7\text{ C atoms}}{1\,\mathrm{C_7H_5NO_3S}\text{ molecule}}\right) = 9.17\times10^{20}\text{ C atoms} \tag{1.13}$$

Check Your Learning

How many C_4H_{10} molecules are contained in 9.213 g of this compound? How many hydrogen atoms?

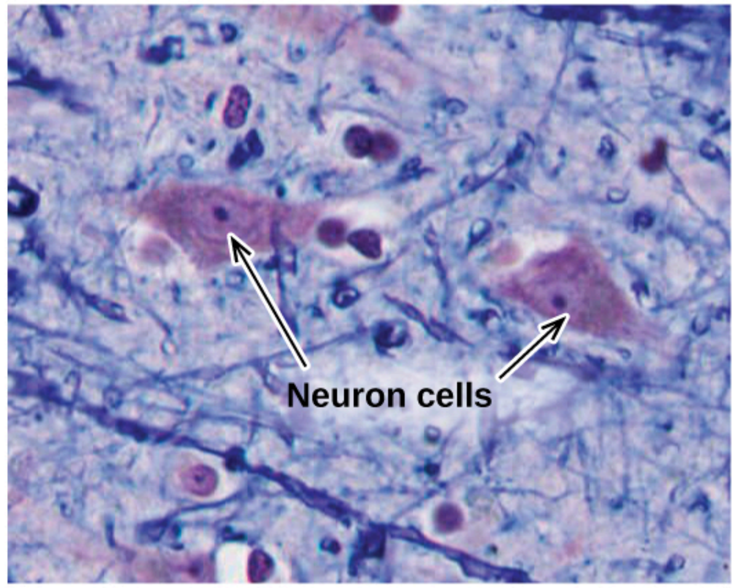
Answer:

9.545×10^{22} molecules C_4H_{10}; 9.545×10^{23} atoms H

Counting Neurotransmitter Molecules in the Brain

The brain is the control center of the central nervous system (**Figure 1.15**). It sends and receives signals to and from muscles and other internal organs to monitor and control their functions; it processes stimuli detected by sensory organs to guide interactions with the external world; and it houses the complex physiological processes that give rise to our intellect and emotions. The broad field of neuroscience spans all aspects of the structure and function of the central nervous system, including research on the anatomy and physiology of the brain. Great progress has been made in brain research over the past few decades, and the BRAIN Initiative, a federal initiative announced in 2013, aims to accelerate and capitalize on these advances through the concerted efforts of various industrial, academic, and government agencies (more details available at www.whitehouse.gov/share/brain-initiative).

(a)



(b)

Figure 1.15 (a) A typical human brain weighs about 1.5 kg and occupies a volume of roughly 1.1 L. (b) Information is transmitted in brain tissue and throughout the central nervous system by specialized cells called neurons (micrograph shows cells at 1600× magnification).

Specialized cells called neurons transmit information between different parts of the central nervous system by way of electrical and chemical signals. Chemical signaling occurs at the interface between different neurons when one of the cells releases molecules (called neurotransmitters) that diffuse across the small gap between the cells (called the synapse) and bind to the surface of the other cell. These neurotransmitter molecules are stored in small intracellular structures called vesicles that fuse to the cell wall and then break open to release their contents when the neuron is appropriately stimulated. This process is called exocytosis (see **Figure 1.16**). One neurotransmitter that has been very extensively studied is dopamine, $C_8H_{11}NO_2$. Dopamine is involved in various neurological processes that impact a wide variety of human behaviors. Dysfunctions in the dopamine systems of the brain underlie serious neurological diseases such as Parkinson's and schizophrenia.

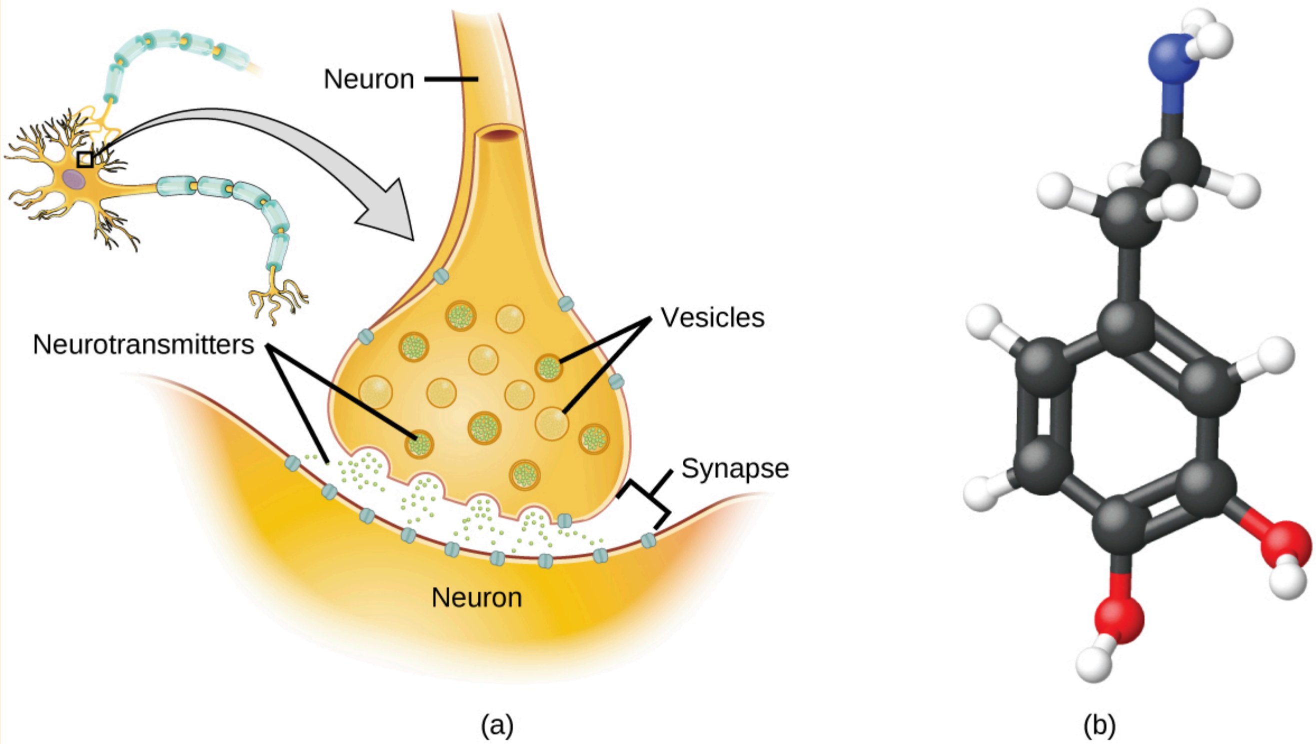


Figure 1.16 (a) Chemical signals are transmitted from neurons to other cells by the release of neurotransmitter molecules into the small gaps (synapses) between the cells. (b) Dopamine, $C_8H_{11}NO_2$, is a neurotransmitter involved in a number of neurological processes.

One important aspect of the complex processes related to dopamine signaling is the number of neurotransmitter molecules released during exocytosis. Since this number is a central factor in determining neurological response (and subsequent human thought and action), it is important to know how this number changes with certain controlled stimulations, such as the administration of drugs. It is also important to understand the mechanism responsible for any changes in the number of neurotransmitter molecules released—for example, some dysfunction in exocytosis, a change in the number of vesicles in the neuron, or a change in the number of neurotransmitter molecules in each vesicle.

Significant progress has been made recently in directly measuring the number of dopamine molecules stored in individual vesicles and the amount actually released when the vesicle undergoes exocytosis. Using miniaturized probes that can selectively detect dopamine molecules in very small amounts, scientists have determined that the vesicles of a certain type of mouse brain neuron contain an average of 30,000 dopamine molecules per vesicle (about 5×10^{-20} mol or 50 zmol). Analysis of these neurons from mice subjected to various drug therapies shows significant changes in the average number of dopamine molecules contained in individual vesicles, increasing or decreasing by up to three-fold, depending on the specific drug used. These studies also indicate that not all of the dopamine in a given vesicle is released during exocytosis, suggesting that it may be possible to regulate the fraction released using pharmaceutical therapies.[1]

Key Concepts and Summary

The formula mass of a substance is the sum of the average atomic masses of each atom represented in the chemical formula and is expressed in atomic mass units. The formula mass of a covalent compound is also called the molecular mass. A convenient amount unit for expressing very large numbers of atoms or molecules is the mole. Experimental measurements have determined the number of entities composing 1 mole of substance to be 6.022×10^{23}, a quantity called Avogadro's number. The mass in grams of 1 mole of substance is its molar mass. Due to the use of the same reference substance in defining the atomic mass unit and the mole, the formula mass (amu) and molar mass (g/mol) for any substance are numerically equivalent (for example, one H_2O molecule weighs approximately18 amu and 1 mole of H_2O molecules weighs approximately 18 g).

1. Omiatek, Donna M., Amanda J. Bressler, Ann-Sofie Cans, Anne M. Andrews, Michael L. Heien, and Andrew G. Ewing. "The Real Catecholamine Content of Secretory Vesicles in the CNS Revealed by Electrochemical Cytometry." *Scientific Report* 3 (2013): 1447, accessed January 14, 2015, doi:10.1038/srep01447.

Chemistry End of Chapter Exercises

Exercise 1.18

What is the total mass (amu) of carbon in each of the following molecules?

(a) CH_4

(b) $CHCl_3$

(c) $C_{12}H_{10}O_6$

(d) $CH_3CH_2CH_2CH_2CH_3$

Solution
(a) 12.01 amu; (b) 12.01 amu; (c) 144.12 amu; (d) 60.05 amu

Exercise 1.19

What is the total mass of hydrogen in each of the molecules?

(a) CH_4

(b) $CHCl_3$

(c) $C_{12}H_{10}O_6$

(d) $CH_3CH_2CH_2CH_2CH_3$

Exercise 1.20

Calculate the molecular or formula mass of each of the following:

(a) P_4

(b) H_2O

(c) $Ca(NO_3)_2$

(d) CH_3CO_2H (acetic acid)

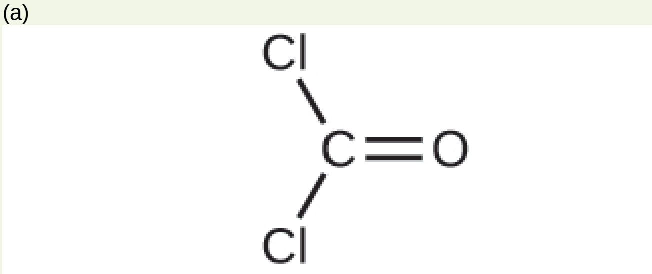
(e) $C_{12}H_{22}O_{11}$ (sucrose, cane sugar).

Solution
(a) 123.896 amu; (b) 18.015 amu; (c) 164.086 amu; (d) 60.052 amu; (e) 342.297 amu

Exercise 1.21

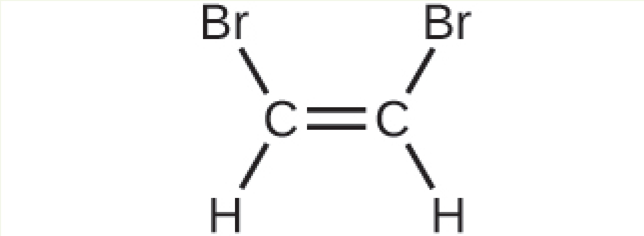
Determine the molecular mass of the following compounds:

(a)

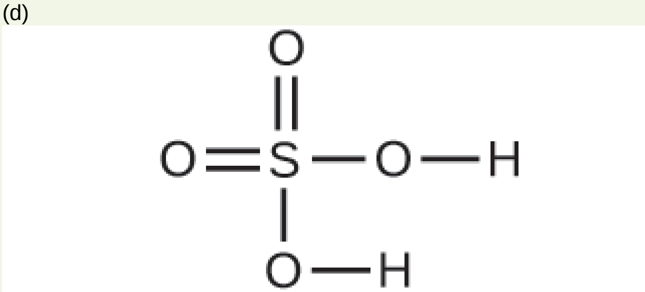


(b)

(c)



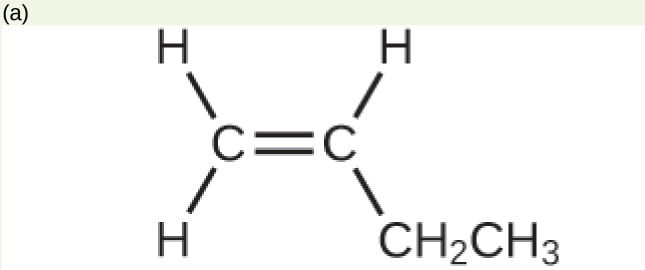
(d)



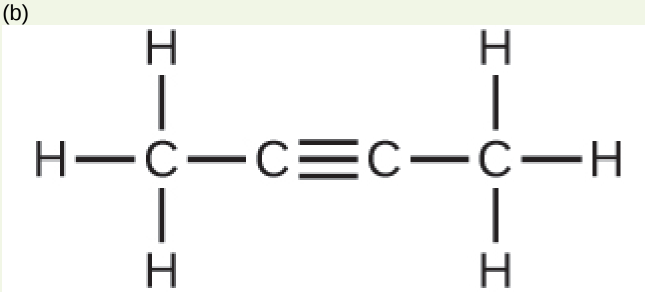
Exercise 1.22

Determine the molecular mass of the following compounds:

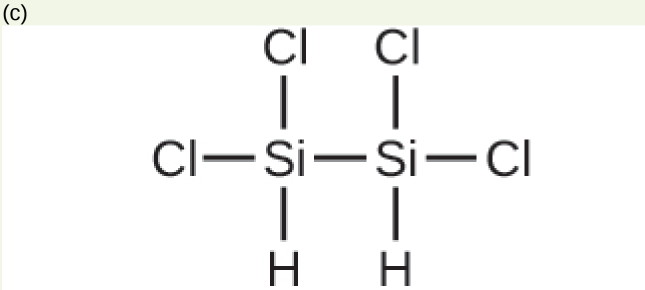
(a)



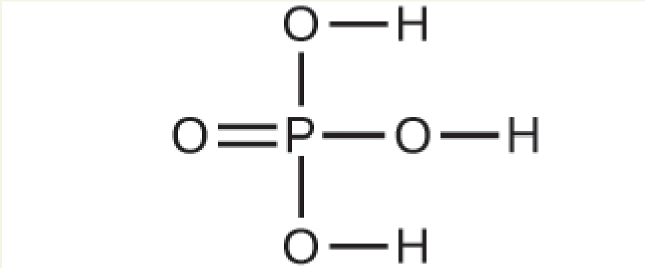
(b)



(c)



(d)



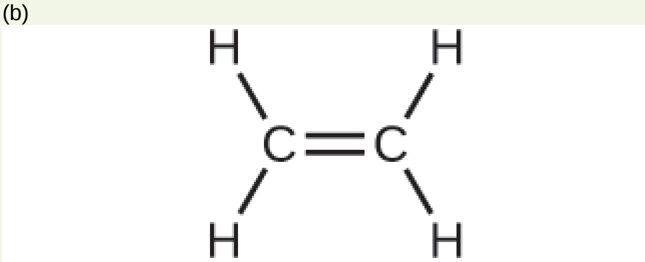
Solution
(a) 56.107 amu;
(b) 54.091 amu;
(c) 199.9976 amu;
(d) 97.9950 amu

Exercise 1.23

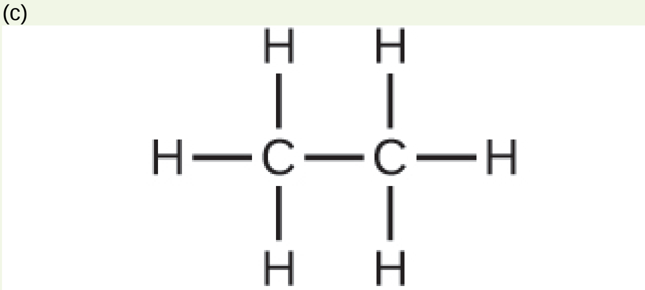
Which molecule has a molecular mass of 28.05 amu?

(a)

(b)



(c)



Exercise 1.24

Write a sentence that describes how to determine the number of moles of a compound in a known mass of the compound if we know its molecular formula.

Solution
Use the molecular formula to find the molar mass; to obtain the number of moles, divide the mass of compound by the molar mass of the compound expressed in grams.

Exercise 1.25

Compare 1 mole of H_2, 1 mole of O_2, and 1 mole of F_2.

(a) Which has the largest number of molecules? Explain why.

(b) Which has the greatest mass? Explain why.

Exercise 1.26

Which contains the greatest mass of oxygen: 0.75 mol of ethanol (C_2H_5OH), 0.60 mol of formic acid (HCO_2H), or 1.0 mol of water (H_2O)? Explain why.

Solution
Formic acid. Its formula has twice as many oxygen atoms as the other two compounds (one each). Therefore, 0.60 mol of formic acid would be equivalent to 1.20 mol of a compound containing a single oxygen atom.

Exercise 1.27

Which contains the greatest number of moles of oxygen atoms: 1 mol of ethanol (C_2H_5OH), 1 mol of formic acid (HCO_2H), or 1 mol of water (H_2O)? Explain why.

Exercise 1.28

How are the molecular mass and the molar mass of a compound similar and how are they different?

Solution
The two masses have the same numerical value, but the units are different: The molecular mass is the mass of 1 molecule while the molar mass is the mass of 6.022×10^{23} molecules.

Exercise 1.29

Calculate the molar mass of each of the following compounds:

(a) hydrogen fluoride, HF

(b) ammonia, NH_3

(c) nitric acid, HNO_3

(d) silver sulfate, Ag_2SO_4

(e) boric acid, $B(OH)_3$

Exercise 1.30

Calculate the molar mass of each of the following:

(a) S_8

(b) C_5H_{12}

(c) $Sc_2(SO_4)_3$

(d) CH_3COCH_3 (acetone)

(e) $C_6H_{12}O_6$ (glucose)

Solution
(a) 256.528 g/mol; (b) 72.150 g mol^{-1}; (c) 378.103 g mol^{-1}; (d) 58.080 g mol^{-1}; (e) 180.158 g mol^{-1}

Exercise 1.31

Calculate the empirical or molecular formula mass and the molar mass of each of the following minerals:

(a) limestone, $CaCO_3$

(b) halite, NaCl

(c) beryl, $Be_3Al_2Si_6O_{18}$

(d) malachite, $Cu_2(OH)_2CO_3$

(e) turquoise, $CuAl_6(PO_4)_4(OH)_8(H_2O)_4$

Exercise 1.32

Calculate the molar mass of each of the following:

(a) the anesthetic halothane, $C_2HBrClF_3$

(b) the herbicide paraquat, $C_{12}H_{14}N_2Cl_2$

(c) caffeine, $C_8H_{10}N_4O_2$

(d) urea, $CO(NH_2)_2$

(e) a typical soap, $C_{17}H_{35}CO_2Na$

Solution

(a) 197.382 g mol^{-1}; (b) 257.163 g mol^{-1}; (c) 194.193 g mol^{-1}; (d) 60.056 g mol^{-1}; (e) 306.464 g mol^{-1}

Exercise 1.33

Determine the number of moles of compound and the number of moles of each type of atom in each of the following:

(a) 25.0 g of propylene, C_3H_6

(b) 3.06×10^{-3} g of the amino acid glycine, $C_2H_5NO_2$

(c) 25 lb of the herbicide Treflan, $C_{13}H_{16}N_2O_4F$ (1 lb = 454 g)

(d) 0.125 kg of the insecticide Paris Green, $Cu_4(AsO_3)_2(CH_3CO_2)_2$

(e) 325 mg of aspirin, $C_6H_4(CO_2H)(CO_2CH_3)$

Exercise 1.34

Determine the mass of each of the following:

(a) 0.0146 mol KOH

(b) 10.2 mol ethane, C_2H_6

(c) 1.6×10^{-3} mol Na_2SO_4

(d) 6.854×10^{3} mol glucose, $C_6H_{12}O_6$

(e) 2.86 mol $Co(NH_3)_6Cl_3$

Solution

(a) 0.819 g;
(b) 307 g;
(c) 0.23 g;
(d) 1.235×10^{6} g (1235 kg);
(e) 765 g

Exercise 1.35

Determine the number of moles of the compound and determine the number of moles of each type of atom in each of the following:

(a) 2.12 g of potassium bromide, KBr

(b) 0.1488 g of phosphoric acid, H_3PO_4

(c) 23 kg of calcium carbonate, $CaCO_3$

(d) 78.452 g of aluminum sulfate, $Al_2(SO_4)_3$

(e) 0.1250 mg of caffeine, $C_8H_{10}N_4O_2$

Exercise 1.36

Determine the mass of each of the following:

(a) 2.345 mol LiCl

(b) 0.0872 mol acetylene, C_2H_2

(c) 3.3×10^{-2} mol Na_2CO_3

(d) 1.23 $\times$ 10^3 mol fructose, $C_6H_{12}O_6$

(e) 0.5758 mol $FeSO_4(H_2O)_7$

Solution
(a) 99.41 g;
(b) 2.27 g;
(c) 3.5 g;
(d) 222 kg;
(e) 160.1 g

Exercise 1.37

The approximate minimum daily dietary requirement of the amino acid leucine, $C_6H_{13}NO_2$, is 1.1 g. What is this requirement in moles?

Exercise 1.38

Determine the mass in grams of each of the following:

(a) 0.600 mol of oxygen atoms

(b) 0.600 mol of oxygen molecules, O_2

(c) 0.600 mol of ozone molecules, O_3

Solution
(a) 9.60 g; (b) 19.2 g; (c) 28.8 g

Exercise 1.39

A 55-kg woman has 7.5 $\times$ 10^{-3} mol of hemoglobin (molar mass = 64,456 g/mol) in her blood. How many hemoglobin molecules is this? What is this quantity in grams?

Exercise 1.40

Determine the number of atoms and the mass of zirconium, silicon, and oxygen found in 0.3384 mol of zircon, $ZrSiO_4$, a semiprecious stone.

Solution
zirconium: 2.038 $\times$ 10^{23} atoms; 30.87 g; silicon: 2.038 $\times$ 10^{23} atoms; 9.504 g; oxygen: 8.151 $\times$ 10^{23} atoms; 21.66 g

Exercise 1.41

Determine which of the following contains the greatest mass of hydrogen: 1 mol of CH_4, 0.6 mol of C_6H_6, or 0.4 mol of C_3H_8.

Exercise 1.42

Determine which of the following contains the greatest mass of aluminum: 122 g of $AlPO_4$, 266 g of Al_2C1_6, or 225 g of Al_2S_3.

Solution
$AlPO_4$: 1.000 mol, or 26.98 g Al
Al_2Cl_6: 1.994 mol, or 53.74 g Al
Al_2S_3: 3.00 mol, or 80.94 g Al
The Al2S3 sample thus contains the greatest mass of Al.

Exercise 1.43

Diamond is one form of elemental carbon. An engagement ring contains a diamond weighing 1.25 carats (1 carat = 200 mg). How many atoms are present in the diamond?

Exercise 1.44

The Cullinan diamond was the largest natural diamond ever found (January 25, 1905). It weighed 3104 carats (1 carat = 200 mg). How many carbon atoms were present in the stone?

Solution

3.113×10^{25} C atoms

Exercise 1.45

One 55-gram serving of a particular cereal supplies 270 mg of sodium, 11% of the recommended daily allowance. How many moles and atoms of sodium are in the recommended daily allowance?

Exercise 1.46

A certain nut crunch cereal contains 11.0 grams of sugar (sucrose, $C_{12}H_{22}O_{11}$) per serving size of 60.0 grams. How many servings of this cereal must be eaten to consume 0.0278 moles of sugar?

Solution

0.865 servings, or about 1 serving.

Exercise 1.47

A tube of toothpaste contains 0.76 g of sodium monofluorophosphate (Na_2PO_3F) in 100 mL.

(a) What mass of fluorine atoms in mg was present?

(b) How many fluorine atoms were present?

Exercise 1.48

Which of the following represents the least number of molecules?

(a) 20.0 g of H_2O (18.02 g/mol)

(b) 77.0 g of CH_4 (16.06 g/mol)

(c) 68.0 g of CaH_2 (42.09 g/mol)

(d) 100.0 g of N_2O (44.02 g/mol)

(e) 84.0 g of HF (20.01 g/mol)

Solution

20.0 g H_2O represents the least number of molecules since it has the least number of moles.

1.5 The Molecular Picture of Gasses

UMASS AMHERST Instructor's Notes

The section on ideal gasses vs. real gasses is also available as a video **here (https://www.youtube.com/watch?v=3vGfWoPIstk)** .

Figure 1.17 The air inside this hot air balloon flying over Putrajaya, Malaysia, is hotter than the ambient air. As a result, the balloon experiences a buoyant force pushing it upward. (credit: Kevin Poh, Flickr)

In this section, we continue to explore the thermal behavior of gases. In particular, we examine the characteristics of atoms and molecules that compose gases. (Most gases, for example nitrogen, N_2, and oxygen, O_2, are composed of two or more atoms. We will primarily use the term "molecule" in discussing a gas because the term can also be applied to monatomic gases, such as helium.)

Gases are easily compressed. We can see evidence of this in **m42215 (https://legacy.cnx.org/content/m42215/latest/#import-auto-id1814176)** , where you will note that gases have the *largest* coefficients of volume expansion. The large coefficients mean that gases expand and contract very rapidly with temperature changes. In addition, you will note that most gases expand at the *same* rate, or have the same β. This raises the question as to why gases should all act in nearly the same way, when liquids and solids have widely varying expansion rates.

The answer lies in the large separation of atoms and molecules in gases, compared to their sizes, as illustrated in **Figure 1.18**. Because atoms and molecules have large separations, forces between them can be ignored, except when they collide with each other during collisions. The motion of atoms and molecules (at temperatures well above the boiling temperature) is fast, such that the gas occupies all of the accessible volume and the expansion of gases is rapid. In contrast, in liquids and solids, atoms and molecules are closer together and are quite sensitive to the forces between them.

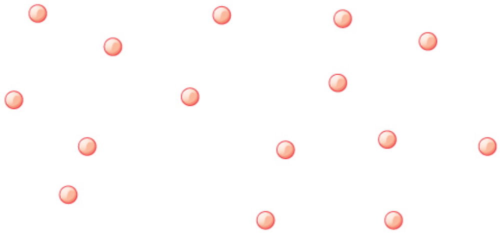


Figure 1.18 Atoms and molecules in a gas are typically widely separated, as shown. Because the forces between them are quite weak at these distances, the properties of a gas depend more on the number of atoms per unit volume and on temperature than on the type of atom.

To get some idea of how pressure, temperature, and volume of a gas are related to one another, consider what happens when you pump air into an initially deflated tire. The tire's volume first increases in direct proportion to the amount of air injected, without much increase in the tire pressure. Once the tire has expanded to nearly its full size, the walls limit volume expansion. If we continue to pump air into it, the pressure increases. The pressure will further increase when the car is driven and the tires move. Most manufacturers specify optimal tire pressure for cold tires. (See **Figure 1.19**.)

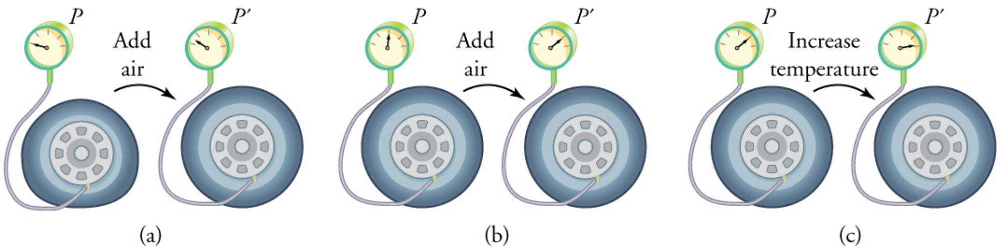


Figure 1.19 (a) When air is pumped into a deflated tire, its volume first increases without much increase in pressure. (b) When the tire is filled to a certain point, the tire walls resist further expansion and the pressure increases with more air. (c) Once the tire is inflated, its pressure increases with temperature.

At room temperatures, collisions between atoms and molecules can be ignored. In this case, the gas is called an ideal gas, in which case the relationship between the pressure, volume, and temperature is given by the equation of state called the ideal gas law.

Ideal Gas Law

The **ideal gas law** states that

$$PV = NkT, \tag{1.14}$$

where P is the absolute pressure of a gas, V is the volume it occupies, N is the number of atoms and molecules in the gas, and T is its absolute temperature. The constant k is called the **Boltzmann constant** in honor of Austrian physicist Ludwig Boltzmann (1844–1906) and has the value

$$k = 1.38\times10^{-23}\text{ J/K}. \tag{1.15}$$

The ideal gas law can be derived from basic principles, but was originally deduced from experimental measurements of Charles' law (that volume occupied by a gas is proportional to temperature at a fixed pressure) and from Boyle's law (that for a fixed temperature, the product PV is a constant). In the ideal gas model, the volume occupied by its atoms and molecules is a negligible fraction of V. The ideal gas law describes the behavior of real gases under most conditions. (Note, for example, that N is the total number of atoms and molecules, independent of the type of gas.)

Let us see how the ideal gas law is consistent with the behavior of filling the tire when it is pumped slowly and the temperature is constant. At first, the pressure P is essentially equal to atmospheric pressure, and the volume V increases in direct proportion to the number of atoms and molecules N put into the tire. Once the volume of the tire is constant, the equation $PV = NkT$ predicts that the pressure should increase in proportion to *the number N of atoms and molecules*.

Example 1.12 Calculating Pressure Changes Due to Temperature Changes: Tire Pressure

Suppose your bicycle tire is fully inflated, with an absolute pressure of $7.00\times10^5\text{ Pa}$ (a gauge pressure of just under 90.0 lb/in^2) at a temperature of 18.0ºC. What is the pressure after its temperature has risen to 35.0ºC? Assume that there are no appreciable leaks or changes in volume.

Strategy

The pressure in the tire is changing only because of changes in temperature. First we need to identify what we know and what we want to know, and then identify an equation to solve for the unknown.

We know the initial pressure $P_0 = 7.00\times10^5\text{ Pa}$, the initial temperature $T_0 = 18.0\text{ºC}$, and the final temperature $T_\text{f} = 35.0\text{ºC}$. We must find the final pressure P_f. How can we use the equation $PV = NkT$? At first, it may seem that not enough information is given, because the volume V and number of atoms N are not specified. What we can do is use the equation twice: $P_0V_0 = NkT_0$ and $P_\text{f}V_\text{f} = NkT_\text{f}$. If we divide $P_\text{f}V_\text{f}$ by P_0V_0 we can come up with an equation that allows us to solve for P_f.

$$\frac{P_\text{f}V_\text{f}}{P_0V_0} = \frac{N_\text{f}kT_\text{f}}{N_0kT_0} \tag{1.16}$$

Since the volume is constant, V_f and V_0 are the same and they cancel out. The same is true for N_f and N_0, and k, which is a constant. Therefore,

$$\frac{P_{\text{f}}}{P_0} = \frac{T_{\text{f}}}{T_0}. \tag{1.17}$$

We can then rearrange this to solve for P_{f} :

$$P_{\text{f}} = P_0 \frac{T_{\text{f}}}{T_0}, \tag{1.18}$$

where the temperature must be in units of kelvins, because T_0 and T_{f} are absolute temperatures.

Solution

1. Convert temperatures from Celsius to Kelvin.

$$\begin{aligned} T_0 &= (18.0 + 273)\text{K} = 291\ \text{K} \\ T_{\text{f}} &= (35.0 + 273)\text{K} = 308\ \text{K} \end{aligned} \tag{1.19}$$

2. Substitute the known values into the equation.

$$P_{\text{f}} = P_0 \frac{T_{\text{f}}}{T_0} = 7.00\times10^5\ \text{Pa}\left(\frac{308\ \text{K}}{291\ \text{K}}\right) = 7.41\times10^5\ \text{Pa} \tag{1.20}$$

Discussion

The final temperature is about 6% greater than the original temperature, so the final pressure is about 6% greater as well. Note that *absolute* pressure and *absolute* temperature must be used in the ideal gas law.

Making Connections: Take-Home Experiment—Refrigerating a Balloon

Inflate a balloon at room temperature. Leave the inflated balloon in the refrigerator overnight. What happens to the balloon, and why?

Example 1.13 Calculating the Number of Molecules in a Cubic Meter of Gas

How many molecules are in a typical object, such as gas in a tire or water in a drink? We can use the ideal gas law to give us an idea of how large N typically is.

Calculate the number of molecules in a cubic meter of gas at standard temperature and pressure (STP), which is defined to be 0°C and atmospheric pressure.

Strategy

Because pressure, volume, and temperature are all specified, we can use the ideal gas law $PV = NkT$, to find N .

Solution

1. Identify the knowns.

$$\begin{aligned} T &= 0^\circ\text{C} = 273\ \text{K} \\ P &= 1.01\times10^5\ \text{Pa} \\ V &= 1.00\ \text{m}^3 \\ k &= 1.38\times10^{-23}\ \text{J/K} \end{aligned} \tag{1.21}$$

2. Identify the unknown: number of molecules, N .

3. Rearrange the ideal gas law to solve for N .

$$\begin{aligned} PV &= NkT \\ N &= \frac{PV}{kT} \end{aligned} \tag{1.22}$$

4. Substitute the known values into the equation and solve for N .

$$N = \frac{PV}{kT} = \frac{\left(1.01\times10^5\ \text{Pa}\right)\left(1.00\ \text{m}^3\right)}{\left(1.38\times10^{-23}\ \text{J/K}\right)(273\ \text{K})} = 2.68\times10^{25}\ \text{molecules} \tag{1.23}$$

Discussion

This number is undeniably large, considering that a gas is mostly empty space. N is huge, even in small volumes. For example, $1\ \text{cm}^3$ of a gas at STP has 2.68×10^{19} molecules in it. Once again, note that N is the same for all types or mixtures of gases.

The Ideal Gas Law Restated Using Moles

A very common expression of the ideal gas law uses the number of moles, n, rather than the number of atoms and molecules, N. We start from the ideal gas law,

$$PV = NkT, \tag{1.24}$$

and multiply and divide the equation by Avogadro's number N_A. This gives

$$PV = \frac{N}{N_\text{A}} N_\text{A} kT. \tag{1.25}$$

Note that $n = N/N_\text{A}$ is the number of moles. We define the universal gas constant $R = N_\text{A}k$, and obtain the ideal gas law in terms of moles.

Ideal Gas Law (in terms of moles)

The ideal gas law (in terms of moles) is

$$PV = nRT. \tag{1.26}$$

The numerical value of R in SI units is

$$R = N_\text{A}k = \left(6.02\times10^{23}\ \text{mol}^{-1}\right)\left(1.38\times10^{-23}\ \text{J/K}\right) = 8.31\ \text{J/mol}\cdot\text{K}. \tag{1.27}$$

In other units,

$$\begin{aligned} R &= 1.99\ \text{cal/mol}\cdot\text{K} \\ R &= 0.0821\ \text{L}\cdot\text{atm/mol}\cdot\text{K}. \end{aligned} \tag{1.28}$$

You can use whichever value of R is most convenient for a particular problem.

Example 1.14 Calculating Number of Moles: Gas in a Bike Tire

How many moles of gas are in a bike tire with a volume of $2.00\times10^{-3}\ \text{m}^3(2.00\ \text{L})$, a pressure of $7.00\times10^5\ \text{Pa}$ (a gauge pressure of just under $90.0\ \text{lb/in}^2$), and at a temperature of 18.0°C?

Strategy

Identify the knowns and unknowns, and choose an equation to solve for the unknown. In this case, we solve the ideal gas law, $PV = nRT$, for the number of moles n.

Solution

1. Identify the knowns.

$$\begin{aligned} P &= 7.00\times10^5\ \text{Pa} \\ V &= 2.00\times10^{-3}\ \text{m}^3 \\ T &= 18.0^\circ\text{C} = 291\ \text{K} \\ R &= 8.31\ \text{J/mol}\cdot\text{K} \end{aligned} \tag{1.29}$$

2. Rearrange the equation to solve for n and substitute known values.

$$\begin{aligned} n &= \frac{PV}{RT} = \frac{\left(7.00\times10^5\ \text{Pa}\right)\left(2.00\times10^{-3}\ \text{m}^3\right)}{(8.31\ \text{J/mol}\cdot\text{K})(291\ \text{K})} \\ &= 0.579\ \text{mol} \end{aligned} \tag{1.30}$$

Discussion

The most convenient choice for R in this case is $8.31\ \text{J/mol}\cdot\text{K},$ because our known quantities are in SI units. The pressure and temperature are obtained from the initial conditions in **Example 1.12**, but we would get the same answer if we used the final values.

The ideal gas law can be considered to be another manifestation of the law of conservation of energy (see **Conservation of Energy (https://legacy.cnx.org/content/m42151/latest/)**). Work done on a gas results in an increase in its energy, increasing pressure and/or temperature, or decreasing volume. This increased energy can also be viewed as increased internal kinetic energy, given the gas's atoms and molecules.

Ideal Gasses vs Real Gasses

To understand the ideal gas approximation, it's probably best to compare it to a real gas, such as the oxygen or nitrogen in the room around you. In a real gas, the gas is made up of particles with some small, but not exactly zero volume. In the ideal gas approximation, we assume that the molecules have exactly zero size. Similarly, in a real gas the molecules attract each other ever so slightly through van der Waals forces and the like. An ideal gas molecule, however, does not attract another ideal gas molecule, they don't interact with each other at all. Also in our analysis of the ideal gas, we will typically ignore gravity. We'll just turn it off and pretend it doesn't have an effect.

These assumptions may seem to have no connection to reality. I mean, a gas of particles of zero size that don't interact with each other and have no impact by gravity? Those seem like fairly, you know, unphysical conditions. However, if you think about an oxygen molecule, its volume is 10^-30 m^2, which is very small compared to the room you're in, so the approximation that a gas has zero size for its particles is not actually that bad. Similarly, the force of the hydrogen bond is something on the neighborhood of 10^{-10} N, which again, is a very small force, so the idea that ideal gas molecules don't attract each other at all is not a terrible approximation. In fact, carbon dioxide gas behaves like an ideal gas to 0.5% accuracy at everyday pressures and temperatures. So, we can use the ideal gas to get a very good idea of how real gases are behaving under a variety of typical conditions.

Let's explore the consequences of this ideal gas approximation. So first we've turned off gravity, what effect does that have? Well, without gravity the particles have no gravitational potential energy, only kinetic energies are possible. There's no g for the mgh. Furthermore, without gravity, particles travel in a straight line until they run into something. They do not follow the parabolic arcs that we've been studying in our third unit. Now let's explore the consequences of the fact that ideal gas particles have zero size and zero intermolecular force. Well, two particles of zero size cannot collide. Think about that one for a minute, but you'll see it's true. Since they can't collide with each other, the particles only hit the walls. The combination of these two approximations means that ideal gas particles travel in straight lines until they hit a wall.

Section Summary

- The ideal gas law relates the pressure and volume of a gas to the number of gas molecules and the temperature of the gas.
- The ideal gas law can be written in terms of the number of molecules of gas:

$$PV = NkT, \tag{1.31}$$

where P is pressure, V is volume, T is temperature, N is number of molecules, and k is the Boltzmann constant

$$k = 1.38\times10^{-23}\text{ J/K}. \tag{1.32}$$

- A mole is the number of atoms in a 12-g sample of carbon-12.
- The number of molecules in a mole is called Avogadro's number N_{A} ,

$$N_{\text{A}} = 6.02\times10^{23}\text{ mol}^{-1}. \tag{1.33}$$

- A mole of any substance has a mass in grams equal to its molecular weight, which can be determined from the periodic table of elements.
- The ideal gas law can also be written and solved in terms of the number of moles of gas:

$$PV = nRT, \tag{1.34}$$

where n is number of moles and R is the universal gas constant,

$$R = 8.31\text{ J/mol}\cdot\text{K}. \tag{1.35}$$

- The ideal gas law is generally valid at temperatures well above the boiling temperature.

Conceptual Questions

Exercise 1.49

Find out the human population of Earth. Is there a mole of people inhabiting Earth? If the average mass of a person is 60 kg, calculate the mass of a mole of people. How does the mass of a mole of people compare with the mass of Earth?

Exercise 1.50

Under what circumstances would you expect a gas to behave significantly differently than predicted by the ideal gas law?

Exercise 1.51

A constant-volume gas thermometer contains a fixed amount of gas. What property of the gas is measured to indicate its temperature?

Problems & Exercises

Exercise 1.52

The gauge pressure in your car tires is $2.50 \times 10^5\ \text{N/m}^2$ at a temperature of 35.0ºC when you drive it onto a ferry boat to Alaska. What is their gauge pressure later, when their temperature has dropped to -40.0ºC ?

Solution

1.62 atm

Exercise 1.53

Convert an absolute pressure of $7.00 \times 10^5\ \text{N/m}^2$ to gauge pressure in lb/in^2. (This value was stated to be just less than $90.0\ \text{lb/in}^2$ in **Example 1.14**. Is it?)

Exercise 1.54

Suppose a gas-filled incandescent light bulb is manufactured so that the gas inside the bulb is at atmospheric pressure when the bulb has a temperature of 20.0ºC . (a) Find the gauge pressure inside such a bulb when it is hot, assuming its average temperature is 60.0ºC (an approximation) and neglecting any change in volume due to thermal expansion or gas leaks. (b) The actual final pressure for the light bulb will be less than calculated in part (a) because the glass bulb will expand. What will the actual final pressure be, taking this into account? Is this a negligible difference?

Solution

(a) 0.136 atm

(b) 0.135 atm. The difference between this value and the value from part (a) is negligible.

Exercise 1.55

Large helium-filled balloons are used to lift scientific equipment to high altitudes. (a) What is the pressure inside such a balloon if it starts out at sea level with a temperature of 10.0ºC and rises to an altitude where its volume is twenty times the original volume and its temperature is -50.0ºC ? (b) What is the gauge pressure? (Assume atmospheric pressure is constant.)

Exercise 1.56

Confirm that the units of nRT are those of energy for each value of R : (a) $8.31\ \text{J/mol} \cdot \text{K}$, (b) $1.99\ \text{cal/mol} \cdot \text{K}$, and (c) $0.0821\ \text{L} \cdot \text{atm/mol} \cdot \text{K}$.

Solution

(a) $nRT = (\text{mol})(\text{J/mol} \cdot \text{K})(\text{K}) = \text{J}$

(b) $nRT = (\text{mol})(\text{cal/mol} \cdot \text{K})(\text{K}) = \text{cal}$

(c)
$$\begin{aligned} nRT &= (\text{mol})(\text{L} \cdot \text{atm/mol} \cdot \text{K})(\text{K}) \\ &= \text{L} \cdot \text{atm} = (\text{m}^3)(\text{N/m}^2) \\ &= \text{N} \cdot \text{m} = \text{J} \end{aligned}$$

Exercise 1.57

In the text, it was shown that $N / V = 2.68 \times 10^{25}\ \text{m}^{-3}$ for gas at STP. (a) Show that this quantity is equivalent to $N / V = 2.68 \times 10^{19}\ \text{cm}^{-3}$, as stated. (b) About how many atoms are there in one μm^3 (a cubic micrometer) at STP? (c) What does your answer to part (b) imply about the separation of atoms and molecules?

Exercise 1.58

Calculate the number of moles in the 2.00-L volume of air in the lungs of the average person. Note that the air is at 37.0°C (body temperature).

Solution

$7.86 \times 10^{-2}\ \text{mol}$

Exercise 1.59

An airplane passenger has $100\ \text{cm}^3$ of air in his stomach just before the plane takes off from a sea-level airport. What volume will the air have at cruising altitude if cabin pressure drops to $7.50 \times 10^4\ \text{N/m}^2$?

Exercise 1.60

(a) What is the volume (in km^3) of Avogadro's number of sand grains if each grain is a cube and has sides that are 1.0 mm long? (b) How many kilometers of beaches in length would this cover if the beach averages 100 m in width and 10.0 m in depth? Neglect air spaces between grains.

Solution

(a) $6.02 \times 10^5\ \text{km}^3$

(b) $6.02 \times 10^8\ \text{km}$

Exercise 1.61

An expensive vacuum system can achieve a pressure as low as $1.00 \times 10^{-7}\ \text{N/m}^2$ at 20°C. How many atoms are there in a cubic centimeter at this pressure and temperature?

Exercise 1.62

The number density of gas atoms at a certain location in the space above our planet is about $1.00 \times 10^{11}\ \text{m}^{-3}$, and the pressure is $2.75 \times 10^{-10}\ \text{N/m}^2$ in this space. What is the temperature there?

Solution

-73.9°C

Exercise 1.63

A bicycle tire has a pressure of $7.00 \times 10^5\ \text{N/m}^2$ at a temperature of 18.0°C and contains 2.00 L of gas. What will its pressure be if you let out an amount of air that has a volume of $100\ \text{cm}^3$ at atmospheric pressure? Assume tire temperature and volume remain constant.

Exercise 1.64

A high-pressure gas cylinder contains 50.0 L of toxic gas at a pressure of $1.40 \times 10^7\ \text{N/m}^2$ and a temperature of 25.0°C.

Its valve leaks after the cylinder is dropped. The cylinder is cooled to dry ice temperature (-78.5°C) to reduce the leak rate and pressure so that it can be safely repaired. (a) What is the final pressure in the tank, assuming a negligible amount of gas leaks while being cooled and that there is no phase change? (b) What is the final pressure if one-tenth of the gas escapes? (c) To what temperature must the tank be cooled to reduce the pressure to 1.00 atm (assuming the gas does not change phase and that there is no leakage during cooling)? (d) Does cooling the tank appear to be a practical solution?

Solution

(a) $9.14\times10^{6}\ \text{N/m}^2$

(b) $8.23\times10^{6}\ \text{N/m}^2$

(c) 2.16 K

(d) No. The final temperature needed is much too low to be easily achieved for a large object.

Exercise 1.65

Find the number of moles in 2.00 L of gas at 35.0°C and under $7.41\times10^{7}\ \text{N/m}^2$ of pressure.

Exercise 1.66

Calculate the depth to which Avogadro's number of table tennis balls would cover Earth. Each ball has a diameter of 3.75 cm. Assume the space between balls adds an extra 25.0% to their volume and assume they are not crushed by their own weight.

Solution

41 km

Exercise 1.67

(a) What is the gauge pressure in a 25.0°C car tire containing 3.60 mol of gas in a 30.0 L volume? (b) What will its gauge pressure be if you add 1.00 L of gas originally at atmospheric pressure and 25.0°C? Assume the temperature returns to 25.0°C and the volume remains constant.

Exercise 1.68

(a) In the deep space between galaxies, the density of atoms is as low as $10^{6}\ \text{atoms/m}^3$, and the temperature is a frigid 2.7 K. What is the pressure? (b) What volume (in m^3) is occupied by 1 mol of gas? (c) If this volume is a cube, what is the length of its sides in kilometers?

Solution

(a) $3.7\times10^{-17}\ \text{Pa}$

(b) $6.0\times10^{17}\ \text{m}^3$

(c) $8.4\times10^{2}\ \text{km}$

1.6 The Molecular Picture of Solids

What is a Solid?

UMASS AMHERST Instructor's Notes

This subsection is based on
umdberg / Solids (2013). Available at: **http://umdberg.pbworks.com/w/page/68393164/Solids%20(2013)**

(http://umdberg.pbworks.com/w/page/68393164/Solids%20(2013)) . (Accessed: 2nd August 2017)

Much of the matter we typically deal with in our everyday lives is a solid. The idea of "solid" is something that retains is shape. This is an idealization -- it really means "it doesn't change its shape while I'm watching it." As we know, a solid can change its shape if you push on it hard enough (by deforming or breaking). Some things that appear solid will change their shape slowly under the gentle pull of gravity. Some glass in very old windows shows a small amount of sagging. And what is it with butter, anyway?

A *uniform solid* is one where every part of it is the same as every other part. This is a model statement, since all matter is composed of atoms and at the atomic level no matter is uniform since there are spaces between the atoms. When we say something is a "uniform solid" we mean that we are going to be examining it on a scale large enough that we can ignore atomic discontinuities.

Some uniform solids consist of atoms or ions in a regular pattern, such as diamonds composed only of carbon atoms in a regular lattice, or a salt crystal composed of sodium and chloride ions in a regular array. However, these are relatively uncommon and so for the most part we will consider solids to made up of molecules. Solids can be formed from a single compound where all the molecules are the same, or multiple compounds, with a mixture of molecules. A multi-component solid is sometimes harder to characterize as it can be a non-uniform mixture or conglomerate.

Biological solids can be quite complex. They are often not uniform, but instead are composites, made of particles embedded in layers composed of different materials. Your skin is a good example. It is composed of a thick layer of connective tissue (dermis) underlain by a membrane and overlaid by a surface (epidermal) layer. The dermis is actually a three dimensional network of collagen fibers that are embedded in a protein-polysaccaride matrix. The stretchiness of your skin is provided by additional elastin fibers which are distributed throughout the dermis.*

When most liquids are cooled, they eventually freeze and form **crystalline solids**, solids in which the atoms, ions, or molecules are arranged in a definite repeating pattern. It is also possible for a liquid to freeze before its molecules become arranged in an orderly pattern. The resulting materials are called **amorphous solids** or noncrystalline solids (or, sometimes, glasses). The particles of such solids lack an ordered internal structure and are randomly arranged (**Figure 1.20**).

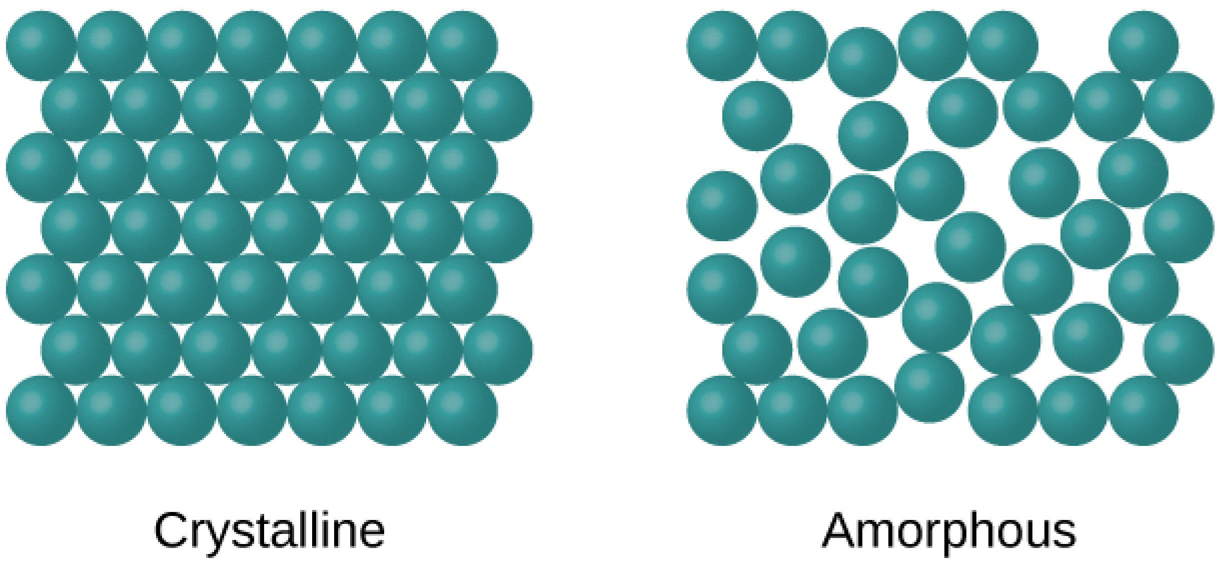


Figure 1.20 The entities of a solid phase may be arranged in a regular, repeating pattern (crystalline solids) or randomly (amorphous).

Metals and ionic compounds typically form ordered, crystalline solids. Substances that consist of large molecules, or a mixture of molecules whose movements are more restricted, often form amorphous solids. For examples, candle waxes are amorphous solids composed of large hydrocarbon molecules. Some substances, such as boron oxide (shown in **Figure 1.21**), can form either crystalline or amorphous solids, depending on the conditions under which it is produced. Also, amorphous solids may undergo a transition to the crystalline state under appropriate conditions.

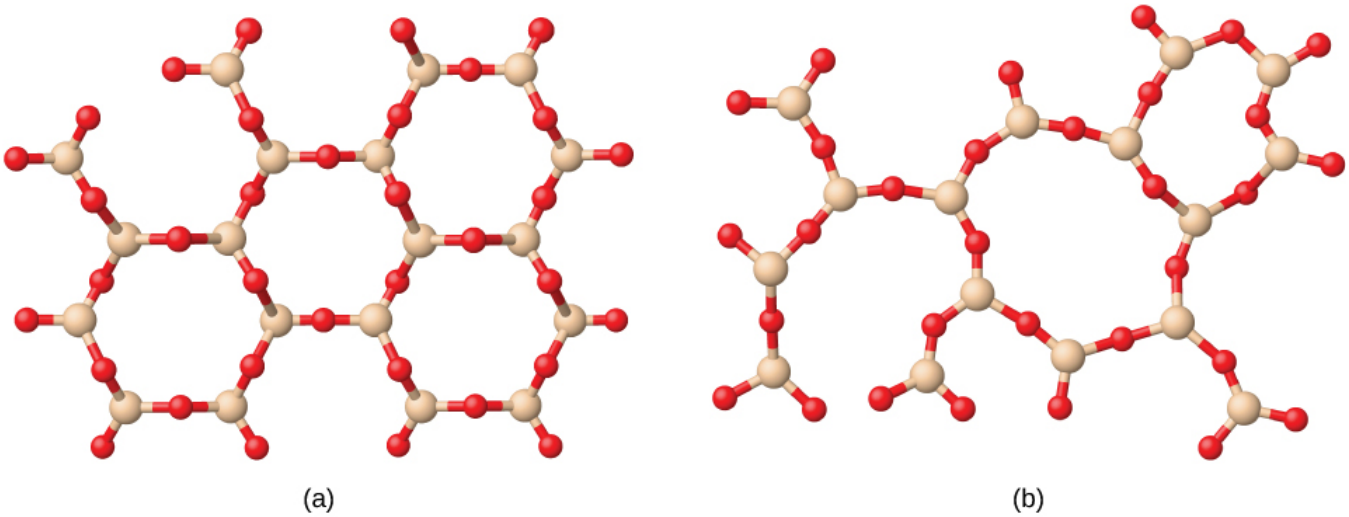


Figure 1.21 (a) Diboron trioxide, B_2O_3, is normally found as a white, amorphous solid (a glass), which has a high degree of disorder in its structure. (b) By careful, extended heating, it can be converted into a crystalline form of B_2O_3, which has a very ordered arrangement.

Crystalline solids are generally classified according the nature of the forces that hold its particles together. These forces are primarily responsible for the physical properties exhibited by the bulk solids. The following sections provide descriptions of the major types of crystalline solids: ionic, metallic, covalent network, and molecular.

Ionic Solids

Ionic solids, such as sodium chloride and nickel oxide, are composed of positive and negative ions that are held together by electrostatic attractions, which can be quite strong (**Figure 1.22**). Many ionic crystals also have high melting points. This is due to the very strong attractions between the ions—in ionic compounds, the attractions between full charges are (much) larger than those between the partial charges in polar molecular compounds. This will be looked at in more detail in a later discussion of lattice energies. Although they are hard, they also tend to be brittle, and they shatter rather than bend. Ionic solids do not conduct electricity; however, they do conduct when molten or dissolved because their ions are free to move. Many simple compounds formed by the reaction of a metallic element with a nonmetallic element are ionic.

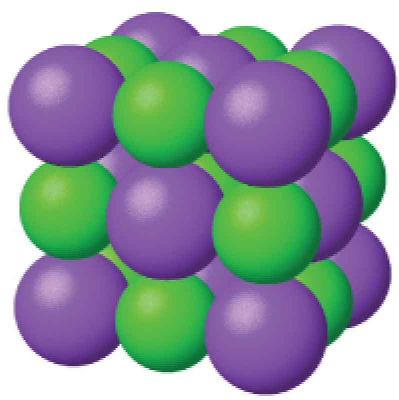


Figure 1.22 Sodium chloride is an ionic solid.

Metallic Solids

Metallic solids such as crystals of copper, aluminum, and iron are formed by metal atoms **Figure 1.23**. The structure of metallic crystals is often described as a uniform distribution of atomic nuclei within a “sea” of delocalized electrons. The atoms within such a metallic solid are held together by a unique force known as *metallic bonding* that gives rise to many useful and varied bulk properties. All exhibit high thermal and electrical conductivity, metallic luster, and malleability. Many are very hard and quite strong. Because of their malleability (the ability to deform under pressure or hammering), they do not shatter and, therefore, make useful construction materials. The melting points of the metals vary widely. Mercury is a liquid at room temperature, and the alkali metals melt below 200 °C. Several post-transition metals also have low melting points, whereas the transition metals melt at temperatures above 1000 °C. These differences reflect differences in strengths of metallic bonding among the metals.

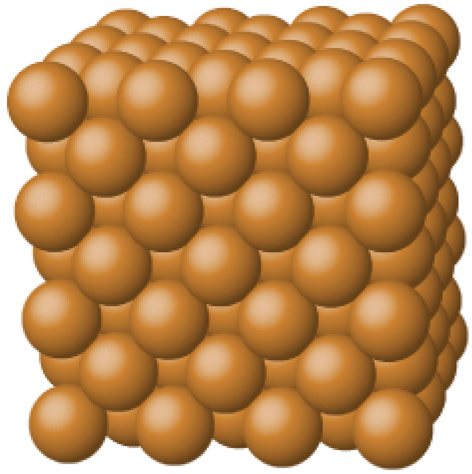


Figure 1.23 Copper is a metallic solid.

Covalent Network Solid

Covalent network solids include crystals of diamond, silicon, some other nonmetals, and some covalent compounds such as silicon dioxide (sand) and silicon carbide (carborundum, the abrasive on sandpaper). Many minerals have networks of covalent bonds. The atoms in these solids are held together by a network of covalent bonds, as shown in Figure 1.24. To break or to melt a covalent network solid, covalent bonds must be broken. Because covalent bonds are relatively strong, covalent network solids are typically characterized by hardness, strength, and high melting points. For example, diamond is one of the hardest substances known and melts above 3500 °C.

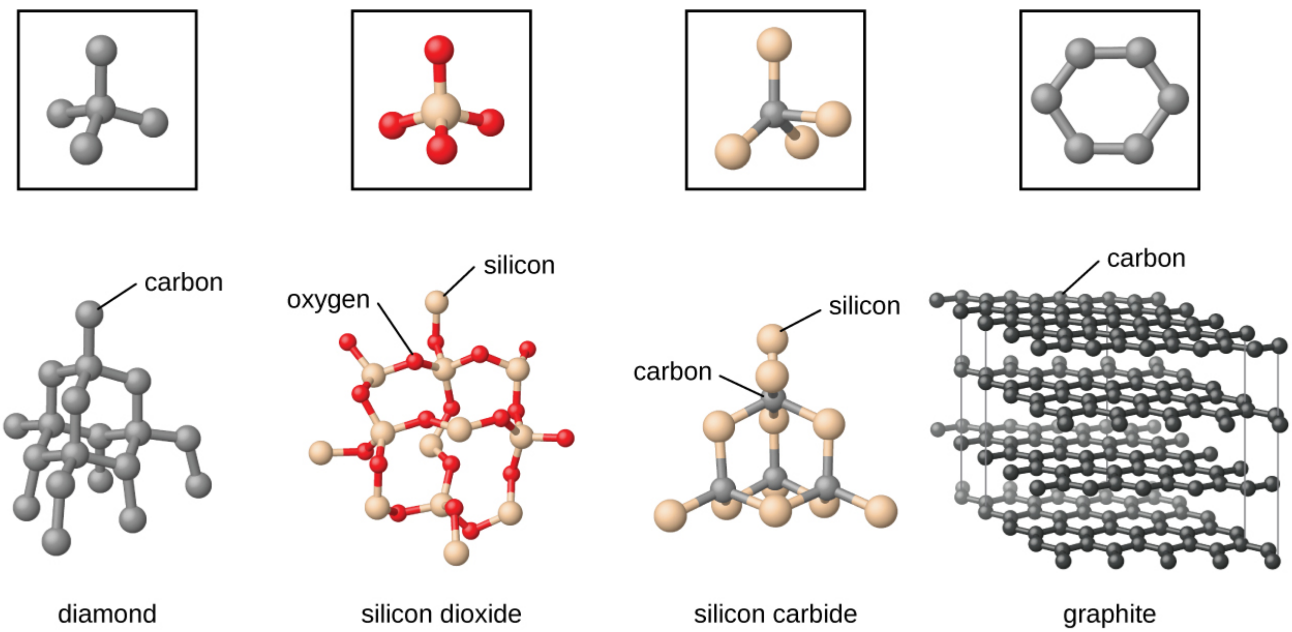


Figure 1.24 A covalent crystal contains a three-dimensional network of covalent bonds, as illustrated by the structures of diamond, silicon dioxide, silicon carbide, and graphite. Graphite is an exceptional example, composed of planar sheets of covalent crystals that are held together in layers by noncovalent forces. Unlike typical covalent solids, graphite is very soft and electrically conductive.

Molecular Solid

Molecular solids, such as ice, sucrose (table sugar), and iodine, as shown in Figure 1.25, are composed of neutral molecules. The strengths of the attractive forces between the units present in different crystals vary widely, as indicated by the melting points of the crystals. Small symmetrical molecules (nonpolar molecules), such as H_2, N_2, O_2, and F_2, have weak attractive forces and form molecular solids with very low melting points (below −200 °C). Substances consisting of larger, nonpolar molecules have larger attractive forces and melt at higher temperatures. Molecular solids composed of molecules with permanent dipole moments (polar molecules) melt at still higher temperatures. Examples include ice (melting point, 0 °C) and table sugar (melting point, 185 °C).

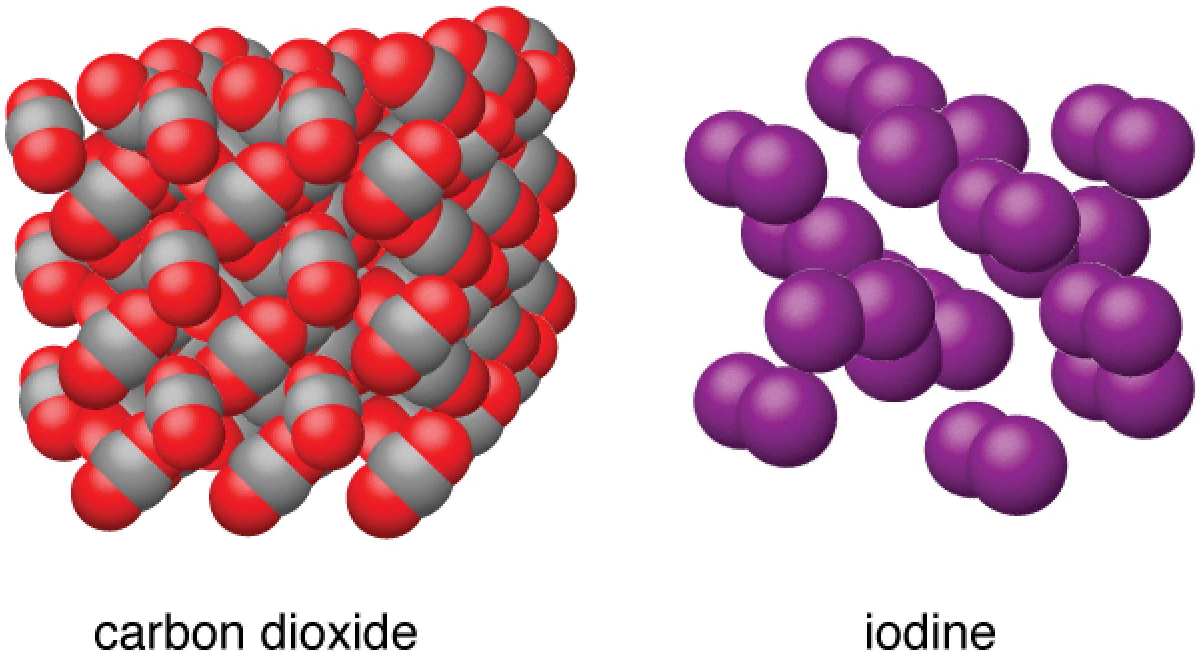


Figure 1.25 Carbon dioxide (CO_2) consists of small, nonpolar molecules and forms a molecular solid with a melting point of −78 °C. Iodine (I_2) consists of larger, nonpolar molecules and forms a molecular solid that melts at 114 °C.

Properties of Solids

A crystalline solid, like those listed in **Table 1.5**, has a precise melting temperature because each atom or molecule of the same type is held in place with the same forces or energy. Thus, the attractions between the units that make up the crystal all have the same strength and all require the same amount of energy to be broken. The gradual softening of an amorphous material differs dramatically from the distinct melting of a crystalline solid. This results from the structural nonequivalence of the molecules in the amorphous solid. Some forces are weaker than others, and when an amorphous material is heated, the weakest intermolecular attractions break first. As the temperature is increased further, the stronger attractions are broken. Thus amorphous materials soften over a range of temperatures.

Table 1.5

Types of Crystalline Solids and Their Properties				
Type of Solid	**Type of Particles**	**Type of Attractions**	**Properties**	**Examples**
ionic	ions	ionic bonds	hard, brittle, conducts electricity as a liquid but not as a solid, high to very high melting points	NaCl, Al_2O_3
metallic	atoms of electropositive elements	metallic bonds	shiny, malleable, ductile, conducts heat and electricity well, variable hardness and melting temperature	Cu, Fe, Ti, Pb, U
covalent network	atoms of electronegative elements	covalent bonds	very hard, not conductive, very high melting points	C (diamond), SiO_2, SiC
molecular	molecules (or atoms)	IMFs	variable hardness, variable brittleness, not conductive, low melting points	H_2O, CO_2, I_2, $C_{12}H_{22}O_{11}$

Graphene: Material of the Future

Carbon is an essential element in our world. The unique properties of carbon atoms allow the existence of carbon-based life forms such as ourselves. Carbon forms a huge variety of substances that we use on a daily basis, including those shown in **Figure 1.26**. You may be familiar with diamond and graphite, the two most common *allotropes* of carbon. (Allotropes are different structural forms of the same element.) Diamond is one of the hardest-known substances, whereas graphite is soft enough to be used as pencil lead. These very different properties stem from the different arrangements of the carbon atoms in the different allotropes.

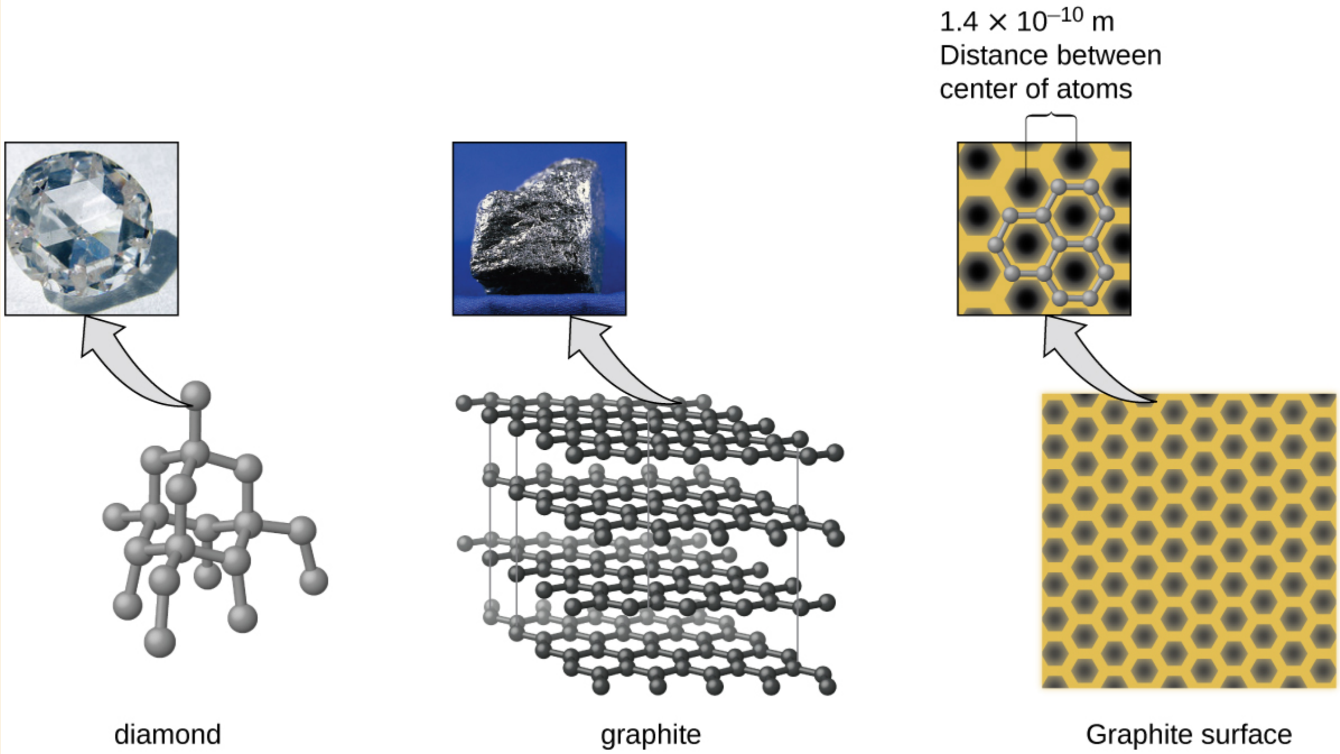


Figure 1.26 Diamond is extremely hard because of the strong bonding between carbon atoms in all directions. Graphite (in pencil lead) rubs off onto paper due to the weak attractions between the carbon layers. An image of a graphite surface shows the distance between the centers of adjacent carbon atoms. (credit left photo: modification of work by Steve Jurvetson; credit middle photo: modification of work by United States Geological Survey)

You may be less familiar with a recently discovered form of carbon: graphene. Graphene was first isolated in 2004 by using tape to peel off thinner and thinner layers from graphite. It is essentially a single sheet (one atom thick) of graphite. Graphene, illustrated in **Figure 1.27**, is not only strong and lightweight, but it is also an excellent conductor of electricity and heat. These properties may prove very useful in a wide range of applications, such as vastly improved computer chips and circuits, better batteries and solar cells, and stronger and lighter structural materials. The 2010 Nobel Prize in Physics was awarded to Andre Geim and Konstantin Novoselov for their pioneering work with graphene.

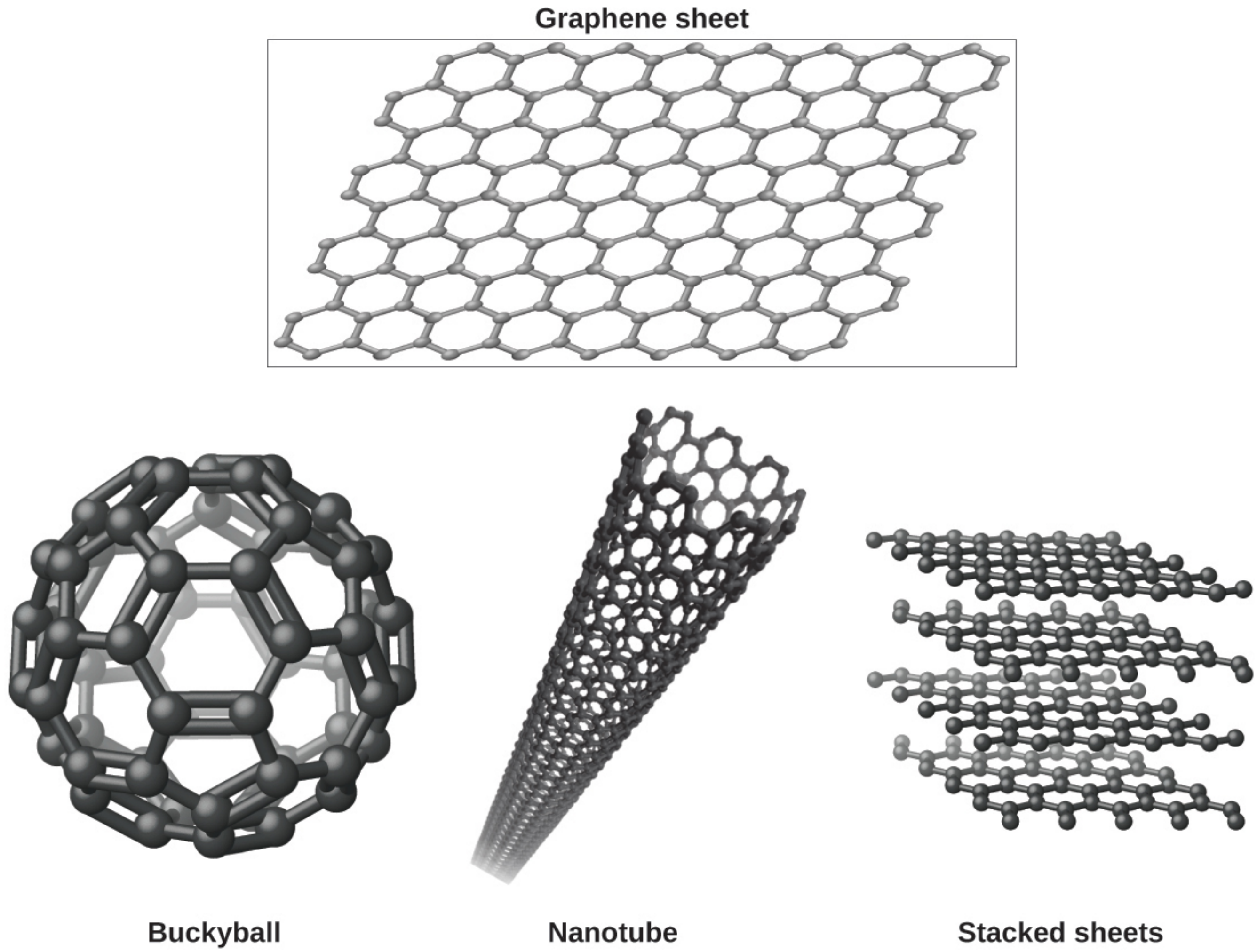


Figure 1.27 Graphene sheets can be formed into buckyballs, nanotubes, and stacked layers.

Crystal Defects

In a crystalline solid, the atoms, ions, or molecules are arranged in a definite repeating pattern, but occasional defects may occur in the pattern. Several types of defects are known, as illustrated in **Figure 1.28**. **Vacancies** are defects that occur when positions that should contain atoms or ions are vacant. Less commonly, some atoms or ions in a crystal may occupy positions, called **interstitial sites**, located between the regular positions for atoms. Other distortions are found in impure crystals, as, for example, when the cations, anions, or molecules of the impurity are too large to fit into the regular positions without distorting the structure. Trace amounts of impurities are sometimes added to a crystal (a process known as *doping)* in order to create defects in the structure that yield desirable changes in its properties. For example, silicon crystals are doped with varying amounts of different elements to yield suitable electrical properties for their use in the manufacture of semiconductors and computer chips.

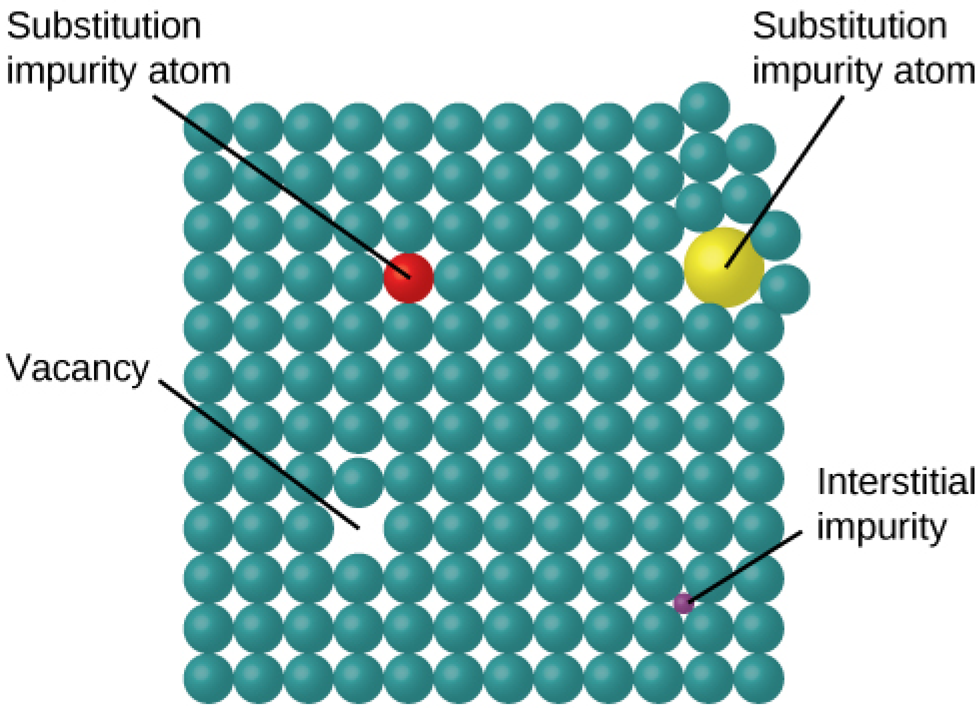


Figure 1.28 Types of crystal defects include vacancies, interstitial atoms, and substitutions impurities.

Einstein Solid

In this class, we will be focused on a model called the *Einstein solid* model of simple crystalline solids, particularly those where the atoms are arranged in simple cube-based structures such as table salt (NaCl) and copper as shown in Figure 10. This model, named after Albert Einstein, is like the ideal gas law discussed in the previous section: almost no solid behaves exactly like an Einstein solid. However, the behavior of many solids is approximately Einsteinian and the principles behind the model can be used successfully to understand the more complex solids that you will probably study in your other science courses.

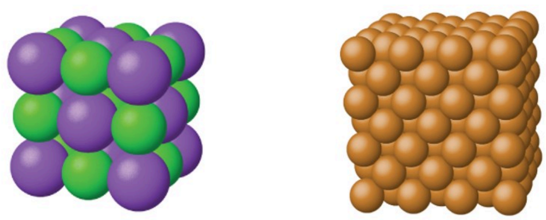


Figure 1.29 Both table salt (NaCl) on the left and solid copper on the right are simple crystalline solids with a cube-based structure.

In the Einstein solid model, we model the bonds between each atom as tiny springs that obey the Hooke's Law F=kx that we discussed in Unit 2. This model is shown in Figure 11. While chemical bonds, like everything else, do NOT exactly obey Hooke's Law, the modelling of atomic bonds as springs is very common throughout physics, chemistry, and biology. The reason this model is so common is that, for small shifts, atoms do behave as if they are on springs: pull two atoms in a crystal apart and the atomic bonds will pull them back together. If you push two atoms too close together, then the two positively charged nuclei will repel each other pushing them apart. The result is that each atom vibrates about its equilibrium position or *lattice site* as if it were attached to tiny springs. Given that calculations with Hooke's Law are so simple to do in comparison to full calculations of atomic forces, you can see why this Einstein model is a useful approximation.

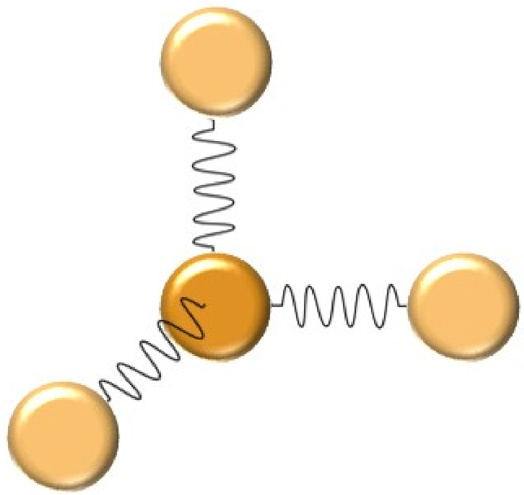


Figure 1.30 The model of an Einstein solid, the central atom is connected to its neighbors by atomic bonds that are modeled by springs that obey Hooke's Law

Key Concepts and Summary

Some substances form crystalline solids consisting of particles in a very organized structure; others form amorphous (noncrystalline) solids with an internal structure that is not ordered. The main types of crystalline solids are ionic solids, metallic solids, covalent network solids, and molecular solids. The properties of the different kinds of crystalline solids are due to the types of particles of which they consist, the arrangements of the particles, and the strengths of the attractions between them. Because their particles experience identical attractions, crystalline solids have distinct melting temperatures; the particles in amorphous solids experience a range of interactions, so they soften gradually and melt over a range of temperatures. Some crystalline solids have defects in the definite repeating pattern of their particles. These defects (which include vacancies, atoms or ions not in the regular positions, and impurities) change physical properties such as electrical conductivity, which is exploited in the silicon crystals used to manufacture computer chips.

Chemistry End of Chapter Exercises

Exercise 1.69

What types of liquids typically form amorphous solids?

Exercise 1.70

At very low temperatures oxygen, O_2, freezes and forms a crystalline solid. Which best describes these crystals?

(a) ionic

(b) covalent network

(c) metallic

(d) amorphous

(e) molecular crystals

Solution
(e) molecular crystals

Exercise 1.71

As it cools, olive oil slowly solidifies and forms a solid over a range of temperatures. Which best describes the solid?

(a) ionic

(b) covalent network

(c) metallic

(d) amorphous

(e) molecular crystals

Exercise 1.72

Explain why ice, which is a crystalline solid, has a melting temperature of 0 °C, whereas butter, which is an amorphous solid, softens over a range of temperatures.

Solution
Ice has a crystalline structure stabilized by hydrogen bonding. These intermolecular forces are of comparable strength and thus require the same amount of energy to overcome. As a result, ice melts at a single temperature and not over a range of temperatures. The various, very large molecules that compose butter experience varied van der Waals attractions of various strengths that are overcome at various temperatures, and so the melting process occurs over a wide temperature range.

Exercise 1.73

Identify the type of crystalline solid (metallic, network covalent, ionic, or molecular) formed by each of the following substances:

(a) SiO_2

(b) KCl

(c) Cu

(d) CO_2

(e) C (diamond)

(f) $BaSO_4$

(g) NH_3

(h) NH_4F

(i) C_2H_5OH

Exercise 1.74

Identify the type of crystalline solid (metallic, network covalent, ionic, or molecular) formed by each of the following substances:

(a) $CaCl_2$

(b) SiC

(c) N_2

(d) Fe

(e) C (graphite)

(f) $CH_3CH_2CH_2CH_3$

(g) HCl

(h) NH_4NO_3

(i) K_3PO_4

Solution
(a) ionic; (b) covalent network; (c) molecular; (d) metallic; (e) covalent network; (f) molecular; (g) molecular; (h) ionic; (i) ionic

Exercise 1.75

Classify each substance in the table as either a metallic, ionic, molecular, or covalent network solid:

Table 1.6

Substance	Appearance	Melting Point	Electrical Conductivity	Solubility in Water
X	lustrous, malleable	1500 °C	high	insoluble
Y	soft, yellow	113 °C	none	insoluble
Z	hard, white	800 °C	only if melted/dissolved	soluble

Exercise 1.76

Classify each substance in the table as either a metallic, ionic, molecular, or covalent network solid:

Table 1.7

Substance	Appearance	Melting Point	Electrical Conductivity	Solubility in Water
X	brittle, white	800 °C	only if melted/dissolved	soluble
Y	shiny, malleable	1100 °C	high	insoluble
Z	hard, colorless	3550 °C	none	insoluble

Solution
X = ionic; Y = metallic; Z = covalent network

Exercise 1.77

Identify the following substances as ionic, metallic, covalent network, or molecular solids:

Substance A is malleable, ductile, conducts electricity well, and has a melting point of 1135 °C. Substance B is brittle, does not conduct electricity as a solid but does when molten, and has a melting point of 2072 °C. Substance C is very hard, does not conduct electricity, and has a melting point of 3440 °C. Substance D is soft, does not conduct electricity, and has a melting point of 185 °C.

Exercise 1.78

Substance A is shiny, conducts electricity well, and melts at 975 °C. Substance A is likely a(n):

(a) ionic solid

(b) metallic solid

(c) molecular solid

(d) covalent network solid

Solution
(b) metallic solid

Exercise 1.79

Substance B is hard, does not conduct electricity, and melts at 1200 °C. Substance B is likely a(n):

(a) ionic solid

(b) metallic solid

(c) molecular solid

(d) covalent network solid

Glossary

amorphous solid: (also, noncrystalline solid) solid in which the particles lack an ordered internal structure

anion: negatively charged atom or molecule (contains more electrons than protons)

atomic mass: average mass of atoms of an element, expressed in amu

atomic mass unit (amu): (also, unified atomic mass unit, u, or Dalton, Da) unit of mass equal to $\frac{1}{12}$ of the mass of a ^{12}C atom

atomic number (Z): number of protons in the nucleus of an atom

Avogadro's number: N_A , the number of molecules or atoms in one mole of a substance; $N_A = 6.02 \times 10^{23}$ particles/mole

Avogadro's number (N_A): experimentally determined value of the number of entities comprising 1 mole of substance, equal to 6.022 $\times$ 10^{23} mol^{-1}

Boltzmann constant: k , a physical constant that relates energy to temperature; $k = 1.38 \times 10^{-23}$ J/K

cation: positively charged atom or molecule (contains fewer electrons than protons)

chemical symbol: one-, two-, or three-letter abbreviation used to represent an element or its atoms

covalent network solid: solid whose particles are held together by covalent bonds

crystalline solid: solid in which the particles are arranged in a definite repeating pattern

Dalton (Da): alternative unit equivalent to the atomic mass unit

formula mass: sum of the average masses for all atoms represented in a chemical formula; for covalent compounds, this is also the molecular mass

fundamental unit of charge: (also called the elementary charge) equals the magnitude of the charge of an electron (e) with e = 1.602 $\times$ 10^{-19} C

ideal gas law: the physical law that relates the pressure and volume of a gas to the number of gas molecules or number of moles of gas and the temperature of the gas

interstitial sites: spaces between the regular particle positions in any array of atoms or ions

ion: electrically charged atom or molecule (contains unequal numbers of protons and electrons)

ionic solid: solid composed of positive and negative ions held together by strong electrostatic attractions

mass number (A): sum of the numbers of neutrons and protons in the nucleus of an atom

metallic solid: solid composed of metal atoms

molar mass: mass in grams of 1 mole of a substance

mole: amount of substance containing the same number of atoms, molecules, ions, or other entities as the number of atoms in exactly 12 grams of ^{12}C

mole: the quantity of a substance whose mass (in grams) is equal to its molecular mass

molecular solid: solid composed of neutral molecules held together by intermolecular forces of attraction

unified atomic mass unit (u): alternative unit equivalent to the atomic mass unit

vacancy: defect that occurs when a position that should contain an atom or ion is vacant

2 INTRODUCTION TO ENERGY

2.1 Introduction to Energy

Up to this point in the course, we've really been talking about forces. Now we're going to change things up and talk about a quantity that you may be familiar with from previous science courses: the idea of energy. This chapter introduces this idea. In the next chapter, we'll begin our formal study of energy by looking at some of the same macroscopic phenomena we have been discussing already: falling objects, sliding with friction, and other such situations. We'll just be looking at these same situations from a different viewpoint. It turns out that you can describe any phenomenon we've talked about in the course up to this point either way, either with energy or with forces, although often, one is easier than the other. In the subsequent chapter we'll move on and use our idea of energy at the microscopic scale to understand completely new phenomena that are very directly related to ideas you might have discussed in the chemistry or a biology course.

So, what is energy? in short, energy is the ability to do work. That's the definition that we'll be using in this course. You can also flip this on its head with the change of perspective, however, and say that work is one way of transferring this quantity known as energy. Before we get too deep into it, it's probably worth comparing briefly the idea of energy and the idea forces. So, up to this point we've been talking about forces, and forces are an idea of instance. We look at the forces that are acting on an object "right now", and "right now" is all that matters, and then we can work instant by instant like we did in our simulations to study the motion of an object. Most physics courses begin with the idea of forces because they're easy to get a feel for what a force is., we've all experienced pushes and pulls.

Energy, on the other hand, is an example of a conservation law. Energy is a quantity that never changes through a process. This allows us to relate two points in time that are not directly next to each other. So, with forces, we had to take very tiny little time steps. With energy, we can go from the beginning of a process all the way to the end, not care too much about the middle, and know that energy is going to be conserved. So, why don't we start with energy? Well, energy can be conceptually a little bit more difficult to get your head around. What exactly is energy; it's a little bit more abstract of an idea, which is why most physics courses, including this one, put it off until after a discussion of forces.

So, what's the big point of the next three chapters? So, you've probably dealt with energy before in a previous science class. Our goal is to develop a coherent picture of energy across different scientific disciplines and across a large variety of different distant scales, from the sizes that we experience in our everyday world of people and cars and trees, all the way down to the atomic and molecular scale. We will, therefore, deal with several different ideas that can seem unrelated. We'll talk about boxes on hills and fish on springs and the kinetic energy of, say, cars, but we'll also talk about the kinetic energy of molecules, which we'll find out is directly related to the idea of temperature, and we'll talk about energy transfer through random collisions of particles on the atomic scale, which we'll talk about as the idea of heat. These different ideas and others including chemical bonds are connected by the idea of energy.

So, again, let's review how the next three chapters are laid out this current chapter provides a basic overview of what energy is, identifies the two main types of energy, kinetic and potential energy, identifies two main scales of energy we'll discuss in this class, the macroscopic scale of people, cars, etc., and the microscopic scale of atoms and molecules. The next chapter discusses energy on this macroscopic scale of people cars and so forth, and the chapter after that really gets into energy on the microscopic scale, atoms, molecules, and temperature.

2.2 Units of Energy

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Converting between the different units of energy

If energy is defined as the ability to do work, then energy and work must have the same units. Thus, the SI unit of the energy is the Joule (recall 1J=1Nm=1kgms2). Energy, however, is one quantity where there are many other units in common use in scientific literature including electron-Volts (eV), kilowatt-hours (kW·hr), calories, and Calories.

UMASS AMHERST Instructor's Notes

In this course, we will be using Joules and electron-Volts exclusively. We are including these other units for your reference.

Electron-Volts

A common quantity in chemistry is the electron-Volt or eV. One electron-Volt is the amount of energy gained by an electron as it travels between the two ends of a 1 Volt battery (a concept that will be discussed in more detail when you study electricity). Numerically, 1eV = 1.602×10^{-19}J. The reason this unit is common in chemistry is that the energies of atomic bonds are typically about 1eV as shown in the table below[1]. The bond-dissociation energy is the energy released when the bond is formed.

Table 2.1

Bond	Bond-dissociation energy at 298K (eV/Bond)	Comment
C-C	3.60-3.69	Strong, but weaker than C–H bonds
Cl-Cl	2.51	Indicated by the yellowish colour of this gas
H-H	4.52	Strong, nonpolarizable bondCleaved only by metals and by strong oxidants
O-H	4.77	Slightly stronger than C–H bonds
OH-H	2.78	Far weaker than C–H bonds
C-O	11.16	Far stronger than C–H bonds
O-CO	5.51	Slightly stronger than C–H bonds
O=O	5.15	Stronger than single bondsWeaker than many other double bonds
N=N	9.79	One of the strongest bondsLarge activation energy in production of ammonia
H_3C-H	4.550	One of the strongest aliphatic C–H bonds

UMASS AMHERST Instructor's Notes

Note that the eV is significantly smaller than the joule; eV will generally be the smallest unit of energy used in this course.

Kilowatt Hours

When you buy electricity from the power company, the bill says how many kilowatt hours you have purchased. A Watt is a unit of a quantity called *power* and 1 Watt is equal to 1 Joule/second: 1W = 1 J/s. Thus, a kilowatt hour is:

$$(1kW * hr)\left(\frac{1000W}{kW}\right)\left(\frac{1J/s}{W}\right)\left(\frac{3600s}{hr}\right) = 3.6106J = 3.6MJs$$

The calorie is an imperial unit of energy that is still in common use in the nutritional sciences in the United States. One calorie (lowercase c) is the amount of energy needed to raise 1g of water 1^{o}C or 1 cal = 4.814J. On food labels, you will see energy listed in Calories (capital C). One Calorie is equal to 1kilocalorie; in other words, 1 Cal = 1000 cal. Thus, one 1 Cal = 4814 J. In other countries, you will see food labels in both Calories and Joules like the one shown in Figure 1.

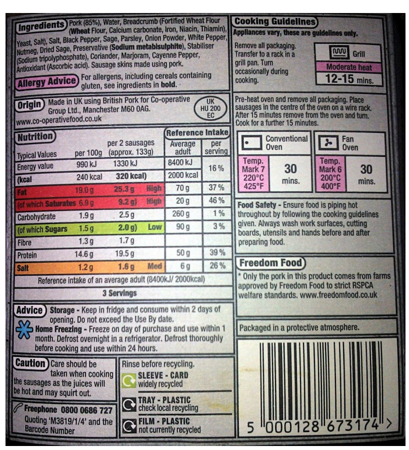


Figure 2.1 A food label from the UK showing the energy of the food in both Joules and kcal (or Calories).

1. "Bond-Dissociation Energy - Wikipedia." Accessed August 1, 2017. https://en.wikipedia.org/wiki/Bond-dissociation_energy.

2.3 Types of Energy and Scales of Energy

UMASS AMHERST Instructor's Notes

This section provides the definitions of the terms kinetic and potential energy and defines the microscopic and macroscopic scales. Details and formulae will be provided in upcoming chapters.

Fundamentally, there are only two kinds of energy: kinetic energy and potential energy. *Kinetic energy (K) is the ability to do work associated with motion and potential energy (U) is the ability to do work arising from the relative positions of two or more objects.* As an example, the car in motion in the left image of Figure 2 has the capability to do work due to its motion - the car has *kinetic energy*. If the car were to crash, then a force would be exerted over a distance deforming the car (right image in Figure 2). The sheer fact that the car is moving means that it *can* do work. Similarly, the larger block in Figure 3 could do work if the system were released. As the large block fell, it would lift the small block. The large block has *potential energy* - an ability to do work due to its position relative to the earth.

Figure 2.2 Figure B: A car traveling down the road (left) has an ability to do work due to its motion - it has kinetic energy. We see that ability to do work when the car crashes (right) - a force acts for a distance deforming the car.[2]

2. SteveBaker. English: A 2007 MINI Cooper'S Car Shown Immediately before - and Soon after - a Severe Car Crash. At: Lat/Long 31.03669,-97.470881, February 9, 2009. Own work. https://commons.wikimedia.org/wiki/File:BeforeAndAfterMINICooperS.png.

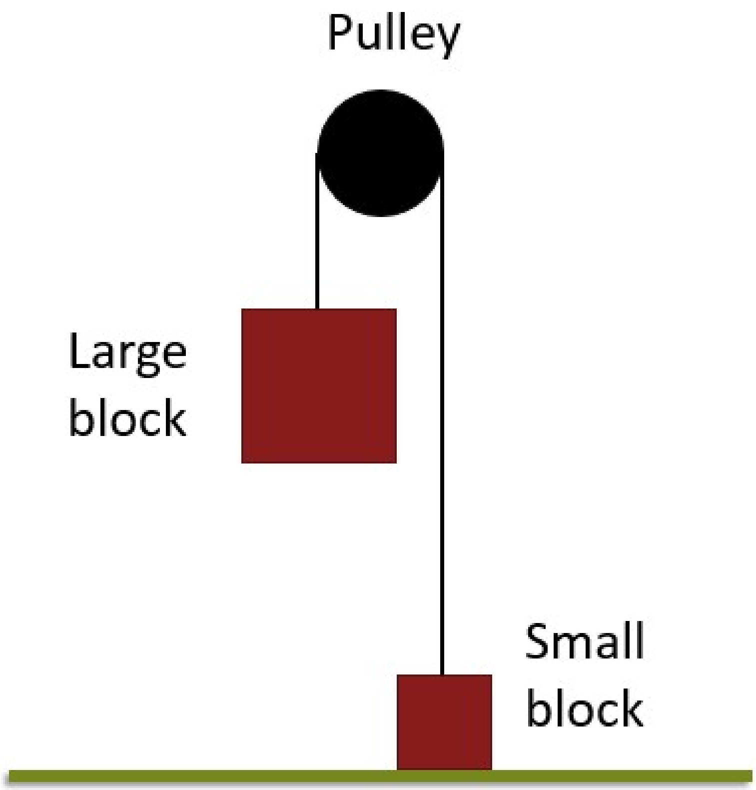


Figure 2.3 A large block connected to a small block over a pulley has an ability to do work due to its position relative to the earth; the large block has potential energy. We see that work when the large block is released exerting, via the rope and pulley, a force on the small block for a distance causing it to accelerate upwards.

All of the many different types of energy that you have heard about in previous courses, thermal, chemical, electrical, etc., all ultimately boil down to these two different types. You may be wondering how chemical and thermal energy can be potential or kinetic. Typically when we think of kinetic energy, we think of the motion of people, cars, and the like! The key is to think about the *scale* of the energy: are we talking about energy at the macroscopic scale (people etc.) or the microscopic scale (atoms and molecules)? As we shall see, thermal energy is just kinetic energy on the microscopic scale and chemical energy is potential energy on the microscopic scale. The relationships between these types of energy can be seen in Figure D. One of our goals throughout these chapters on energy is to develop a coherent picture of energy that applies at both the macroscopic scale of people etc. and at the microscopic scale of atoms and molecules. Thus, while we may present the macroscopic and microscopic scales in two separate chapters, keep in mind that we are talking about the same idea of energy throughout. At the end, we will look at how to transfer energy between these two different scales.

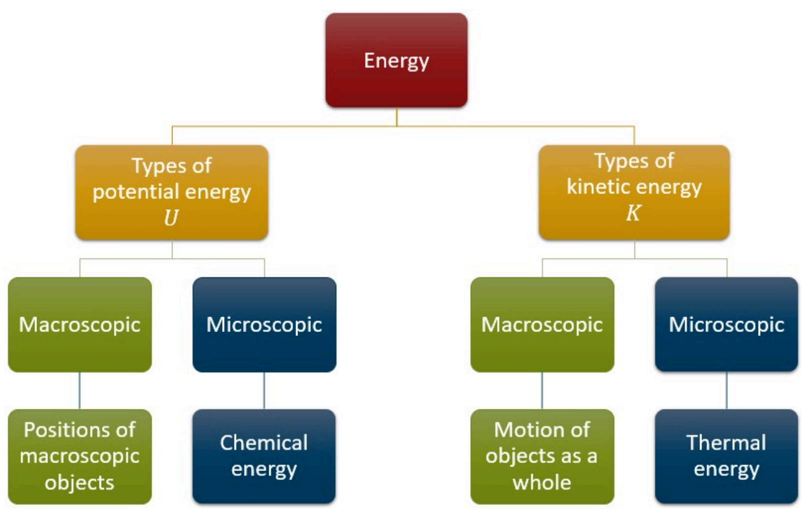


Figure 2.4 The relationships between different types of energy

UMASS AMHERST Instructor's Notes

This chart provides a way to organize the different types of energy. We will flush out this chart in upcoming sections.

2.4 Conservation of Energy

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Understanding that, in a closed system, energy is conserved

You probably have heard from other courses that “energy is conserved.” This statement is true, for the Universe as a whole - the total amount of energy has not changed since the birth of the Universe 13.6Gyr ago. While this is a hugely important fact which gets deep into the heart of physics, it may not seem very useful. However, there is an, equally fundamental, and more useful fact: the amount of energy in any given system is conserved: the energy at the end of some process is what I had at the beginning plus any that came in minus any that went out, or in equation form: $E_i+E_{in}-E_{out}=E_f$.

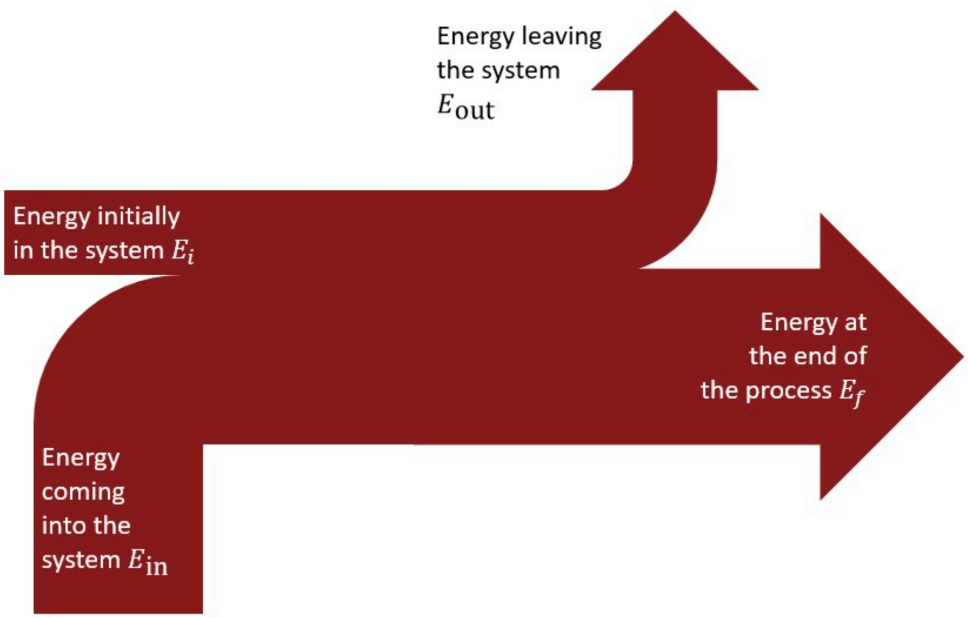


Figure 2.5 The conservation of energy in graphical form $E_i+E_{in}-E_{out}=E_f$

2.5 Ways to Transfer Energy

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Understanding the difference between heat and temperature
- Understanding the differences between the different types of heat transfer

So what ways are there to transfer energy into or out of a system? Well we already know of one way: work. If we do positive work on a system (the force we apply is in roughly the same direction as the displacement), then we will add energy *in*. Conversely, if we do negative work on a system (force essentially opposing the displacement) then energy is leaving the system. This should make some intuitive sense: we expect for positive work that the object will speed up - the object will gain kinetic energy. On the other hand, if we do negative work, we expect that the object will lose kinetic energy.

There is another way to transfer energy into or out of a system: heat which we represent by the letter Q. At its core, *heat is the transfer of energy by collisions at the microscopic scale*.

Heat

We already know that work is one way in which we can transfer energy, but you might be thinking to yourself, is there another way we can transfer energy into or out of a system? We know, for example, that if I place a warm soda in contact with the cold ice cube, the soda cools and some of the ice cube melts until the two come to a constant temperature. It certainly seems that energy is being transferred in this process; how is the energy being transferring in a process like this?

This method of energy transfer is known as heat. Heat is a transfer of energy; an object cannot have heat any more than an object can have work. It's also important to keep in mind the distinction between heat and temperature. The symbol for heat that we will use is the letter Q. Since it is a measure of transferred energy, heat will have the unit of joules, just like work and energy do. The sign convention for heat that we will adopt is that when energy flows into a system, Q is positive, and when energy flows out of a system, heat is negative.

There are three main ways to transfer heat into or out of a system. The first method is conduction, which is essentially when you place a metal stick in a fire, the other end of the stick gets hot, the second method is convection, which occurs when you have say a pot of water over a flame, and the third method is radiation, where light from the Sun warms the earth. Let's now explore these three processes in a little bit more detail.

Conduction

Conduction occurs when atoms in a material collide with each other. So. let's assume we have some sort of solid, and of course inside the solid the atoms are arranged in some regular pattern. When we place one end the solid in the flame, the atoms nearest the flame gain energy and begin to vibrate more vigorously, increasing their temperature. These atoms bump into the atoms next to them, which bump into the atoms next for them, and so on and so on, transferring energy up the rod. This transfer of energy through the molecular collisions of the atoms within the rod is known as heat. In a solid, the atoms are very close together and can easily bump into each other, and therefore solids transfer heat quite well. In liquids and gases, on the other hand, the atoms are a little further apart, and so don't collide as often. They can still transfer energy as heat through conduction, just not as well, because the collisions won't be as frequent. in a metal on the other hand, the C electrons that bind the metal together are free to move around and collide with each other. These C electrons, since they can travel so far, result in a lot of collisions, and result in metals conducting heat very well. Heat will quickly result in the transfer of energy from one end of a piece of metal to the other, which is why it's easy to burn your hand on a piece of metal that's hot on one end.

Radiation

So, in our first example, the fire caused the atoms nearest to it in the solid to begin to move. Well, how did this happen? Well the fire generates particles of light energy called photons that will be discussed in some detail in Physics 132. These photons, some of them travel and collide with the atoms in the solid, giving them their energy and causing them to move, and setting off the chain that we talked about in conduction. This transfer of energy through molecular collisions involving photons is called heat by radiation. This is the mechanism by which the earth gets its energy from the Sun. There is no matter in between the earth and the Sun to provide a mechanism for conduction. The only mechanism of heat transfer is through radiation of photons emitted by the Sun, which can travel through the vacuum of space.

Convection

The final way the transfer heat is called convection. Now, convection is a little bit more complicated than the other mechanisms, so we won't go into it in detail, but in short, convection is the transfer of energy through the bulk motion of a fluid, so a fluid in physics means either a liquid or a gas, and by the bulk motion, we mean giant currents of liquid or gas moving through the material. In our example of a pot of water on a stove, the liquid near the bottom of the pot gains energy through conduction. Flame gets the bottom of the pot warm through radiation, and the bottom of the pot is in contact with the liquid and transfers energy by conduction. Now the water at the bottom of the pot is warmer than the surrounding. Warmer fluids tend to have lower density, so this warm fluid near the bottom rises to the top carrying its energy with it, and cooler liquid settles to the bottom, and thus we get this current up in the middle and down the outside, bulk motion of the water transferring energy through the system. This is something that we see a lot and is of critical importance in the motion of the oceans. Cool water from the poles sinks to the bottom of the ocean, travels down to the equator, where it's warmed up by the Sun, and then rises back to the top, distributing energy and nutrients throughout the ocean.

Summary

So, now we've talked about the three methods of heat transfer, conduction, radiation, and convection. What's the commonality between these three different methods? These three different methods are, at the microscopic level, the transfer of energy through random collisions. In conduction, the atoms are colliding with each other, in radiation you have the collision of photons with atoms, and in convection you have the transfer of energy through the motion of the fluid itself. And this is what heat really is, it's the transfer of energy through collisions at the microscopic scale.

Heat, which we will use the letter Q to represent, is another way of transferring energy. Energy entering the system has positive heat in our convention, and energy leaving the system has negative heat in our convention. This energy transfer can occur in one of three ways, conduction, convection, or radiation, but these methods are, at the molecular level, collisions between particles.

UMASS AMHERST Instructor's Notes

(insert text here)

This paragraph summarizes things nicely and is important for your quiz!
You need to know what heat is and how it is different from temperature. You also need to know that if energy is going into a system, then we count it as positive heat or work. Energy leaving is negative heat or work.

2.6 The Formal Statement of the Conservation of Energy as the First Law of Thermodynamics

UMASS AMHERST Instructor's Notes

While this section is more formally mathematical than most, you should pay attention as this is probably the most important idea in all of science! For your quiz, you will need to be able to use this idea in the abstract. If I give you two of ΔE, Q, or W you should be able to tell me the third.

In section 13.4, we stated that energy must be conserved: E_i+E_{in}-E_{out}=E_f. Moving things around we get E_f-E_i= E_{in}- E_{out}. Recognizing the term on the left as E, we can say E=E_{in}-E_{out}. If we redefine E_{in} and E_{out} as just different directions of transferred energy $E_{transferred}$, then we have E=$E_{transferred}$ where $E_{transferred}$ is positive if energy comes into the system and negative if energy is leaving the system. Now, we know that there are two different ways to transfer energy into or out of a system: heat Q and work W. Thus, $E_{transferred}$ must be the sum of the energy transferred by heat and the energy transferred by work, $E_{transferred}$=Q+W. The statement of the law of conservation of energy can therefore be written as

$$\Delta E = Q + W$$

Written in this form, the law of conservation of energy is called the *First Law of Thermodynamics*, i.e. the First Law of Thermodynamics and the Law of Conservation of Energy are the same thing.

This statement is so fundamental to the idea of physics that it is worth spending a minute to really unpack what it says. Looking again at the First Law of Thermodynamics (with the delta expanded) we see

$$E_f - E_i = Q + W$$

where both heat Q and work W are ways of *transferring* energy into or out of the system. As a first example, say we have some system and we do work on that system without transferring any energy as heat. In this case, W>0 as energy is coming in and Q=0. The result is that E_f>E_i, which makes sense as we have added energy. Similarly, if we had a system that is losing heat to its environment while remaining stationary at constant volume then we know that Q<0 because heat is flowing out and W=0 due to the fact that there is no “distance” for W=Fdcosθ. Therefore, Q+W<0 and E_f<E_i as expected given that energy is flowing out of the system.

2.7 Why the First Law of Thermodynamics May Look Different in Your Other Courses

UMASS AMHERST Instructor's Notes

There are some differences in conventions when expressing the first law of thermodynamics to be aware of.

In some other courses or references, you may see the first law of thermodynamics written as $\Delta E = Q - W$, i.e. the sign of work may be different. This is still the same First Law of Thermodynamics/Law of Conservation of Energy that we are talking about here. The difference is one of perspective. In this class, we are considering energy flowing into the system as positive and energy flowing out of the system as negative. This convention matches our convention for heat Q as well as matching our definition of work from mechanics $W = Fdcos\theta$ which considers only external forces. Physically, we are thinking about work done *on* the system by *external* forces.

To understand the $\Delta E = Q - W$ formulation, you need a bit of history. The Laws of Thermodynamics were formulated during the Industrial Revolution as people were studying the properties of steam engines and the like. When studying the performance of a steam engine, the interesting quantity is not the work done *on the system by external* forces, but instead the work done *by the engine on its environment*. Stated another way, the developers of the Laws of Thermodynamics were not using our idea of object egoism! Instead of thinking about $\vec{F}_{environment \to engine}$, they were thinking about $\vec{F}_{engine \to environment}$. Now by Newton's Third Law, these two forces are equal except for a negative sign. Thus, when you think about work done *by* the engine instead of the work done *on* the system, work flips sign and you end up with $\Delta E = Q - W$ instead of $\Delta E = Q + W$.

In this class, we will stick with $\boldsymbol{\Delta E = Q + W}$, i.e. we will use the same definition for work we have been using. The takeaway from this section is that you may see the First Law of Thermodynamics written with a different sign for work. Different fields use different conventions (it would be nice if we could agree, but oh well). Therefore, you should be aware that writing it as $\Delta E = Q - W$ is just a different perspective born out of the historical development of science. This quirk with the sign of work is a great example of the impact that history and all of its associated socio-economic factors can have on the history of science. One wonders what other ideas could be expressed more coherently? What scientific questions have not been explored because the people in power doing the research did not value them?

2.8 Enthalpy

UMASS AMHERST Instructor's Notes

For your quiz, you will need to know the distinctions between enthalpy, energy, heat, and work

The following is a modification of content from 5.3 in OpenStax Chemistry

Chemists and biologists often use enthalpy (H) to describe the thermodynamics of chemical and physical processes instead of the energy (E) and you may have seen this quantity before. Both enthalpy and energy have the same units and the words are similar so students often confuse these two ideas. However, while enthalpy and energy are related, enthalpy is not the same thing as energy. Energy is the ability to do work. Enthalpy is defined as the sum of a system's microscopic or internal energy (E) and the mathematical product of its pressure (P) and volume (V):

$$H = E + PV$$

Enthalpy values for specific substances cannot be measured directly; only enthalpy *changes* for chemical or physical processes can be determined. For processes that take place at constant pressure (a common condition for many chemical and biological processes due to the fact that they are open to the air which is at a constant pressure of 1 atm), the enthalpy change (ΔH) is:

$$\Delta H = \Delta U + P\Delta V$$

Recall from the chapter on work that the mathematical product $P\Delta V$ represents work (W). If I compress the gas in Figure 1, we see that the work is positive (the applied force is in the same direction as displacement of the piston) and energy is flowing into the system. However, the volume of the gas is shrinking. This example illustrates the general concept that the arithmetic signs of ΔV and w will always be opposite:

$$P\Delta V = -W$$

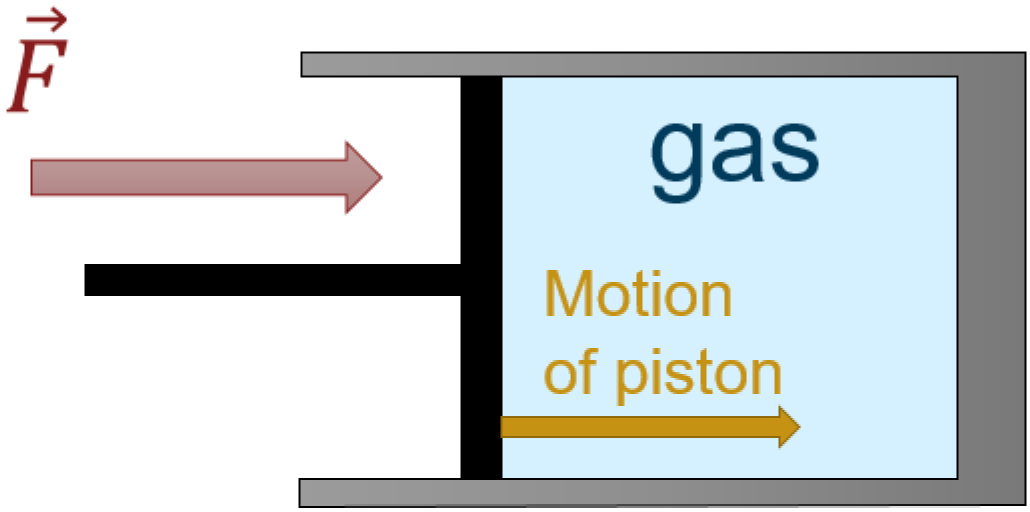


Figure 2.6 A gas being compressed by a piston. The force on the piston is in the same direction as the piston's motion implying positive work. However, the volume of the gas is shrinking meaning $\Delta V < 0$

Substituting this equation and the definition of internal energy into the enthalpy-change equation yields:

$$\Delta H = \Delta E + P\Delta V$$

$$\Delta H = (QP + W) - W$$

$$\Delta H = QP$$

where *QP* is the heat of reaction under conditions of constant pressure. **Thus, if a chemical, biological, or physical process is carried out at constant pressure with the only work done caused by expansion or contraction, then the heat flow (*QP*) and enthalpy change (Δ*H*) for the process are equal.**

This condition, while it may seem restrictive, covers a significant fraction of the situations you will encounter. For example, the heat given off when you operate a Bunsen burner is equal to the enthalpy change of the methane combustion reaction that takes place, since it occurs at the essentially constant pressure of the atmosphere. On the other hand, the heat produced by a reaction measured in a bomb calorimeter (**[link] (https://cnx.org/contents/85abf193-2bd2-4908-8563-90b8a7ac8df6@9.480:0d364b67-be96-44fc-bee5-a368a42c2c82@11#CNX_Chem_05_02_BombCalor)**) is not equal to ΔH because the closed, constant-volume metal container prevents expansion work from occurring. Chemists and biologists usually perform experiments under normal atmospheric conditions, which results in a constant external pressure making $Q = \Delta H$.

3 ENERGY OF OBJECTS AS A WHOLE (MACROSCOPIC SCALE)

3.1 Introduction

For starters, it is simpler to think about energy of objects as a whole at the macroscopic scale separate from the collective energy of the constituent molecules. This thinking is in line with the physics problem solving approach of starting simple and adding complications later. The next chapter deals with energy at the microscopic realm. We will get into connecting these two realms in class. We shall see that there are only specific ways of transferring energy between the macroscopic world and the microscopic world so separating these two regimes makes sense. As you know from the previous chapter, heat is the transfer of energy by microscopic collisions. *Thus, heat is really only important at the microscopic scale and will not be considered in this chapter.*

3.2 Kinetic Energy of an Object

UMASS AMHERST Instructor's Notes

Our goal in this section is to figure out an expression for the kinetic energy. While you should try to understand where the expression for kinetic energy comes from as it will help you understand how energy and work are related, the key is the expression we get to at the end.

Our goal in this section is to figure out an expression for the kinetic energy.

Figuring Out the Expression for Kinetic Energy

To achieve this objective, let's begin our study of energy with, as usual the simplest possible situation. Consider a one-dimensional situation where a force is used to accelerate an object in a direction parallel to its initial velocity. Such a situation occurs for the package on the roller belt conveyor system shown in Figure 1.

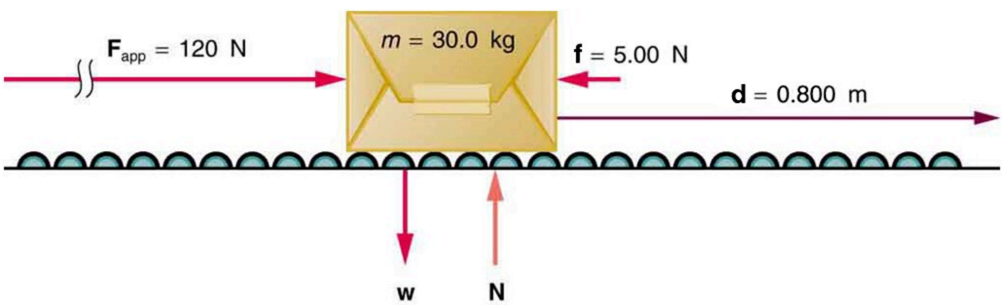


Figure 3.1 A package on a roller belt is pushed horizontally through a distance d

In this case, there is no transfer of energy by molecular collisions, i.e. there is no heat and Q=0. Meaning that our statement of conservation of energy goes from

$$E = Q + W$$

to

$$E = W$$

Similarly, there is no ability to do work due to position; the box cannot fall because of the rollers. Thus, there is no potential energy in this problem and all of our energy is kinetic energy: E=K. Therefore, our statement of conservation of energy for this situation is just

$$\Delta K = W$$

$$K_f - K_i = W$$

The effect of the net force Fnet is to accelerate the package from v_0 to v. The kinetic energy of the package increases, indicating that the net work done on the system is positive. (See **(https://cnx.org/contents/Ax2o07Ul@9.86:P_-6tVsN@5/Kinetic-Energy-and-the-Work-En#fs-id1703845) Example (https://cnx.org/contents/Ax2o07Ul@9.86:P_-6tVsN@5/Kinetic-Energy-and-the-Work-En#fs-id1703845)** .) By using Newton's second law, and doing some algebra, we can reach an expression for kinetic energy.

The force of gravity and the normal force acting on the package are perpendicular to the displacement and do no work. Moreover, they are also equal in magnitude and opposite in direction so they cancel in calculating the net force. The net force arises solely from the horizontal applied force Fapp and the horizontal friction force f. Thus, as expected, the net force is parallel to the displacement, so that θ=0º and cosθ=1, and the net work is given by

Wnet=Fnet$d$.

Substituting Fnet=ma from Newton's second law gives

$$Wnet = mad$$

To get a relationship between net work and the speed given to a system by the net force acting on it, we take

$$d = x - x_0$$

and use the equation studied in **(https://cnx.org/contents/031da8d3-b525-429c-80cf-6c8ed997733a@9.86:ea2bb23c-4fce-4e9d-a46b-3754125da988@9) Motion Equations for Constant Acceleration in One Dimension (https://cnx.org/contents/031da8d3-b525-429c-80cf-6c8ed997733a@9.86:ea2bb23c-4fce-4e9d-a46b-3754125da988@9)** for the change in speed over a distance d if the acceleration has the constant value a; namely, $v^2=v_0^2+2ad$ (note that a appears in the expression for the net work). Solving for acceleration gives a=(v^2−$v02$)/2d. When a is substituted into the preceding expression for W_{net}, we obtain

$$W_{net} = m(\frac{v^2 - v_0^2}{2d})d$$

The d cancels, and we rearrange this to obtain

$$W_{net} = \frac{1}{2}mv^2 - \frac{1}{2}mv_0^2$$

UMASS AMHERST Instructor's Notes

The following subsection is based on
umdberg / Kinetic energy and the work-energy theorem (2013). Available at: **http://umdberg.pbworks.com/w/page/68405433/Kinetic%20energy%20and%20the%20work-energy%20theorem%20(2013). (http://umdberg.pbworks.com/w/page/68405433/Kinetic%20energy%20and%20the%20work-energy%20theorem%20(2013))** (Accessed: 9th August 2017)

Interpreting the Result: Kinetic Energy

UMASS AMHERST Instructor's Notes

While, this section is short, it is arguably more important than the text above. This section is about interpreting our expression for kinetic energy and helping you understand how it is different from the, at first glance, similar looking momentum.

This section is based upon **http://umdberg.pbworks.com/w/page/68405433/Kinetic%20energy%20and%20the%20work-energy%20theorem%20(2013) (http://umdberg.pbworks.com/w/page/68405433/Kinetic%20energy%20and%20the%20work-energy%20theorem%20(2013))**

What has come out after all our manipulations is a that the work in this case is related to a change in a quantity associated with motion, 1/2mv^2. This is kind of like momentum in that it counts both the mass and the velocity, but it differs in that momentum is proportional to the velocity vector -- so it is very directional. Reversing momentum is a big deal even if the speed doesn't change. For our new quantity, since it is proportional to v^2 instead of to v, the direction of motion doesn't matter. You get the same v^2 whether v is positive or negative. If our general result turns out to only depend on the magnitude of v and not the direction (it will), we will have solved our problem and learned what it is that changes an object's speed (not caring about direction).

When you compare the result of our manipulations to our analysis in terms of energy, you can see that 1/2mv^2 must be the *kinetic energy*. It is a measure of "the energy associated with how much an object is moving".

3.3 Examples Applying Conservation of Energy with only Kinetic Energy

UMASS AMHERST Instructor's Notes

For your quiz, you are expected to be able to solve problems such as these.

Example 3.1 Calculating the Kinetic Energy of a Package

Suppose a 30.0-kg package on the roller belt conveyor system in ??? is moving at 0.500 m/s. What is its kinetic energy?

Strategy

Because the mass m and speed v are given, the kinetic energy can be calculated from its definition as given in the equation $\text{KE} = \frac{1}{2}mv^2$.

Solution

The kinetic energy is given by

$$\text{KE} = \frac{1}{2}mv^2. \tag{3.1}$$

Entering known values gives

$$\text{KE} = 0.5(30.0\text{ kg})(0.500\text{ m/s})^2, \tag{3.2}$$

which yields

$$\text{KE} = 3.75\text{ kg}\cdot\text{m}^2/\text{s}^2 = 3.75\text{ J}. \tag{3.3}$$

Discussion

Note that the unit of kinetic energy is the joule, the same as the unit of work, as mentioned when work was first defined. It is also interesting that, although this is a fairly massive package, its kinetic energy is not large at this relatively low speed. This fact is consistent with the observation that people can move packages like this without exhausting themselves.

Example 3.2 Kinetic Energy of a Car

A car travels at 5m/s when it accelerates to 10m/s. After the car has finished accelerating, by what factor did its kinetic energy increase?

Solution

We are interested in the ratio of the final to the initial kinetic energies

$$\frac{K_f}{K_i}$$

Substituting our definition for kinetic energy we get

$$\frac{K_f}{K_i} = \frac{\frac{1}{2}mv_f^2}{\frac{1}{2}mv_i^2}$$

Since the mass of the car does not change, m and the $\frac{1}{2}$ cancel leaving

$$\frac{K_f}{K_i} = \frac{v_f^2}{v_i^2}$$

Substituting our values in we get

$$\frac{K_f}{K_i} = \frac{10^2}{5^2} = \frac{100}{25} = 4$$

Analysis

The speed went up by two, but the kinetic energy went up by a factor of four, a result consistent with the fact that kinetic energy depends upon the square of the velocity. Speed matters a lot when thinking about energy!

Example 3.3 Deep Space 1

Deep Space 1 was a space probe, launched on October 24, 1998, and it used a type of engine called a ion propulsion drive. This engine generates a weak force, but it can do so over a long period of time and using only a small amount of fuel. The probe has a mass of 474 kg and is traveling with an initial speed of 275 m/s. The only force acting on the probe is from the ion drive, at a force of 0.056N parallel to the probe's displacement, which is 2.42 million km. What is the final speed of the probe?

Solution

In this problem, we're looking for a change in speed, which tells us that we're looking for a change in the kinetic energy. With the only force on the probe being the ion drive, we can use the work-energy theorem to find the change in energy:

$$W = \Delta E$$

Then, we can expand the energy: $W = E_f - E_i$

Since the only energy in this system is the kinetic energy, we can substitute the energies with our definition of kinetic energy:

$$W = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2$$

Since it's the final velocity that we're looking for, we can solve this equation for the final velocity:

$$v_f = \sqrt{\frac{W + \frac{1}{2}mv_i^2}{\frac{1}{2}m}} = \sqrt{\frac{2W}{m} + v_i^2}$$

The mass and the initial velocity are both given, but we need to find the work done on the probe. We can use our definition of work to calculate it:

$$W = F\Delta x = 0.056N * 2.42x10^9 m = 135520000J$$

Substituting our values and calculating the final velocity: $v_f = \sqrt{\frac{2(135520000J)}{474kg} + (275m/s)^2} = 804m/s$

3.4 Macroscopic Potential Energy

Work Done Against Gravity

Climbing stairs and lifting objects is work in both the scientific and everyday sense—it is work done against the gravitational force. When there is work, there is a transformation of energy. The work done against the gravitational force goes into an important form of stored energy that we will explore in this section.

Let us calculate the work done in lifting an object of mass m through a height h, such as in **Figure 3.2**. If the object is lifted straight up at constant speed, then the force needed to lift it is equal to its weight mg. The work done on the mass is then $W = Fd = mgh$. We define this to be the **gravitational potential energy** $(\mathrm{U_g})$ put into (or gained by) the object-Earth system. This energy is associated with the state of separation between two objects that attract each other by the gravitational force. For convenience, we refer to this as the $\mathrm{U_g}$ gained by the object, recognizing that this is energy stored in the gravitational field of Earth. Why do we use the word "system"? Potential energy is a property of a system rather than of a single object—due to its physical position. An object's gravitational potential is due to its position relative to the surroundings within the Earth-object system. The force applied to the object is an external force, from outside the system. When it does positive work it increases the gravitational potential energy of the system. Because gravitational potential energy depends on relative position, we need a reference level at which to set the potential energy equal to 0. We usually choose this point to be Earth's surface, but this point is arbitrary; what is important is the *difference* in gravitational potential energy, because this difference is what relates to the work done. The difference in gravitational potential energy of an object (in the Earth-object system) between two rungs of a ladder will be the same for the first two rungs as for the last two rungs.

Converting Between Potential Energy and Kinetic Energy

Gravitational potential energy may be converted to other forms of energy, such as kinetic energy. If we release the mass, gravitational force will do an amount of work equal to mgh on it, thereby increasing its kinetic energy by that same amount (by the work-energy theorem). We will find it more useful to consider just the conversion of $\mathrm{U_g}$ to KE without explicitly considering the intermediate step of work. (See ???.) This shortcut makes it is easier to solve problems using energy (if possible) rather than explicitly using forces.

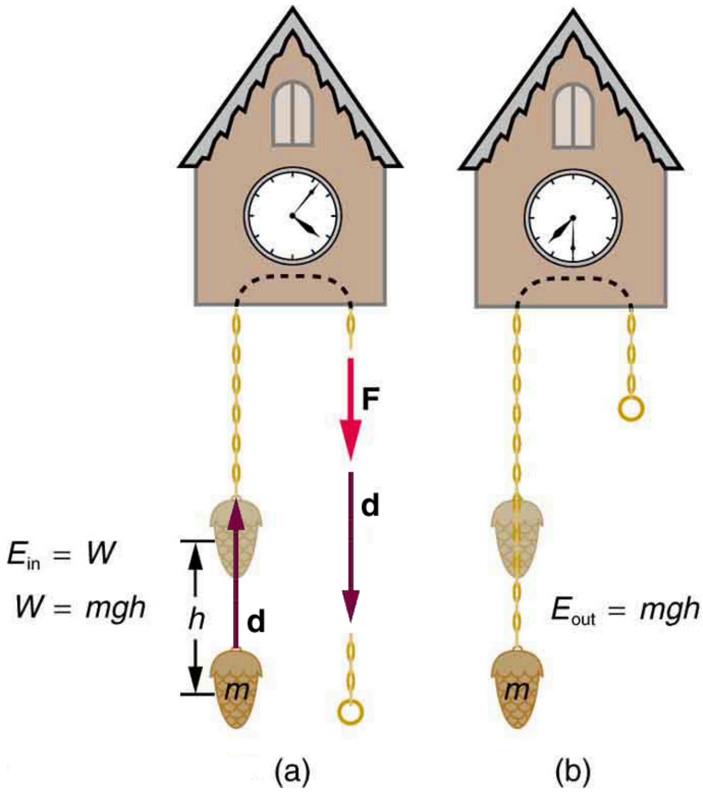


Figure 3.2 (a) The work done to lift the weight is stored in the mass-Earth system as gravitational potential energy. (b) As the weight moves downward, this gravitational potential energy is transferred to the cuckoo clock.

More precisely, we define the *change* in gravitational potential energy $\Delta\mathrm{U_g}$ to be

$$\Delta\mathrm{U_g} = mgh, \tag{3.4}$$

where, for simplicity, we denote the change in height by h rather than the usual Δh. Note that h is positive when the final height is greater than the initial height, and vice versa. For example, if a 0.500-kg mass hung from a cuckoo clock is raised 1.00 m, then its change in gravitational potential energy is

$$\begin{aligned} mgh &= (0.500\ \text{kg})\left(9.80\ \text{m/s}^2\right)(1.00\ \text{m}) \\ &= 4.90\ \text{kg}\cdot\text{m}^2/\text{s}^2 = 4.90\ \text{J}. \end{aligned} \tag{3.5}$$

Note that the units of gravitational potential energy turn out to be joules, the same as for work and other forms of energy. As the clock runs, the mass is lowered. We can think of the mass as gradually giving up its 4.90 J of gravitational potential energy, *without directly considering the force of gravity that does the work*.

Using Potential Energy to Simplify Calculations

The equation $\Delta\mathrm{U_g} = mgh$ applies for any path that has a change in height of h, not just when the mass is lifted straight up. (See **Figure 3.3**.) It is much easier to calculate mgh (a simple multiplication) than it is to calculate the work done along a complicated path. The idea of gravitational potential energy has the double advantage that it is very broadly applicable and it makes calculations easier. From now on, we will consider that any change in vertical position h of a mass m is accompanied by a change in gravitational potential energy mgh, and we will avoid the equivalent but more difficult task of calculating work done by or against the gravitational force.

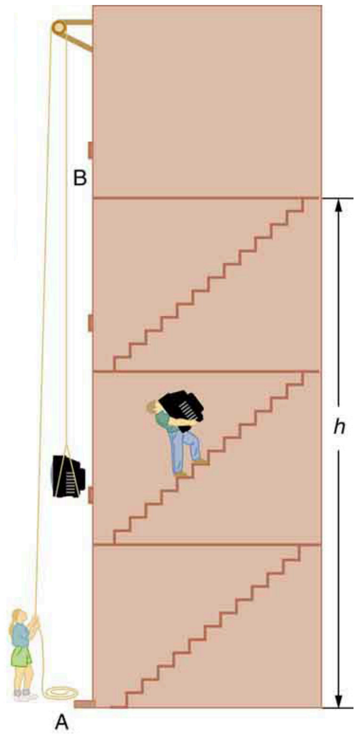


Figure 3.3 The change in gravitational potential energy (ΔU_g) between points A and B is independent of the path. $\Delta U_g = mgh$ for any path between the two points. Gravity is one of a small class of forces where the work done by or against the force depends only on the starting and ending points, not on the path between them.

Example 3.4 The Force to Stop Falling

A 60.0-kg person jumps onto the floor from a height of 3.00 m. If he lands stiffly (with his knee joints compressing by 0.500 cm), calculate the force on the knee joints.

Strategy

This person's energy is brought to zero in this situation by the work done on him by the floor as he stops. The initial U_g is transformed into KE as he falls. The work done by the floor reduces this kinetic energy to zero.

Solution

The work done on the person by the floor as he stops is given by

$$W = Fd\cos\theta = -Fd, \tag{3.6}$$

with a minus sign because the displacement while stopping and the force from floor are in opposite directions $(\cos\theta = \cos 180^\circ = -1)$. The floor removes energy from the system, so it does negative work.

The kinetic energy the person has upon reaching the floor is the amount of potential energy lost by falling through height h:

$$KE = -\Delta U_g = -mgh, \tag{3.7}$$

The distance d that the person's knees bend is much smaller than the height h of the fall, so the additional change in gravitational potential energy during the knee bend is ignored.

The work W done by the floor on the person stops the person and brings the person's kinetic energy to zero:

$$W = -KE = mgh. \tag{3.8}$$

Combining this equation with the expression for W gives

$$-Fd = mgh. \tag{3.9}$$

Recalling that h is negative because the person fell *down*, the force on the knee joints is given by

$$F = -\frac{mgh}{d} = -\frac{(60.0\text{ kg})(9.80\text{ m/s}^2)(-3.00\text{ m})}{5.00\times10^{-3}\text{ m}} = 3.53\times10^5\text{ N}. \tag{3.10}$$

Discussion

Such a large force (500 times more than the person's weight) over the short impact time is enough to break bones. A much better way to cushion the shock is by bending the legs or rolling on the ground, increasing the time over which the force acts. A bending motion of 0.5 m this way yields a force 100 times smaller than in the example. A kangaroo's hopping shows this method in action. The kangaroo is the only large animal to use hopping for locomotion, but the shock in hopping is cushioned by the bending of its hind legs in each jump.(See **Figure 3.4**.)

Figure 3.4 The work done by the ground upon the kangaroo reduces its kinetic energy to zero as it lands. However, by applying the force of the ground on the hind legs over a longer distance, the impact on the bones is reduced. (credit: Chris Samuel, Flickr)

We have seen that work done by or against the gravitational force depends only on the starting and ending points, and not on the path between, allowing us to define the simplifying concept of gravitational potential energy. We can do the same thing for a few other forces, and we will see that this leads to a formal definition of the law of conservation of energy.

The Zero of Potential Energy

This is perhaps a slightly subtler topic than you might first imagine. Let's think about this quantity of gravitational potential energy written as mgh. Here we have Mr. clumsy dropping a ball.

Figure 3.5 Mr. Clumsy dropping the ball.

Where is the gravitational potential energy of the ball equal to zero? Well, m and g are both numbers, m is the mass of the ball and g is 9.8 m/s^2, so essentially this is the same question as where is height equal to 0, h. Well, the logical place you might think for h to be equal to 0 would be the ground. We define the ground to be h=0, then the gravitational potential energy on the ground, U_g, is going to be zero when the ball is on the ground. Now let's move Mr. Clumsy to a platform on the top of a skyscraper.

Figure 3.6 Mr. Clumsy dropping the ball on top of a skyscraper, with a subway tunnel below.

Now where is h=0? This is a perhaps a little bit trickier question; do we define h=0 to be at the platform, or do we define it to be in the ground, or do we even define it to be in the subway tunnel underneath the skyscraper? Which of these should we choose for h=0, or which of these should we choose for the zero of gravitational potential energy? Well, the universe doesn't care where we choose h to be equal to 0. so due to that, any of these choices, the platform, the ground, the subway tunnel, they're all fine, doesn't matter which we pick. You should just be very explicit with your choice, so when you're approaching a problem with gravitational potential energy, explicitly write down that I am going to choose the zero of gravitational potential energy to be the platform, for example. If I choose the ground, say, to have zero potential gravitational potential energy, then points in the subway tunnel below the ground have negative gravitational potential energy. There's absolutely nothing wrong with that.

Why is there nothing wrong with negative potential energy? Well, the work done by gravity is equal to the negative of the change in gravitational potential energy, or

$$W = -\Delta U$$

ΔU is always U_f-U_i, so the work done is

$$W = U_i - U_f$$

or using mgh for potential energy

$$W = mgh_i - mgh_f$$

The key point in all of this is that the work done does not depend upon the value of potential energy itself, only the change in the potential energy is a relevant quantity, and the change in potential.

Potential Energy of a Spring

First, let us obtain an expression for the potential energy stored in a spring ($\mathrm{U_s}$). We calculate the work done to stretch or compress a spring that obeys Hooke's law. (Hooke's law was examined in **Elasticity: Stress and Strain (https://legacy.cnx.org/content/m42081/latest/)** , and states that the magnitude of force F on the spring and the resulting deformation ΔL are proportional, $F = k\Delta L$.) (See **Figure 3.7**.) For our spring, we will replace ΔL (the amount of deformation produced by a force F) by the distance x that the spring is stretched or compressed along its length. So the force needed to stretch the spring has magnitude $F = kx$, where k is the spring's force constant. The force increases linearly from 0 at the start to kx in the fully stretched position. The average force is $kx/2$. Thus the work done in stretching or compressing the spring is $W_\mathrm{s} = Fd = \left(\frac{kx}{2}\right)x = \frac{1}{2}kx^2$. Alternatively, we noted in **Kinetic Energy and the Work-Energy Theorem (https://legacy.cnx.org/content/m42147/latest/)** that the area under a graph of F vs. x is the work done by the force. In **Figure 3.7**(c) we see that this area is also $\frac{1}{2}kx^2$. We therefore define the **potential energy of a spring**, $\mathrm{U_s}$, to be

$$\mathrm{U_s} = \frac{1}{2}kx^2, \tag{3.11}$$

where k is the spring's force constant and x is the displacement from its undeformed position. The potential energy represents the work done *on* the spring and the energy stored in it as a result of stretching or compressing it a distance x . The potential energy of the spring $\mathrm{U_s}$ does not depend on the path taken; it depends only on the stretch or squeeze x in the final configuration.

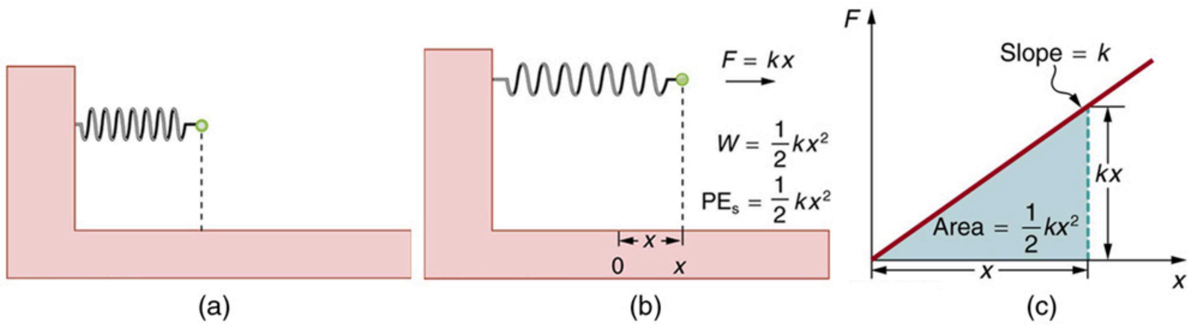


Figure 3.7 (a) An undeformed spring has no $\mathrm{U_s}$ stored in it. (b) The force needed to stretch (or compress) the spring a distance x has a magnitude $F = kx$, and the work done to stretch (or compress) it is $\frac{1}{2}kx^2$. Because the force is conservative, this work is stored as potential energy $(\mathrm{U_s})$ in the spring, and it can be fully recovered. (c) A graph of F vs. x has a slope of k , and the area under the graph is $\frac{1}{2}kx^2$. Thus the work done or potential energy stored is $\frac{1}{2}kx^2$.

The equation $\mathrm{U_s} = \frac{1}{2}kx^2$ has general validity beyond the special case for which it was derived. Potential energy can be stored in any elastic medium by deforming it. Indeed, the general definition of **potential energy** is energy due to position, shape, or configuration. For shape or position deformations, stored energy is $\mathrm{U_s} = \frac{1}{2}kx^2$, where k is the force constant of the particular system and x is its deformation. Another example is seen in **Figure 3.8** for a guitar string.

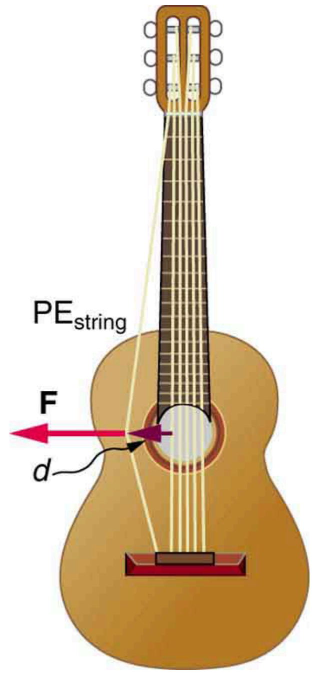


Figure 3.8 Work is done to deform the guitar string, giving it potential energy. When released, the potential energy is converted to kinetic energy and back to potential as the string oscillates back and forth. A very small fraction is dissipated as sound energy, slowly removing energy from the string.

3.5 Conservative vs Non-Conservative Forces

Let's begin by considering a ball near the surface of the earth. The ball begins three meters above the ground, falls, and bounces back up to its original height. How much work is done by gravity in this case? Well, work is always defined as the force of interest times the displacement times the cosine of the angle between the force and the displacement. In this case the force we're interested in is gravity, mg, so the work done is $mgdcos(\theta)$. On the way down, the force is parallel to the displacement. The ball is moving down and the force is down, so the angle between the force and the displacement is zero degrees. The cosine of zero degrees is 1, so the work done by gravity as the ball travels down is positive mgd on the way up. However, the force is down and the displacement is up, so the force and the displacement are antiparallel, or the angle between them is 180 degrees. Now the cosine of 180 degrees is -1, and so the work done by gravity as the ball travels back up is $-mgd$. The total work done by gravity on the entire loop is the sum of the work done by gravity on the way down plus the work done by gravity on the way back up, which in this case is equal to zero.

The ball in this example is moving on what is called a closed path. The ball starts and stops in the same place. We've just seen that for this closed path, the work done by the gravitational force is equal to zero. It turns out that the work done by gravity on any closed path is equal to zero. This leads to a question: is this statement true for all forces? Is the work done by any force on any closed path always equal to zero?

To answer this question, let's think about a different force: friction. In this example, we have a 2kg box as it's dragged three meters across a table, with a coefficient of kinetic friction of 0.2, and back, and we're interested in the work done by the force of kinetic friction over this closed path. Once again, the work as always is the force times the displacement times the cosine of the angle in between. In this case, the force of interest is the force of kinetic friction, which we know to write as the coefficient of kinetic friction, μk , times the normal force. The box is not moving in the vertical direction, which I've called y in this example, so we know that the acceleration in the y direction is equal to zero. Thus, by Newton's second law, we know that the force in the y direction is equal to zero, the net force is equal to zero.

Thus, we can conclude that the normal force in this problem, is equal to the weight of the box, mg . Thus, we have the force of kinetic friction, $\mu k * mg$. As the box is dragged to the left, the force of friction is opposite the displacement, thus the angle between them is 180 degrees. The box is moving to the left, but the force is opposing the motion, the angle between them is 180 degrees. The cosine of 180 degrees is, once again, -1. And so, the work done by the force of friction as it moves to the left is $-\mu k(mgd)$. On the way to the right, the force is still opposite the displacement. Now the block is going to the right, but the friction is still opposing the motion, so the angle between the force and the displacement is still 180 degrees, which means that the work done by friction as the block goes to the right is also $-\mu k(mgd)$. Thus, we see in this case that the work done by the force of friction around this closed path, the box starts and stops in the same place, so this is a closed path, the work done on this closed path is $-2\mu k(mgd)$, which is not equal to 0.

Thus, it seems that we have two different types of forces. We have forces for which the work done over a closed path is always equal to 0, an example of this is gravity, we call such forces conservative. We have another type of force for which the work done over a closed path is not equal to 0. The frictional force that we just saw is an example of this. Forces for which the work done over a closed path is not equal to 0 are called non-conservative forces. The definition of a conservative force is a force for which the work done by the force over a closed path is equal to 0. So, this is the statement you can use to determine if a force is conservative or non-conservative. Why do we care about this distinction between conservative and non-conservative forces? Because only conservative forces are associated with the potential energy.

To explore this idea, let's consider a block sliding down a frictionless ramp and a block just falling to the ground. Both blocks travel the same vertical distance h. In both cases, the work done by gravity is equal to the change in gravitational potential energy. The work done by gravity is written mathematically as a $-\Delta U$, or $-\Delta mgh$. The reason we can write the work done by gravity in terms of a potential energy is because gravity is a conservative force. Friction is not a conservative force; thus, we cannot write the work done by friction as the change in some type of frictional potential energy. So, let's summarize. We've seen that there are two classes of forces. Forces over which the work done over a closed path is zero, such as gravity, such forces are known as conservative, and for these types of forces we can describe the work done by the force as a change in the potential energy. $W = -\Delta U$. We also have forces for which the work done over a closed path is not equal to zero, such as friction. For these non- conservative forces, we cannot describe the work as in terms of a change in potential energy. There is no such thing as a potential energy for friction.

3.6 Organizing the Different Types of Macroscopic Energy (Mechanical Energy)

Now we have all of the different types of macroscopic energy that we will talk about in this course: kinetic energy $K = \frac{1}{2}mv^2$, gravitational potential energy $U_g = mgy$, and the potential energy of a spring $U_s = \frac{1}{2}kx^2$. These different types of energy can be organized as in the chart in the **figure**. Collectively these types of energy are called *mechanical energy.*

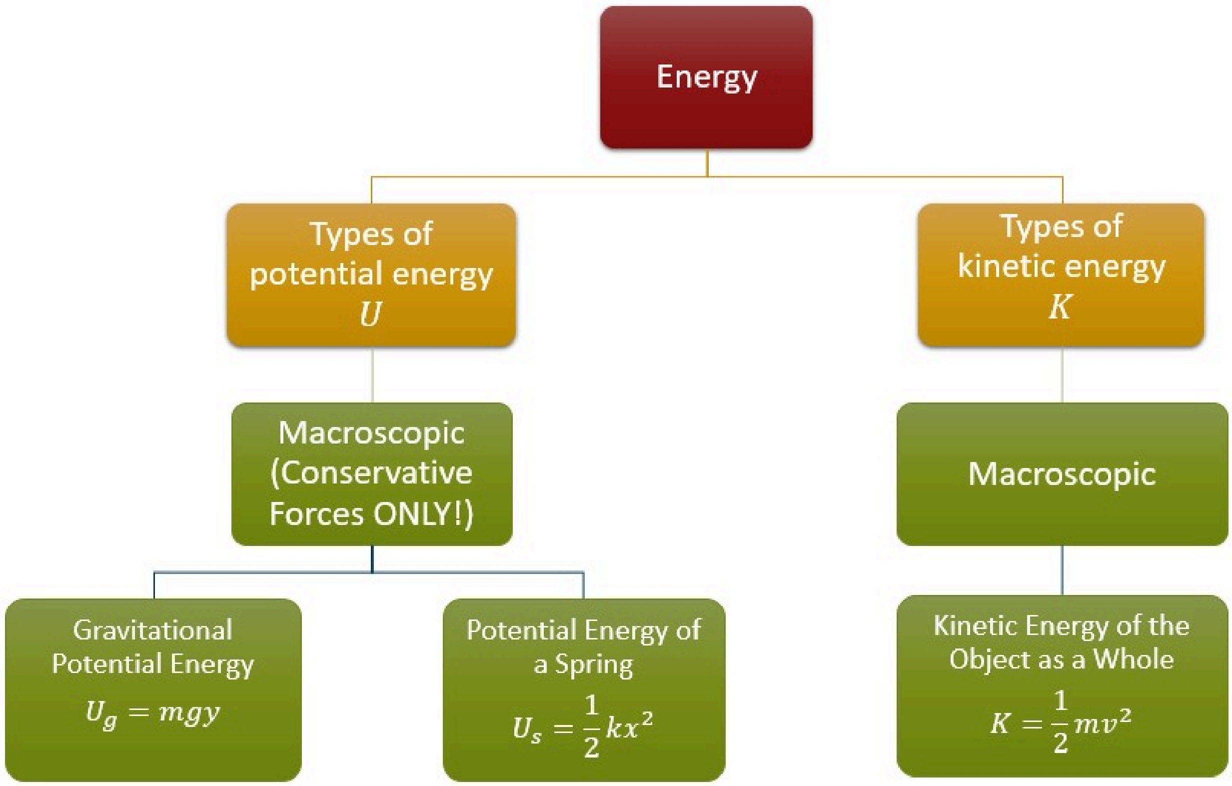


Figure 3.9 The different types of macroscopic (also called mechanical) energy

Glossary

gravitational potential energy: the energy an object has due to its position in a gravitational field

4 ENERGY OF CONSTITUENT ATOMS (MICROSCOPIC SCALE)

4.1 Introduction

In this chapter, we will be exploring the idea of energy at the microscopic scale. Instead of talking about the energy of the object as a whole we will be talking about the kinetic and potential energies contained within the molecules themselves. This energy is generally MUCH larger than the mechanical energy at the macroscopic scale and is of fundamental importance to our modern world and the subjects of biology and chemistry which are greatly concerned with the conversion of microscopic chemical and thermal energy into useful work. This chapter will ONLY deal with the microscopic world; just as the last chapter dealt solely with the macroscopic realm. We will look at how to connect these two different distance scales in class.

In this chapter, we are still dealing with the First Law of Thermodynamics $\Delta E = Q + W$ where the total energy E is still the sum of the potential energies U and kinetic energies K: $E = U + K$. The only difference is that now the work as well as the types of potential and kinetic energies will be microscopic. Since we are looking at the microscopic scale, heat Q will play more of a role (recall that heat is the transfer of energy by *microscopic* collisions!) than it did at the macroscopic scale of the last chapter. The form of potential energy that we will be mostly concerned with at this scale is the potential in molecular bonds: so-called chemical potential energy U_{chem}. In terms of kinetic energy, we shall see in this chapter that kinetic energy at the microscopic scale is related to the temperature of the object: K_{therm} is related to T. The relationships between these different forms of energy are shown in Figure 1. The total energy E at the microscopic scale is sometimes called the *internal energy* of the system.

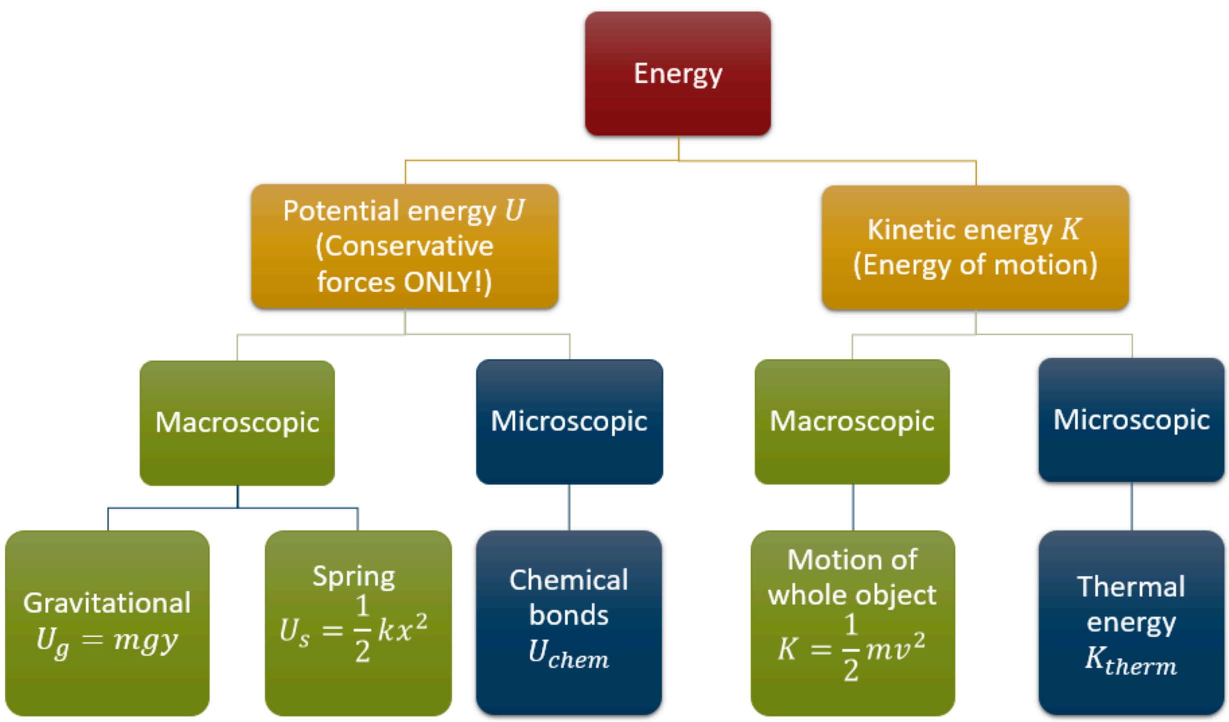


Figure 4.1 The different types of energy classified as microscopic and macroscopic

4.2 The Potential Energy of Molecules

As stated in the introduction, the primary source of microscopic potential energy with which we shall concern ourselves is the potential energy stored in chemical bonds or chemical potential energy U_{chem}. This potential energy is a result of the force of electrical attraction between different atoms (recall electricity and magnetism was one of our fundamental forces). As you shall see in the next course, the electrical force is a conservative force and thus we can associate a potential energy with it.

The strength of chemical bonds is typically quoted in one of two ways: either the energy in the bond, called the *bond dissociation energy*, is quoted directly (typically in eV) or the enthalpy per mole will be quoted. For example, the Cl-Cl bond has a bond dissociation energy of 2.51 eV/bond or a bond dissociation enthalpy per mole of $\Delta H = 242 kJ / mol$. How do we interpret these numbers in terms of potential energy? We use the same freedom to choose the zero of potential energy that we discussed in section 14.3 when we discussed potential energy at the macroscopic scale.

Thinking about gravity, we tend to put the zero of potential energy at ground level; objects above the ground then have positive potential energy while objects underground have negative potential energy. This use of negative potential energy makes sense, an object at ground level will fall to below ground level if allowed to do so and lose energy in the process.

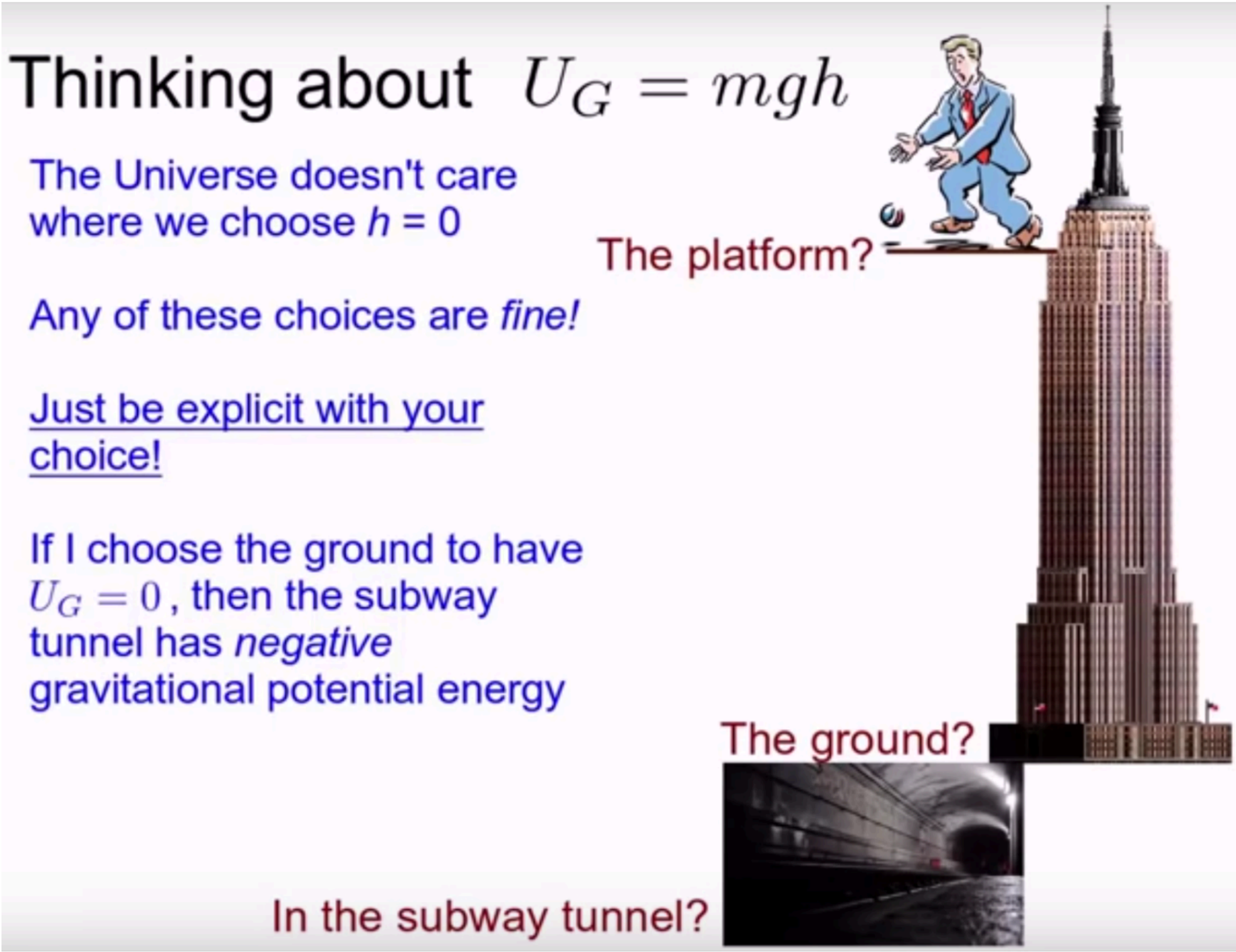


Figure 4.2 A clip from the video on zero of potential energy showing that negative potential energies are possible!

For atoms and molecules, we have a similar freedom to choose where to put zero potential energy. The standard convention is to say that free atoms that are far apart have *zero* potential energy. Atoms in most bonds have lower potential energy than free atoms (that is why the bonds form!). Therefore the potential energy of the atoms is less than zero: the potential energy of atoms in bonds is *negative*. This may seem like a weird choice for the zero of potential energy, but it is the convention and it makes sense when you think about it!

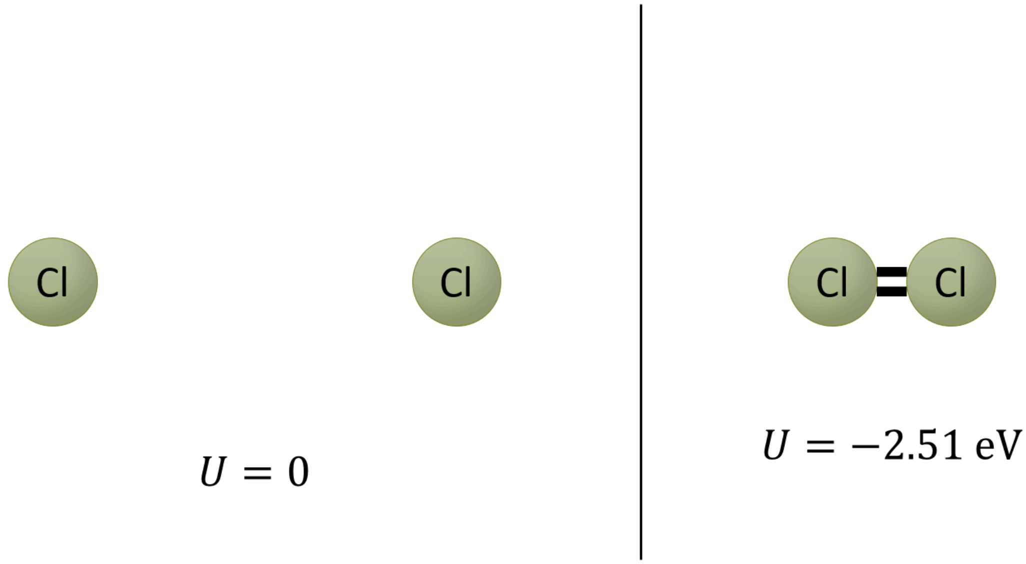


Figure 4.3 Two Cl atoms separated by a great distance have zero potential energy while two bonded Cl atoms have a potential energy of -2.51 eV. Remember, potential energy is the due to the relative position of two objects, so it does not make sense to ask which atom in the bonded pair has the potential energy. The potential energy is due to the two of them!

Let's return to the quoted Cl-Cl bond with dissociation energy of 2.51 eV/bond. What does this value mean? It means that two Cl atoms bonded together have a potential energy of 2.51 eV *less* than if they were free. Said another way, the potential energy of Cl atoms in Cl_2 is -2.51 eV, while the potential energy of free Cl atoms is 0 eV. This is consistent with what you probably already know about Chlorine: Cl_2 is the lower energy state than free Cl atoms. I would get 2.51 eV of energy for every Cl-Cl bond that is formed, as the atoms move from zero potential energy to -2.51 eV. Similarly, I would need to add 2.51 eV of energy to break a Cl-Cl bond and move the two atoms *up* to zero potential energy.

4.3 Application of Bond Energies

Let's see how we can use these two different ways of quoting chemical energy and convert the result into something we can use.

Example 4.1 Bond energy expressed in eV

Back in chapter 13, we saw that the bond dissociation energy of the H-H bond was 4.52 eV/bond. How much energy is released by converting 6.000g of monotomic H into 6.000g of H_2?

Solution

Also at **https://youtu.be/jBQGwjokX14 (https://youtu.be/jBQGwjokX14)**

We have 6.000g of monotomic hydrogen. The molar mass of hydrogen is 1.008g/mol. Therefore we have

$$(6.000gH)\cdot\left(\frac{1molH}{1.008gH}\right)\cdot\left(\frac{6.022\times10^{23}Hatoms}{1molH}\right) = 3.584\times10^{24}Hatoms$$

Each molecule of H_2 requires two H atoms so we can make

$$\frac{3.584\times10^{24}}{2} = 1.79\times10^{24}H_2molecules$$

We know, from the table that each bond releases 4.52 eV, so

$$(1.79\times10^{24}H-Hbonds)\cdot\left(\frac{4.52eV}{H-Hbond}\right) = 8.10\times10^{24}eV$$

which we can convert to Joules

$$(8.10\times10^{24}eV)\cdot\left(\frac{1.602\times10^{-19}J}{1eV}\right) = 1.30\times10^{6}J = 1.30MJ$$

Example 4.2 Bond energies expressed as enthalpies

The enthalpy of dissociation of $CO_2 \rightarrow CO + O$ is $\Delta H = +532 kJ/mol$. How much *energy* is required to break up 20g of CO_2 into CO and O assuming the reaction occurs at constant pressure?

Solution

Also at **https://youtu.be/SE_SN3plzbY (https://youtu.be/SE_SN3plzbY)**

We need to convert from enthalpy *H* into energy. The definition of enthalpy is (from section 13.8)

$$H = E + PV$$

At constant pressure, the change in enthalpy is

$$\Delta H = \Delta E + P\Delta V$$

Using the first Law of Thermodynamics $\Delta E = Q + W$

$$\Delta H = (Q + W) + P\Delta V$$

Recalling from section 13.8 the argument that, under constant pressure, the work W and $P\Delta V$ are of equal magnitude, but opposite sign, we have

$$\Delta H = Q$$

(This is the exact same reasoning we did in 13.8 to show that, under constant pressure $\Delta H = Q$, we are just repeating the steps to reinforce the logic).

Now, we know that the *enthalpy* change $\Delta H = +532 kJ/mol$ is the same as the *heat* $Q = +532 kJ/mol$. Since the heat is positive we must add heat to make the process go; one mol of CO_2 going to CO and O will need 532 kJ of energy.

UMASS AMHERST Instructor's Notes

There are three different quantities in that last sentence: enthalpy, energy, and heat. Furthermore, all three have units of Joules! Make sure you understand the distinctions between these three concepts!

Now, all we need to do is determine the amount required to break up the 20g of CO_2 in the problem. From the periodic table we see that the molar mass of CO_2 is:

$$m_C = 12.01 g/mol$$

$$m_O = 16.00 g/mol$$

$$m_{CO_2} = (12.01 g/mol) + 2(16.00 g/mol)$$

$$m_{CO_2} = 44.01 g/mol$$

Now to get the total energy that we must add as heat:

$$(20 g CO_2)\cdot\left(\frac{1 mol CO_2}{44.01 g CO_2}\right)\cdot\left(\frac{532 kJ heat required}{mol CO_2}\right) = 241 kJ$$

We must add 241 kJ to break up 20g CO_2 into CO and O.

4.4 Kinetic energy at the Microscopic Scale

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Using the relationship between kinetic energy, temperature, and degrees of freedom.

Now, we have talked about potential energy at the microscopic scale. What about kinetic energy? Back in chapter 12, you were refreshed on the idea of temperature. What is temperature *really*? It turns out that the ideas of temperature and the kinetic energy at the microscopic scale are deeply related. Along the way, we will introduce the idea of *degrees of freedom* or places to put energy. This concept will also be important in our next unit on entropy.

Microscopic Temperature

So what is temperature? Temperature is, in essence, a macroscopic measurement of the average kinetic energy of molecules - a microscopic quantity. Molecules are always moving and vibrating around. The more kinetic energy the molecules have, the higher their temperature as shown in **FIGURE.**

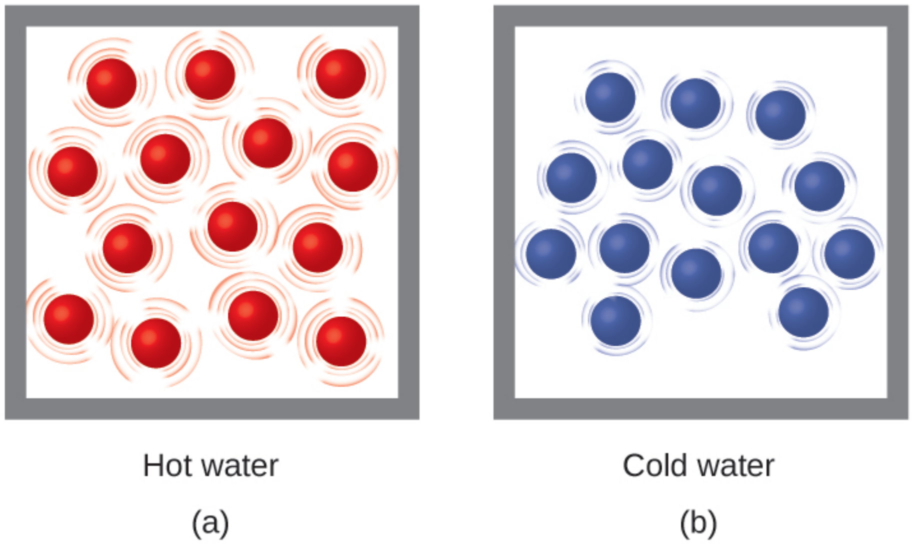


Figure 4.4 (a) The molecules in a sample of hot water move more rapidly than (b) those in a sample of cold water

However, there is a bit of a complication. When I add energy to a molecule I can put it in a bunch of different places or *degrees of freedom*. Temperature is then formally related to the average energy per degree of freedom. In equation form

$$k_B T = \frac{2\langle E\rangle}{n_{d.o.f.}}$$

or more commonly written

$$\langle K\rangle = \frac{n_{d.o.f.}}{2} k_B T\ .$$

In this expression $\langle K\rangle$ is the average energy, $n_{d.o.f.}$ is the number of degrees of freedom (places to put energy), and T is the temperature *in Kelvin*. The factor of 2 comes out of a formal calculus-based derivation, but as far as we are concerned, it is just there for convenience purposes. The symbol k_B is the Boltzmann constant introduced in 12.4 $k_B = 1.38\times10^{-23} J/K$ and appears anytime you try to connect the microscopic world to the macroscopic world. In this case, we are connecting microscopic kinetic energy, measured in Joules, to the macroscopic quantity known as temperature; thus the units J/K. We will see this constant again playing a similar role connecting the microscopic and macroscopic worlds in our discussion of entropy in the next unit.

15.2.2 Degrees of Freedom

What do we mean by a "place to put energy?" Consider a simple ideal gas molecule. Now add some energy to it. Where can that energy go? Well we know from section 3.1 that the *x*, *y*, and *z* directions are independent of each other. Thus, the energy I add could go into either kinetic energy associated with the *x*-direction, kinetic energy associated with the *y*-direction, or kinetic energy associated with the *z*-direction as shown in the **FIGURE**. Thus, an ideal gas molecule has *three degrees of freedom*. Remember, kinetic energy itself does not have direction. We are just talking about different places *on each atom* that we could put that energy - a different concept. We will talk about counting degrees of freedom for other gases in class.

Also at: **https://youtu.be/QXjZpUDIC6A (https://youtu.be/QXjZpUDIC6A)**

For an ideal gas, there were in fact three degrees of freedom, as I could either put energy into motion in the X direction, motion in the Y direction, or motion in the Z direction. So, three places I can put energy, three degrees of freedom. What about in a solid? In a solid, the atoms are not free to move around. Where can I put the energy in this case? I can clearly add energy to a solid, I can put it over a flame, what are the different microscopic places, different degrees of freedom that are possible in a solid?

We are going to consider the Einsteinian solid, where we treat each atom as connected to its neighbors in three dimensions by springs. In an Einsteinian solid, the atoms can't move anywhere, but they can vibrate on these springs that we're using to represent the atomic bonds. You might think, there are three bonds so there are three places I can put energy, I could put it in any of the three bonds, and this is a very reasonable assumption. but not quite right. In fact, there are six degrees of freedom in a solid. There are two for each bond, not just one. When I add energy to a vibrating bond in a solid, there are two places I can put it. I can make the vibrations larger, increase their amplitude, or I can make it vibrate more quickly. Thus, there are two degrees of freedom per bond in a three-dimensional solid, and therefore, in a 3D solid, there are two degrees of freedom per bond, size of vibration and speed of vibration, times three bonds, gives us a total of six degrees of freedom for a three-dimensional Einsteinian solid.

In summary, on degrees of freedom, what you're looking at are the number of places within the individual atoms you can put energy. For an ideal gas, there are three places, I can put energy either into the motion in the X Direction, motion in the Y direction, or I could put energy into motion in the Z direction, and these three directions, as we've been discussing since the beginning of class are independent. For an atom in a solid, however, there are actually two degrees of freedom per bond, one for the size of the vibration and one for the speed of the vibrations. For a standard solid in three dimensions where each atom has three independent bonds, one in X, one in Y, and one in Z, there are 3 times 2, or 6, degrees of freedom.

Example using temperature, kinetic energy, and degrees of freedom

(a) What is the average kinetic energy of an ideal gas molecule at 20.0ºC (room temperature)? (b) Find the average speed of a helium atom (He) at this temperature.

Solution for (a)

We know that the average kinetic energy of a molecule is related to the temperature by the expression

$$\langle K \rangle = \frac{n_{dof}}{2} k_B T$$

For an ideal gas molecule, the number of degrees of freedom is 3 so we have

$$\langle K \rangle = \frac{3}{2} k_B T$$

Before we can use this expression, we need to convert the temperature to Kelvin

$$20^{\circ} C = 293.15K$$

Now substituting it all together we have

$$\langle K \rangle = \frac{3}{2}(1.38 \times 10^{-23} J / K)(293.15K)$$

$$\langle K \rangle = 6.07 \times 10^{-21} J = 0.038eV = 38meV$$

Solution for (b)

Now we know the average kinetic energy, we can think about the average speed. We begin with the definition of kinetic energy

$$\langle K \rangle = \left\langle \frac{1}{2} m v^2 \right\rangle .$$

Both ½ and m are constants and can come out of the average

$$\langle K \rangle = \frac{1}{2} m \left\langle v^2 \right\rangle$$

$$\left\langle v^2 \right\rangle = \frac{2\langle K \rangle}{m}$$

$$\langle v \rangle = \sqrt{\frac{2\langle K \rangle}{m}}$$

The mass of helium is, from our periodic table, 4.002 amu which is $6.626 \times 10^{-27} kg$. Putting this value and our result from (a) in our expression we get

$$v > \sqrt{\frac{2(6.07 \times 10^{-21} J)}{(6.626 \times 10^{-27} kg)}} = 1350 m / s$$

Pretty quick!

*In this calculation, we actually calculated not the average speed but the root-mean-square speed. This calculation is pretty much exactly what it says: we calculated the average of all the velocities squared (square each, add them up, divide by the number) and then took the square root. The average velocity of the He atoms is zero (velocity is a vector so moving left cancels moving right)! We will ignore this distinction in this class but it may be important in a future class!

4.5 The Total Kinetic Energy in and Object Associated with its Temperature

We now know that temperature is the average kinetic energy in an atom per degree of freedom (with a factor of 2 and a k_B

thrown in). How much energy in total is in an object just due to the kinetic energy of all of its molecules wiggling around due to their temperature? If the average energy per atom is

$$\langle K\rangle = \frac{n_{dof}}{2}k_B T,$$

then to get the total kinetic energy due to temperature K_{therm}, we take the average and multiply by the number of atoms N

$$K_{therm} = N\langle K\rangle.$$

Making a substitution we have

$$K_{therm} = N(\frac{n_{dof}}{2}k_B T).$$

Dealing with numbers of molecules can be unwieldy, so let's convert to moles n. You will recall from section 12.4 that

$$Nk_B = nR$$

where R is the gas constant 8.314 J/(mol·K), meaning we can rewrite our expression as

$$K_{therm} = \frac{n_{dof}}{2}nRT$$

Example 4.3 Comparing total thermal energy to gravitational potential energy

Consider a block of lead with about the same mass as a person, 70kg, sitting on the second floor of a building, 3m above the ground, at room temperature of 20^{o}C. What is the ratio of gravitational potential energy to the total thermal energy due to the block's temperature?

Solution

The gravitational potential energy of the block is given by

$$U_g = mgh$$

$$U_g = (70kg)(9.8N/kg)(3m)$$

$$U_g = 2000J = 2kJ$$

The thermal energy is given by

$$K_{therm} = \frac{n_{dof}}{2}nRT$$

Since lead is a 3-D solid, we know that the number of degrees of freedom is 6. To get the number of moles, we look up the molar mass of lead and find it to be 207 g/mol which means that 70kg of lead is

$$(70kgPb)\cdot\left(\frac{1000g}{kg}\right)\cdot\left(\frac{1molPb}{207.2gPb}\right) = 337.8molPb$$

Putting it all together (and converting 20^{o}C to Kelvin) we get a total thermal energy of

$$K_{therm} = \frac{6}{2}(337.8mol)\cdot\left(8.314\frac{J}{mol\cdot K}\right)(293.15K)$$

$$K_{therm} = 2.47\times10^{6}J = 2.47MJ$$

Giving us a ratio of

$$\frac{U_g}{K_{therm}} = 8\times10^{-4}$$

i.e. the thermal energy is over 1000x bigger than the gravitational potential energy!

Analysis

We see that the internal thermal energy in this case is MUCH bigger than the gravitational potential energy, a common result. This disparity is why, when considering the energies at the microscopic scale, the macroscopic energies can usually be ignored. However, there are situations where energy moves from one scale to another such as a car engine which converts microscopic potential energy in the gasoline into microscopic kinetic energy of hot gas in the engine, into motion of the car. Clearly in these situations, you need to think about both the microscopic and macroscopic scales.

Example 4.4 Argon Gas

Say we have a container with 0.25mol of argon gas which we can treat as being an ideal gas. How much energy do we need to add as heat to raise the temperature 10^{o}C assuming no work is done?

Solution

We begin with the First Law of Thermodynamics (Conservation of Energy)

$$\Delta E = Q + W$$

which, given that no work is done goes to

$$\Delta E = Q$$

$$E_f - E_i = Q \, .$$

All of the energy in this case is thermal kinetic energy. Replacing E with K_{therm}, therefore, we have

$$K_{therm\,f} - K_{therm\,i} = Q \, .$$

Now, we put in the definition of total thermal kinetic energy to get

$$\frac{n_{dof}}{2} RnT_f - \frac{n_{dof}}{2} RnT_i = Q$$

which after factorization becomes (note this equation for the next section!)

$$\frac{n_{dof}}{2} Rn(T_f - T_i) = Q \, .$$

Argon is an ideal gas, so $n_{dof} = 3$. We know the number of moles $n = 0.25 mol$ and R is the gas constant $R = 8.314 \frac{J}{mol{\cdot}K}$. Moreover, we know the temperature difference $T_f - T_i = 10^{\circ} C$. We, as always, need to convert to Kelvins. However, the size of $1^{\circ} C$ is the same as $1K$, only the zero points are different. Therefore $T_f - T_i = 10K$.

Substituting these values in, we get

$$\frac{3}{2}\left(8.314 \frac{J}{mol{\cdot}K}\right)(0.25 mol)(10K) = +31J$$

The result is positive, so we need to add 31J of heat energy to the system to get the temperature to increase 10^{o}C (which makes sense intuitively).

4.6 Connections to Chemistry

Instructor's Notes

This section is a bit of a derivation, which we try to avoid. However, in this case it is worth it. Spend some time trying to follow along; get out a piece of paper and work it for yourself. The result will be a “woah” moment that explains numbers you had to look up in a table in chemistry!

If you have taken chemistry before, then the context of the previous problem may seem familiar to you. Problems about how much energy you need to raise or lower the temperature of a substance are often described in chemistry books as being calorimetry problems, such as **here (https://legacy.cnx.org/content/m51034/latest/)** , to be solved using the expression

$$mc\Delta T = Q \, .$$

In this expression, m is the mass of substance, ΔT is the change in temperature, and c is a quantity called *specific heat* with units $\frac{J}{kg{\cdot}K}$. If you have done these types of problems, then you have probably either solve for the specific heat c or looked it up in a table like the one below for the substance in which you were interested.

Table 4.1 *A table of specific heats measured while holding the volume constant taken from* **http://www.engineeringtoolbox.com/specific-heat-capacity-gases-d_159.html (http://www.engineeringtoolbox.com/specific-heat-capacity-gases-d_159.html)**

Substance	State	Specific Heat at constant volume (J/gK)
Helium	He(g)	3.12
Water	H_2O(l)	4.19
Ethanol	C_2H_6O(l)	2.3
Nitrogen	N_2(g)	0.743
Aluminum	Al(s)	0.87
Argon	Ar(g)	0.315
Iron	Fe(s)	0.46
Copper	Cu(s)	0.39

It turns out that, with what you know now, you can actually *predict* these specific heats to a high degree of accuracy based upon the properties of the molecule!

Let's compare the chemistry calorimetry formula

$$mc\Delta T = Q$$

with the equation you were asked to note in the last example

$$\frac{n_{dof}}{2}Rn(T_f - T_i) = Q$$

which after some rearranging becomes

$$n(\frac{n_{dof}}{2}R)\Delta T = Q\,.$$

In both expressions we have a quantity telling us how much material we have (mass or number of moles), multiplied by some number, and then multiplied again by the change in temperature to get the amount of heat.

To further see the comparison, let's multiply our result by a clever form of the number 1, M/M where M is the molar mass of the substance in g/mol (I know this seems dumb, but bear with me for just a few lines):

$$\left(\frac{M}{M}\right)n\left(\frac{n_{dof}}{2}R\right)\Delta T = Q\left(\frac{M}{M}\right)$$

This is one of a physicist's favorite tricks and is totally legit: I did the same thing to both sides. One the right-hand side of our equation, where the heat Q is, nothing happens, I multiplied by 1 after all. On the left-hand side, however, I can rearrange things in an interesting way:

$$(Mn)\left(\frac{n_{dof}}{2M}R\right)\Delta T = Q\,.$$

All I did is take the top molar mass M and put it next to the number of moles n, and took the bottom molar mass M and put it with the 2. Now, nM is the mass of the substance m by definition; M has units g/mol and n is the number of moles leaving us with grams. Making this substitution we get

$$m\left(\frac{n_{dof}}{2M}R\right)\Delta T = Q$$

which when we compare with the calorimetry formula from chemistry

$$mc\Delta T = Q$$

we see that the only way this can work is if

$$c = \frac{n_{dof}}{2M}R$$

Example: Let's test this result for two materials

We shall test this result for two different materials (a) argon and (b) copper and compare with the table above

Solution for argon

Argon is an ideal gas so $n_{dof} = 3$. The molar mass for argon is 39.948 g/mol. Substituting into our expression we get

$$c = \frac{3}{2(39.948 g/mol)}\left(8.314\frac{J}{mol{\cdot}K}\right) = 0.312\frac{J}{g{\cdot}K}$$

which matches the table value exactly!

Solution for copper

Copper is a solid with $n_{dof} = 6$ and a molar mass of 63.546 g/mol. Thus, we expect the specific heat of copper to be

$$c = \frac{6}{2(63.546 g/mol)}\left(8.314\frac{J}{mol{\cdot}K}\right) = 0.392\frac{J}{g{\cdot}K}$$

Again, a perfect match!

Takeaway

This is our first example where microscopic properties of atoms can be used to get macroscopically measured phenomena: we went from the microscopic structure of atoms to predicting the macroscopic specific heat. We will do more of connecting these two worlds in class throughout this unit. The next unit on entropy continues this pattern.

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Unit V
Entropy

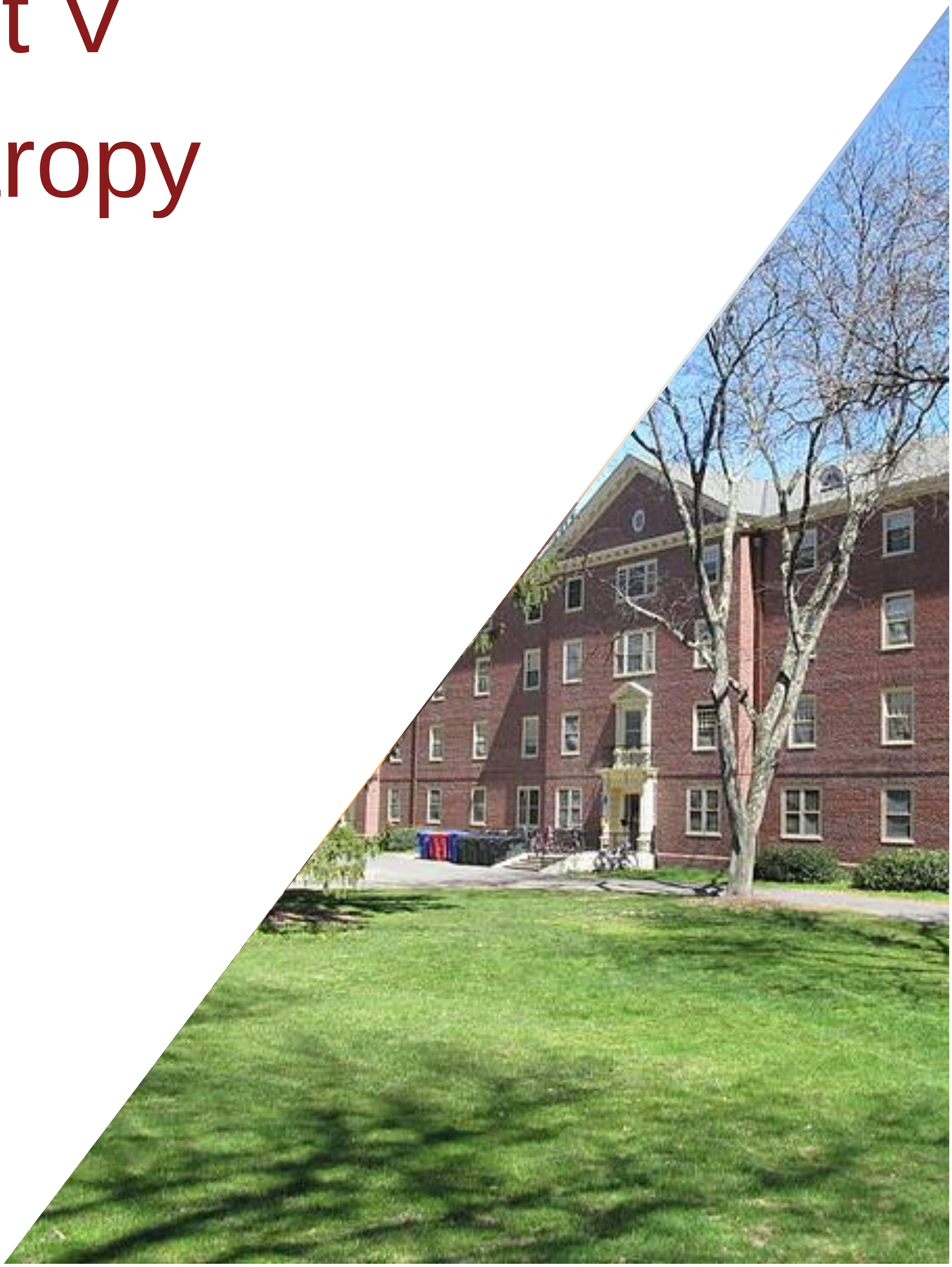


Photo credit: Phelan, John. *English: Baker Hall, University of Massachusetts Amherst, Amherst Massachusetts*, April 7, 2012. Own work. https://commons.wikimedia.org/wiki/File:Baker_Hall,_UMass,_Amherst_MA.jpg.

Table of Contents

UNIT 5 OVERVIEW

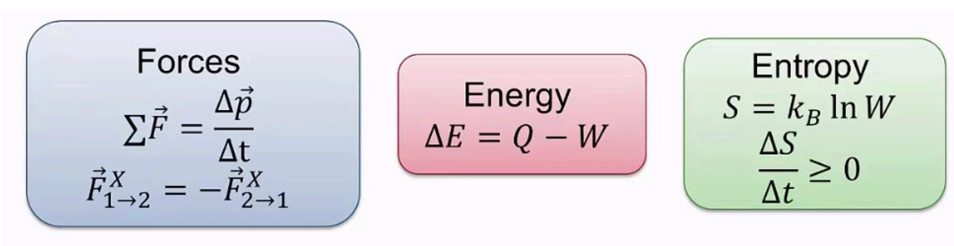
Instructor's Notes

This unit overview is also available as a video **here (https://www.youtube.com/watch?v=5R00SVtK4po)** .

This is our last unit of the course. Throughout this course, we have been progressing from more concrete concepts to more abstract. If you think back, we began with the concept of velocity, which people are familiar with from speedometers on their cars. Then, we move to the idea of force which people can physically feel, you can feel a push or pull, then we moved into “forces and...”, torque, impulse, work. These ideas are a little bit more abstract but rooted heavily in the idea of force. Energy is a bit more abstract of an idea, most people have some experience with energy from previous science courses, but trying to define what energy really is and getting used to thinking about it on a huge variety of distance scales can be a bit of a challenge. Now we are going to cover the one last idea of this course, which is yet still a little bit more abstract, this idea of entropy.

So, why are we covering entropy? Well, I have two answers. First, a practical answer. Many of you have seen entropy in a previous class, typically chemistry. For example, here is a slide from Chem 112 at UMass, and many of you will see this idea again. Entropy changes can help determine if reactions proceed spontaneously or not, and appears in the all-important Gibbs free energy. Typically, up to this point, you’ve either done qualitative arguments about entropy increases or decreases, or looked up standard entropies of formation in tables, but what is this quantity that you’re using, in, say, Chem 112? Well, the typical answer in many courses that introduced the idea of entropy is disorder, but what is disorder? How do you quantify it? And disordered by whose perspective? Disorder, that’s a very nebulous idea. Who gets to decide what’s an ordered state and what’s a disordered state? And it turns out that this definition isn’t even correct, so I think that if you’re going to deal with the topic, as much as many of you will deal with the idea of entropy, you should know what it is.

There’s another, second more physicist answer as to why I think we should cover entropy. In a sense the whole discipline of physics, not just this course, but the whole discipline of physics, from this course to the very frontiers of modern research can really be boiled down to a few key ideas.



Forces and Newton’s laws, here written as Δp/Δt and Newton’s third law, energy and its conservation, and the last one is entropy. So, in this green box, we have the definition of entropy here and what’s known as the second law of thermodynamics. No matter how much physics you study, you’re still looking at how different objects respond to forces, how energy is conserved, what is the entropy in the system and how is it changing. Since forces, energy, and entropy are three of the fundamental pillars of physics, I feel it would be remiss to leave entropy out.

So, what is entropy then? If it’s not disorder what is it? Well, let’s think about what is going on at the microscopic level. At the microscopic level, things are of course always changing. Molecules are moving around, chemical reactions are always proceeding, but many of these changes do not affect the microscopic picture. For example, from chemistry, when you add two reactants, the reaction never really stops, we just reach an equilibrium point where the number of reactions going in one direction equals the number of reactions going in the other direction. The molecules are constantly interacting with each other, forming bonds, dissociating bonds. At the microscopic picture, we have a hubbub of activity but at our macroscopic scale we don’t see a lot of change. So, what do I mean when we say we don’t see a lot of change at the macroscopic level? We mean the total energy in the system, the pressure, if it’s a gas, the volume, all these types of quantities that are easily measured in macroscopic level. Entropy is the number of ways that I can rearrange things on the microscopic level, which we call the number of microstates, which we will indicate by a letter W (no, this is not the work W, this is a different W, conventions are conventions). So, how many ways can I rearrange things, how many different microstates, are there that don’t change the macroscopic world: that is what entropy is. So, it turns out that counting the number of ways energy can be distributed microscopically while leaving the macroscopic world unchanged has important implications, which is weird when you stop and think about it. I mean, the number of possible ways I can arrange things seems like a very theoretical construct, and to make matters more interesting, the numbers we’ll be dealing with will be ginormous. 10 to the 10 to the 23 is not a surprising number to deal with when you start talking about the number of ways to arrange energy amongst all the molecules in a room. These types of numbers start to appear. That is a one with a mole of zeros after it. That’s a big number. These huge and seemingly theoretical numbers are the basis of what entropy is.

So, what do I want you to get out of this unit? I want you to have a beginning of a grasp of what entropy is and how we can quantify it. I want you to understand why some processes proceed spontaneously due to entropy considerations. And finally, I want you to understand how entropy can drive processes in a way that results in final states that might seem more ordered to us, but are in actuality an increase in the number of microstates when you consider the whole system. The following prep videos and reading and homework problems will lay the groundwork of some of the basic mathematics you will need to study this topic. This concludes this video.

1 PROBABILITY

1.1 Why Probability Matters

The topic of probability often receives little emphasis in the math classes that are part of the program for biologists -- until they are required to take a serious course in statistical methods. At that point, the focus may be on the rules and formal tools for generating a statistical result rather than on making sense of what probability and statistics are telling us. This is extremely unfortunate, since the concept of probability is fundamental in a variety of situations that are of extreme importance to many biological professionals. In this essay we discuss briefly why we are interested in probability in the context of a physics class, why researchers (of any ilk) need to know probability, and why medical professionals need to understand probability.

The basic idea of probability

The basic idea of probability is about situations that occur multiple times but have factors that we cannot control. When you flip a coin, the laws of Newtonian physics could tell you where it is going to go – and whether it will land heads or tails. That is, if you knew the initial position, upward speed, initial angular orientation, and rotational velocity to a very high accuracy. AND if you knew that the coin were a fair coin (perfectly symmetric and balanced), AND that there was no breeze, etc., etc.

Even in the context of systems that are well-described by Newtonian physics, there are many systems that we cannot predict well. Their motion is just too sensitive to factors that we cannot control.

In such a situation, what do we do? We might just give up and say: “that’s not predictable,” but another approach has developed, driven by mathematicians responding to questions from gamblers in the 17th century. (Really.) In this approach we carry out the following two steps:

Determine what results are equally probable;

Count the number of ways that a result we are looking for can be made up of the different equally probable results.

For example, consider throwing two cubical dice, each with 6 sides and the sides having 1, 2, 3, 4, 5, and 6 spots respectively. When the dice are thrown so that they bounce around in uncontrollable ways, one result comes up on each. The total will range from 2 (a one comes up on each) to 12 (a six comes up on each). But each total is not equally probable – each face of each die is assumed to be equally probable. As a result there is only one way to create the result of “2” – each die has to show one spot. But there are six ways to create a total of “7” – 1+6, 2+5, 3+4, 4+3, 5+2, and 6+1, with the first number showing the result on the first die, the second the result on the second. This means, that *if we throw the dice many times we expect to get the result 7 six times as often as the result 2.* Understanding this ratio is crucial is you are going to not lose too much money playing dice!

Note a few key ideas:

- The result given by a probabilistic law does NOT tell you what will happen in any given experiment (trial); it will only what will tell you if you REPEAT the experiment many times. And then it will only tell you what fraction of the time you can expect different results.
- The states that are the result of our experiment do not specify every variable. There are “hidden” uncontrolled variables that we do not specify
- The model we have of the system is crucial – what are the hidden variable states (microstates) that are equally probable, and how many different ways can a result state (macrostate) be made up from different hidden variable states.

So the very nature of the “law” we are creating is different from many of the laws we are accustomed to learning in science classes – at least in the intro classes. They only tell the result of many equivalent experiments – an *ensemble* – not of an individual one.

1.2 Probability

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Defining probability in terms of an infinite number of trials
- Calculating the mean and standard deviation for results with different amounts of probability
- Defining microstate
- Defining macrostate

umdberg / Probability (2013). Available at: **http://umdberg.pbworks.com/w/page/68375351/Probability%20%282013%29 (http://umdberg.pbworks.com/w/page/68375351/Probability%20%282013%29)** . (Accessed: 1st August 2017)

Introducing probability

Probability is one of those words for which we all have an intuition, but which is surprisingly hard to define. For example, in our discussion of **diffusion (http://umdberg.pbworks.com/w/page/47846016/Diffusion%20and%20random%20walks)** , we make the assumption that particles move either to the left or right with equal probability. But try to define what this means. Try, more concretely, to define what it means for a coin to have an "equal probability" of coming up heads or tails—but without using words in your definition that are synonymous to "probability" (such as "chance" and "likely"). It's really hard to do! In fact, entire branches of philosophy have been devoted to the question of how to define what is meant by "equal probability". So if you find yourself thinking hard about the probabilities we encounter, you are in good company!

The key idea in probability is **lack of control**. When we flip a coin, it's extremely hard to control which way the coin will come down. The result is very sensitive to the starting conditions you give it at a level of sensitivity greater than you can control. Which, of course, is the point of flipping a coin.

One definition of "equal probability" might look something like this:

As the number of tosses of a fair coin approaches infinity, the number of times that the coin will land heads and the number of times that the coin will land tails approach the same value.

Is that a useful definition? Maybe, but it doesn't seem to capture everything that we intuitively know to be true. We'd like to know what the chances are that the coin will land on "heads" when we toss it just once, without having to toss it an infinite number of times. And we all have the feeling that the answer is obvious - it's ½! - even if we have a hard time expressing it rigorously.

How would we know if it's fair?

Of course determining whether a coin is "fair" or not would require testing it an infinite number of times. And in the real world we expect that no real coin would be perfectly fair. It might be a tiny bit unbalanced so that it consistently, over many many flips, comes out 0.1% more heads than tails. Would we accept that as a "fair" coin?

One of the interesting questions of probability is "how do you know" that a coin is fair, for example? Or better: how well do you know that a coin "appears to be fair"? This subject carries us beyond the scope of this class into the realm of **Bayesian Statistics (http://en.wikipedia.org/wiki/Bayesian_inference)** . We won't discuss that here, though we will note that Bayesian analyses play a large role in the modern approach to medical diagnosis and both medical students and biological researchers will eventually have to master this subject!

A simple model for thinking about probabilities: a fair coin

Rather, we will make a simplified model that we can analyze in detail mathematically. We will assume that we have a *(mathematically) fair coin* -- one that if it we flipped it an infinite number of times would come up an equal number of times heads and tales.

Now we can get back to our story. Let's see if we can make some interesting observations about probabilities by relying on just our intuitions. Suppose, for example, that I toss a (mathematically) fair coin ten times. How many times will it come up "heads"? The correct answer is: who knows! In ten flips, the coin may land on heads seven times, and it may land on heads only twice. We can't predict for sure. But what we *do* know is that if it is a fair coin it is more *likely* that it will land on heads 5 times than it is that it will land on heads all 10 times.

But why do we feel that is the case? Why is the result of 5 heads and 5 tails more likely than the result of 10 heads and 0 tails? If each toss is equally likely to give heads as it is to give tails, why is the 5/5 result more likely than the 10/0 result?

The answer is that there are many more ways for us to arrive at the 5/5 result than there are ways for us to arrive at the 10/0 result. In fact, there is precisely ONE way to arrive at the 10/0 result. Note that in stating "5/5" we are assuming that we don't care in which order the heads and tails appear -- we only care about the total number.

If we only care about the totals: microstates and macrostates

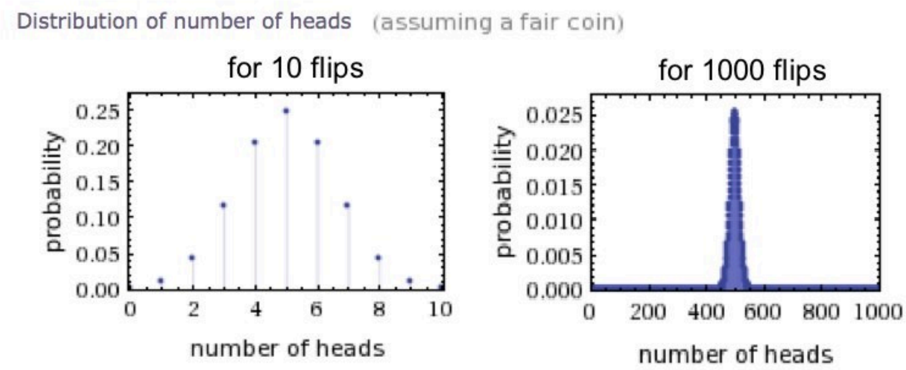
If we only care about the totals there is only ONE way in which you would arrive at the result that the series of tosses produced 10 heads: HHHHHHHHHH. You have a 50% chance the first flip will be a head, a 50% change the second will be a head, and so on. Therefore the probability of 10 heads is $1/2^{10}$ or 1 in 1024.

On the other hand, here are just a few of the 252 ways of arriving at the 5/5 result: HHHHHTTTTT, HTHTHTHTHT, TTTTTHHHHH, HTTHHTTHTH. Each of these particular strings also only has the probability of 1 in 1024 to come up, since there is a 50-50 chance of a head or a tail on each flip. But since there are 252 ways of arriving at 5/5 the chance of finding 5/5 (in any order) is 252/1024 -- much great than finding 10/0 and in fact greater than finding any other specific mix of heads and tails.

Another way of expressing the probabilistic intuition we have been describing is to say that a system is much more likely to be in a state that is consistent with many "arrangements" of the elements comprising the state than it is to be in a state consistent with just a few "arrangements" of the elements comprising the state. An "arrangement" in the coin toss discussion corresponds to a particular ten toss result, say, HTTHTTHHHT. The 10/0 result is consistent with only one such arrangement, while the 5/5 result is consistent with 252 such arrangements.

The difference between a specific string of heads and tails and the total count (in any order) is a model of a very important distinction we use in our development of the 2nd Law of Thermodynamics. The specific string, where every flip is identified, is called a **microstate.** The softer condition -- where we only specify the total number of heads and tails that result -- is called a **macrostate**. In our mathematical fair coin model, what "fair" means is that *every microstate has the same probability of appearing*.

What happens as the number of tosses increases from 10 to, say, 1000? As you might guess, it becomes even more likely to obtain a result near 500/500 than it is to obtain a result near 1000/0. In the jargon of statistics, the probability distribution gets "sharper."



In chemical and biological systems we often deal with HUGE numbers of particles, often on the scale of moles (one mole of molecules contains more than 10^{23} particles!) so one can imagine what the probability distribution looks like in such cases. It is incredibly sharp. The only macrostate that we ever see is the most probable one. Regularities emerge from the probability that are (as long as we are talking about many particles) as strong as laws we consider to be "absolute" instead of "probabilistic".

UMASS AMHERST Instructor's Notes

The following is also available as a video **here (https://www.youtube.com/watch?v=ZFMJWn6tTSg)** .

Probability, Means and Standard Deviations

Let's begin by giving a little bit more thought to the idea of probability that you've explored in some of your readings. The probability of an event is the fraction of time it occurs if the process is repeated an infinite number of times. For a coin, for example, if we flip it fair coin an infinite number of times we expect that one half of them will be heads. Similarly, for a dice, we expect that if we roll it an infinite number of times, one-sixth of the rolls will be a two. Colloquially, the higher the probability, the more likely an outcome is to occur.

If one event does not affect the next, then we say that the events are independent. In this course, we will only be dealing with independent mutually exclusive events. Let's begin by thinking about an example of interpreting the idea of probability. Say you roll a fair dice. What is the probability that you will roll a six? Well, of course the answer is one out of six. If you were to roll the dice an infinite number of times that, you would observe that one sixth of the rolls would in fact be a six. Now, let's say you have rolled a dice three times, and the result of each roll has been a six, i.e. you have rolled three sixes in a row. What is the probability that your next roll will also be a six? Well, the answer to this is still 1/6th. Each roll is independent of the previous, so your probability of the next roll being a six is still one out of six, regardless of what has happened in the past. Dice don't have memory, they don't remember, so the odds of your next roll being a six are one out of six.

Now with this idea of probability, let's move on to thinking about how to calculate means of events with differing probabilities. Consider the following set of measurements for the height of the library, as measured, in meters:

$$88, 87, 88, 90, 90, 88, 85$$

We know how to calculate the average of a set of numbers; you add up all the numbers and then divide by the number of measurements. In this example, we would add up 88, 87, 88, 90, 90, 88, and 85 and divide by 7, to get an average of 88, but we see in this data set that each result appears to not be equally probable. 88 occurs three-sevenths of the time, and 90 occurs two-sevenths of the time. Well, we can deal with this as we just did by adding all the numbers up and counting 88 three times, or we can readjust our definition of average to include the idea of probability:

$$\mu = \sum_{i}^{n} p_i x_i$$

In this new definition, we don't just add up the events, we add up the probability multiplied by the value. So, we take each value multiplied by the probability, and then add to get the mean. In this example, we say that the probability of 88 is three out of seven, so we multiply 88 and 3/7. The probability of 90 is two out of seven, and so we multiply 88 by 2/7. 87 and 85 both have probabilities of one over seven, and so we multiply 87 and 85 by 1/7. If you churn this out in your calculator, you will see that you get the exact same result of 88. So, clearly these two methods yield the same result, however, the second is more powerful if we don't know the full data, but, say, only know the probabilities of different outcomes.

Now let's move on to thinking about calculating standard deviations of events with different probabilities. Here in this table,

Table 1.1

Value	Probability
2	0.2
4	0.4
6	0.1
8	0.3

What is the standard deviation of these data? Well, in our formula for mean, all we did was we change the 1/N to the probability of a given event. You would do the same thing for standard deviation. You do the same thing for standard deviation; instead of multiplying by 1/N out front, you bring it inside the sum, N multiplied by the probability. So now, this equation says take each event, subtract the mean, square it, multiply by the probability, and add them all up, and that will give you the standard deviation squared. Let's test this formula using these data. We would begin by calculating the mean itself, because the mean is an element of calculating the standard deviation. So, to calculate the mean, we say the mean is the sum of the probability of an event multiplied by the value. In this case, let's carry out this calculation for these data.

$$(2)(0.2)+(4)(0.4)+(6)(0.1)+(8)(0.3)$$

Evaluating this expression gives us a mean of 5.

So, now that we have a mean, we can proceed to calculating the standard deviation. The way I'm going to do this is I'm going to add a column to my table, x minus the average, or x- μ, for each value.

Table 1.2

Value	Probability	(x-μ)
2	0.2	-3
4	0.4	-1
6	0.1	1
8	0.3	3

In our definition of standard deviation, we care about this value squared, so, let's continue and add yet another column, squaring, which will get rid of the negatives:

Table 1.3

Value	Probability	(x-μ)	$(x-μ)^2$
2	0.2	-3	9
4	0.4	-1	1
6	0.1	1	1
8	0.3	3	9

Now we want to multiply each value of x minus mu squared by the probability. So, I'm going to add yet another column, probability times $(x-μ)^2$.

Table 1.4

Value	Probability	(x-μ)	$(x-μ)^2$	$p(x-μ)^2$
2	0.2	-3	9	1.8
4	0.4	-1	1	0.4
6	0.1	1	1	0.1
8	0.3	3	9	2.7

Adding these numbers up as instructed gets me a standard deviation squared of 5; turns out that for this data set, the standard deviation squared, and the average are the same. That will not generally be true. I get the standard deviation itself by taking the square root of the standard deviation squared, giving me a standard deviation of 2.24.

In summary, the probability is the frequency something occurs after an infinite number of trials, and colloquially, we say that the higher the probability, the more likely a given event is to occur. With this idea of probability, we can adjust our definitions of mean and standard deviation by swapping out the 1/N out front, and instead multiplying inside the sum by the probability of each occurrence.

1.3 The Meaning of "And" and "Or" in Probability

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Calculating the probability of combinations of events

This section is also available as a video **here (https://www.youtube.com/watch?v=DandAb05ELw)** .

“And” and “or” have important roles in probability and entropy, and we will be exploring them in this section.

Consider a bowl with four balls of different colors, red, green, orange, and blue. If we reach in and grab a ball, the probability of grabbing each color is easy to see; there’s a 1/4 chance of grabbing red, 1/4 chance of grabbing green, 1/4 chance of grabbing orange, and 1/4 chance in grabbing the blue. Regarding the probability, we can say that the probability of grabbing each color is 1/4.

Now let’s think about the probability of grabbing a blue ball or grabbing a green ball. There’s four different possibilities when you pull out a ball, red, green, orange, and blue, and two of them are either blue or green, and so the probability becomes 2/4. Thinking about this mathematically, you’ll notice that the final probability is the sum of the two probabilities. In this problem, we’re looking for the probability of pulling a green ball OR a blue ball, and a general guideline is that when you see “or”, that usually tells you to add.

Moving onto “and”, what is the probability of pulling out a blue ball, putting it back, and pulling out a green ball? Let’s go through all the different possibilities if you pull the blue ball out first:

Table 1.5

blue, red	red, red	green, red	orange, red
blue, green	red, green	green, green	orange, green
blue, blue	red, blue	green, blue	orange, blue
blue, orange	red, orange	green, orange	orange, orange

Out of all these possibilities, there is only one case where you pull out the blue ball, and then the green ball, so the probability is 1/16. Notice that this is the probability of each ball multiplied together. Just like how you add when you see “or”, you multiply when you see “and”.

Now let’s look at a more complicated example that combines these two ideas. What is the probability of pulling out the blue ball and then the green ball, or pulling out the green ball and then the blue ball? Essentially, it’s the same example as above, only we don’t care about the order in which the ball is pulled. First, let’s apply these ideas of adding with “or” and multiplying with “and”. In this scenario, we are looking for blue AND green, OR green AND blue. We can model this as:

$$\left(P_b * P_g\right) + \left(P_g * P_b\right)$$

where P is the probability. Plugging in our values of the probabilities:

$$(1/4 * 1/4) + (1/4 * 1/4)$$

Solving this gives us 2/16, or 1/8. We can confirm this by looking back at table above, we can see that there’s two possibilities out of the 16, so the probability is 2/16 here as well. In general, the “and” and “or” rules will work; there are very few cases where this will not apply.

1.4 Statistical Distributions

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Interpreting the probability of different events using statistical distributions

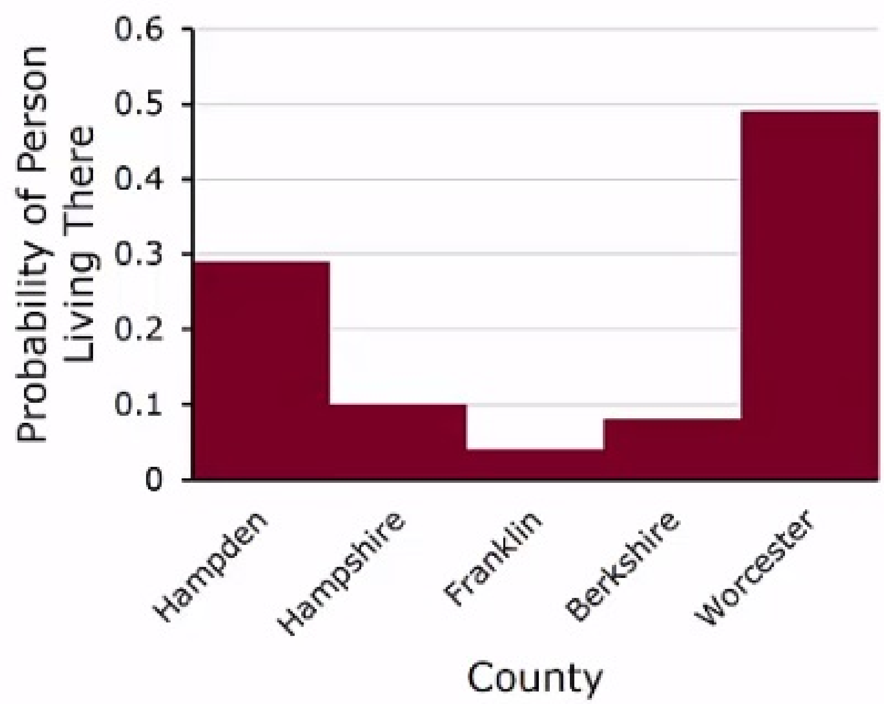
Let’s begin by looking at the distinction between a discrete and a continuous variable. Discrete variables can cannot take on any value. For example, coin tosses are either heads or tails; you can’t be half and half. Similarly, a dice will return one of the values one, two, three, four, five, or six. A dice will never return 2.342, for example. Continuous variables, on the other hand, can take on any value within a given range, and the limit of precision is not from an intrinsic limit within the system but from the measurement technique. Some examples of continuous variables include height. It is completely possible for a person to be 156.03423 centimeters tall, and we can measure a person's height theoretically do any degree of precision that we would like.

Similarly, we often consider mass to be a continuous variable. Now, while strictly, mass is going to be discreet because you can't have less than one electrons worth, the resolution is so small that we generally consider mass to be a continuous variable.

Let's begin by thinking about, now, let's move on to thinking about probability distributions beginning with discrete data.

Western Mass County	Probability of person living there
Berkshire	0.08
Franklin	0.04
Hampden	0.29
Hampshire	0.10
Worcester	0.49

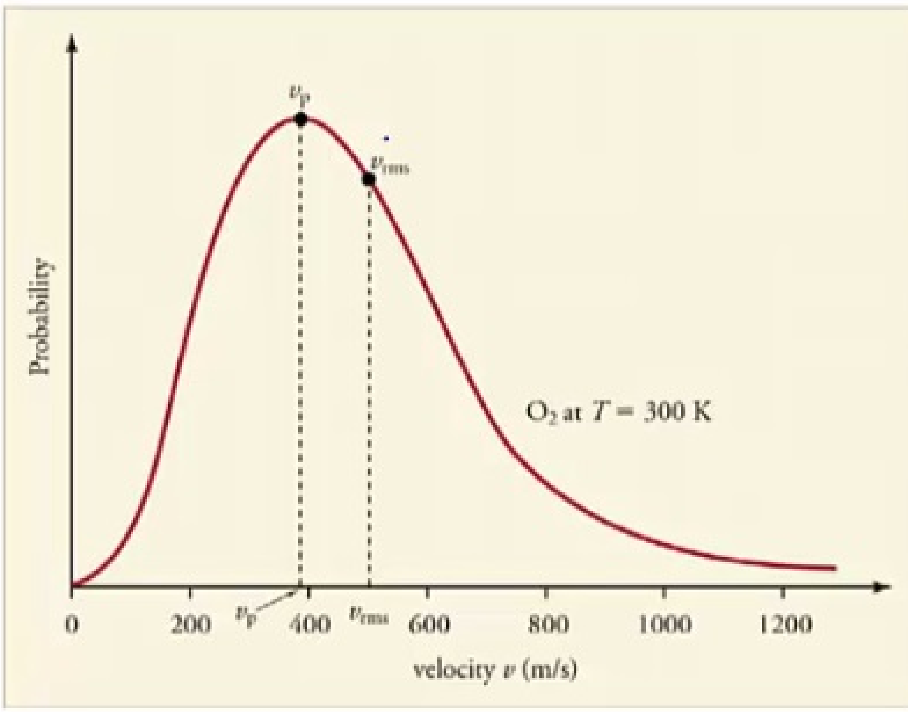
Here, I have a table of the probability of a given person living in one of the five western Massachusetts counties, Berkshire, Franklin, Hampton, Hampshire, and Worcester. A probability distribution is a bar graph where the height of the bar is the probability of the occurrence. So, for these data, a probability distribution would look something like this.



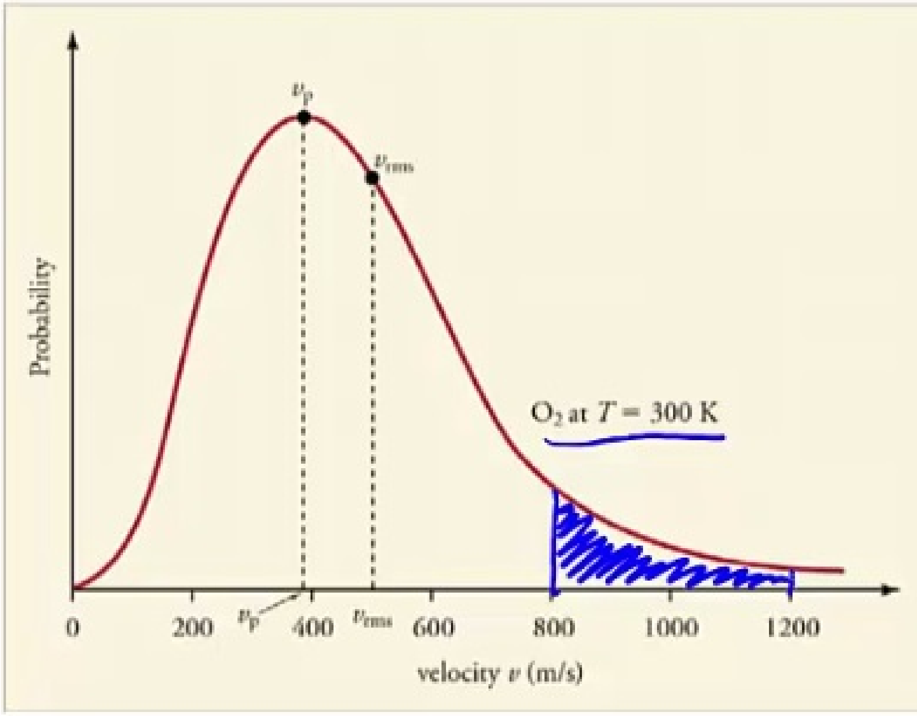
We can see each county listed on the horizontal axis, and the height of the bar indicates the probability of the person living there. All possible outcomes are in fact listed, since all possible outcomes are listed, the heights of all bars together must equal 1, because the probability of a person who lives in western mass living in one of these five counties is of course one hundred percent. Let's think about how to use these types of graphs. So, let's begin with this question as an example. What is the probability of a person who lives in Western Massachusetts to live in either Hampshire or Berkshire counties? We can read off the graph that the probability of a person living in Hampshire County is 10%, 0.10 while the probability of a person living in Berkshire County is a little bit less 0.08. The "or" tells us that we should add, in line with a previous video. So, the probability of a person living in Hampshire or Berkshire is the sum of the probability of a person living in Hampshire and the probability of a person living in Berkshire. Add these two probabilities together, and you get a probability of 0.18 or 18%. This is something that you need to be able to do.

Now let's think about probability distributions for continuous variables. Remember, continuous variables are those quantities that can take any value. An example of a continuous variable might be particle speeds at a given temperature. At a given temperature, particles have a huge variety of different speeds. The expression K= 3/2K_bT tells you the average kinetic energy, or the average speed, v_{rms}. So, let's think about how to interpret probability distributions of continuous variables. The probability of any given number is 0. To understand this, think about what is the probability that a molecule bouncing around the room you're sitting in has exactly 400 meters per second worth of velocity, 400 meters per second to an infinitely high level of precision. Zero, it will always deviate from 400 by a little bit, Because of this, probability is only meaningful if we speak about range of values. Thus, to get the probability, we look at the area under the curve between the values we're interested in.

For example, what is the probability that the velocity of an oxygen molecule at 300k has a velocity between 800 and 1200 meters per second?



Well, we have the probability distribution. The area we are interested in is the area between 800 and 1200. So, this area here tells us the probability that a given oxygen molecule has a speed within this range. Since all possible speeds are represented, the area under the entire curve will be equal to 1. The word associated with this is we say that the curve is normalized. You need to be able to recognize that probability is an area.



1.5 Factorials

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Calculating a factorial
- Approximating the value of a factorial using the Stirling Approximation

- ln(AB)=ln(A)+ln(B)
- ln(AB)=ln(A)−ln(B)
- ln(AN)=Nln(A)

This section is also available as a video **here (https://www.youtube.com/watch?v=VenyEfaeIRA)** .

What is a Factorial?

Let's begin by thinking about what a factorial is. In our next unit on thermodynamics and statistical physics, a lot of our time will be concerned with the study of entropy, what it is, and how can we quantify it. Over the course of this study, we will encounter a mathematical operation which is known as the factorial. The factorial comes up when you are looking at probabilities quite frequently. For example, when answering the question "if I flip a penny, a nickel, a dime, and a quarter, how many possible permutations of two heads are there?", you will need to use the idea of the factorial to solve it. The factorial operation is indicated by the !, such as 3!. How do we calculate factorials? Well, 5! just means to take 5, multiply it by 4, multiply that by 3, multiply that by 2, and finally, multiply by one giving us an answer of 120. Meanwhile, 500! means 500 times 499 times 498 times 497 and so on and so on and so on and so on, until eventually 5 times 4 times 3 times 2 times 1. Now, my calculator has a factorial button; when I try to put in 500!, it essentially burst into flames, but you can see the idea of the calculation here. You take the number, subtract 1, and multiply, and you repeat this process until you get down to 1. Because of this operation, the answer to factorials can get very big very fast.

Before we come to a method on how to deal with such large numbers, there's one last point to bring up; 0! is equal to 1. This is a convention, and we won't go into why necessarily, but you need to know that 0! is equal to 1. Factorials give us very large numbers, so when we take the factorial of a large number, we get a ginormous number, and we need to think of ways to do this without setting our calculators on fire. In our study of thermodynamics and statistical physics will be looking at molecules. Molecules come in moles, which is 10^23 so we will be looking at 10^23! Again, if you try to do this with the calculator's factorial button, you probably will get an error, and so we need a way to handle this.

The Stirling Approximation

Fortunately, in our study, we'll only be interested in taking the natural logarithms of factorials, so if we're interested in n!, what we'll really be interested in is the ln(n!). This will save us quite a bit, because if we're interested in the natural log of the factorial of a number, then we can use what is known as the Stirling approximation.

$$\ln(N!) \approx \frac{1}{2}\ln(2\pi) + \left(N + \frac{1}{2}\right)\ln(N) - N$$

This formula is on your equation sheet, so you don't have to memorize it, however, you need to be able to use it. We're going to try it through a few numbers to see how well it does. So, here is the calculation for two different values of n, 10 and 20.

Table 1.6

N	N!	ln(N!)	Stirling Approximation	% Difference
10	3628800	15.10441	15.09608	-0.0552%
20	2.43×10^{18}	42.33562	42.33145	-0.0098%

You can you see even at 10, 10! is getting very large. The natural logarithm of 10! is about 15.1. If I use the Stirling approximation, I get essentially 15.1. The difference is very tiny, as you can see by the percent difference. So, the difference between the natural logarithm of 10! and the Stirling approximation is 0.05%. Similarly, with the calculation with 20, 20! is already into 10^{18}, which gives us a natural logarithm of 42.3, and now the difference between the natural logarithm of 20! and the result of the Stirling approximation is even smaller, less than one one-hundredth of one percent, so the Stirling approximation is quite accurate.

1.6 Combinations

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Determine the number of ways to arrange n items into subsets of r size

In order to solve combinations, you need to know how to handle factorials, so make sure you understand the material in the previous section before moving on. This section is also available as a video **here (https://www.youtube.com/watch?v=hcZz9Firh6w)** .

What are the types of questions we are looking to answer with combinations? One example is, say that I have five apples of different varieties, how many different combinations of three apples are there? Another example question would be that there are four teams in an NFL division, how many games are necessary for each team to play every other team in their division? Or another example is, say I have 10 molecules in a box, how many different combinations are there with three molecules on the left-hand side and seven on the right-hand side?

So, what is common amongst these different situations? These problems are looking at a large pool of items and trying to choose a subset where the order of the items is not important. In the first example, I'm trying to choose three apples out of five, it doesn't matter what order I choose the apples in. In the second example, I'm choosing two teams to play each other out of the four teams in an NFL division. It doesn't really matter what order the teams play in in this perspective, it just matters how many games do I need. And in the last example, I'm looking to choose three molecules out of ten to be on the right. It doesn't matter which three, just that there are three.

So, now let's move on and try and calculate these different combinations. I will explore this in the Apple example, wherein I have five apples of different varieties and want to know how many different combinations of three apples are there. The way to calculate the number of combinations is given by the formula

$$\frac{n!}{(n-r)!r!}$$

where n is the number of objects in total, in this case five for the five apples, and r is the number of objects at your sub group, in this case three, because I want three apples. Plugging in the numbers into this formula, we see

$$\frac{5!}{(5-3)!3!}$$

which is

$$\frac{5!}{2!3!}$$

Calculating out the factorials, we get

$$\frac{120}{2 * 6}$$

which means that there are 10 different combinations of three apples, given the five that I have.

This calculation is called a binomial coefficient, for reasons that are somewhat sophisticated, and this calculation can be represented in several different ways, and you should be familiar with all the different ways of representing this calculation, as different fields tend to use different notation. The calculation can be represented as:

$$C_r$$

$$C_r$$

$$\binom{n}{r}$$

These all mean the same formula for calculating combinations.

2 ENTROPY

2.1 Introduction

UMASS AMHERST Instructor's Notes

Your Quiz will Cover

- Define entropy in terms of the number of microstates
- State the fundamental principle of statistical mechanics; that all microstates are equally likely
- Calculate the entropy for simple configurations
- From the entropy, determine the number of microstates available

These topics apply to this entire chapter, so keep these in mind as you read on.

Entropy is a weird and abstract concept. However, you might have encountered entropy before. The entropy we will discuss in this class is the same entropy in Gibbs free energy that you may have seen in one of your chemistry classes. You may have also heard of entropy before as being “disorder”. If this is the case, *please forget this idea*, as a lot of confusion surrounding entropy involves this concept of disorder. The idea of entropy as disorder is an outdated one, and is sometimes simply not true.

Consider a bowl of ice sitting in water. Eventually, the ice will melt, and we’ll be left with just a bowl of water. While a bowl of ice inside water may seem more “disorderly” than a uniform bowl of water, the water is a higher entropy state than the water with ice. We’ll discuss why this is in more detail in class.

Getting a handle on the concept of entropy will take some time. We are not expecting you to fully understand it through the readings, and we will work in class to help develop this concept. As always, what we do expect you to know will be listed at the top of each section, under the UMass instructor’s notes.

2.2 Gibbs Free Energy

UMASS AMHERST Instructor's Notes

This sections is also available as a video: **https://www.youtube.com/watch?v=2WlkLcGbbfo (https://www.youtube.com/watch?v=2WlkLcGbbfo)**

The Gibbs free energy is a concept that often comes up in chemistry and biology courses. The definition of the Gibbs free energy is that the change in the Gibbs free energy

$$\Delta G = \Delta H - T\Delta S$$

Let’s deconstruct this equation a minute, ΔG is the change in the Gibbs free energy, this new quantity that we’re introducing, ΔH is the change in enthalpy, not the change in energy, the change in enthalpy that we discussed back in Unit 4, but you may remember from unit 4 that when we’re talking about processes under constant pressure, and most everything is at constant pressure in biology, as most things are open to the air and therefore take place at air pressure, under this situation, enthalpy is the same as heat, so ΔH=Q at constant pressure, which will be the case for most processes that you will encounter and all of the processes that we will discuss.

T in this expression is what you might expect; it’s the temperature. Why do we need the temperature in this equation? Well, is a heat, or enthalpy change, of five joules per mole big or small? Well, you can’t say that unless you have something to compare that five joules per mole to. So, what energy in the system can you compare that five joules per mole to? Well, you could compare it to the average energy available, and as we saw in unit 4 the average energy is the temperature, so that’s why the temperature is in this expression. It provides a comparison point for the enthalpy. The last quantity is ΔS, the change in the entropy. It turns out that the sign, but interestingly not the magnitude of ΔG tells you if a process will happen spontaneously. If ΔG is negative, then the process will occur spontaneously. So, clearly if you want to use this Gibbs free energy to understand if processes will be spontaneous or not, you need to know what entropy is.

How can I get a spontaneous process using this definition of ΔG? Well, there’s two sorts of ways. One, I can be energy-releasing. If I release energy as heat, i.e. I’m an exothermic reaction, then the Q will be negative, and thus if I’m at constant pressure the enthalpy will be negative, and this will push us towards a negative Gibbs free energy change in negative ΔG. The other way is to increase the entropy. If the entropy ΔS increases, then because of this negative sign, the Gibbs free energy change ΔG will actually be pushed negative, so an increase in entropy, i.e. the number of ways to arrange things. will push towards a negative change in Gibbs free energy.

In summary, the Gibbs free energy is an important concept in the chemical and life sciences, at constant pressure which most processes in the chemical and life sciences are, the sign of the Gibbs free energy indicates if a process will be spontaneous or not. If the change in the Gibbs free energy, ΔG, is negative, less than zero, then the process will be spontaneous. The Gibbs free energy looks at a balance of the change in enthalpy ΔH and the average energy in a system, the temperature, and a change in the entropy.

Thus, entropy is an important concept for us to understand to help us understand this idea of the Gibbs free energy. We'll see along the way that this idea of understanding what entropy is will also help us gain a greater insight into certain processes.

2.3 Second Law of Thermodynamics

The following is based off of
umdberg / The 2nd Law of Thermodynamics: A Probabilistic Law (2013). Available at: **http://umdberg.pbworks.com/w/page/68405604/The%202nd%20Law%20of%20Thermodynamics%3A%20%20A%20Probabilistic%20Law%20%282013%29. (http://umdberg.pbworks.com/w/page/68405604/The%202nd%20Law%20of%20Thermodynamics%3A%20%20A%20Probabilistic%20Law%20%282013%29.)** (Accessed: 20th July 2017)

Energy conservation -- **the 1st law of thermodynamics (http://umdberg.pbworks.com/w/page/68405588/The%201st%20law%20of%20thermodynamics%20(2013))** --suffices to rule out a lot of thermal sort of things that don't happen -- like things getting warmer without any source of warmth. But there are a lot of thermal things that don't happen that are perfectly consistent with the 1st law; like thermal energy flowing from a cold object to a hotter object. In order to codify and elaborate our understanding of these results, we turn to the ideas of **probability (http://umdberg.pbworks.com/w/page/68375351/Probability%20(2013))** to understand *how energy tends to be distributed*.

A probabilistic law

That seems a bit strange. What does a discussion about probabilities have to do with a physical law? Physical laws are always true, aren't they? And isn't probability really about things that are only sometimes true? Well, in many ways, molecules in physics are like multi-sided dice, and the likelihood that a particle will be located in a particular location in space (or have a particular energy) is analogous to the likelihood that a multi-sided die will land on a particular side. There are many different ways for the molecules to move, and the details of why they move in one way or another is very sensitive to exactly where they are and how they are moving -- and is very much out of our control.

The likelihood that all the smoke particles in a smoke-filled room will move as a result of their chaotic motions into one corner of the room is analogous to the likelihood that nearly all the coins in a set of 10^{23} tosses will land on heads. It's very, VERY unlikely! If you tossed that many coins over and over again for the lifetime of the universe (14 billion years) the odds that you would see all heads is still minuscule -- totally ignorable. This extremely low probability is what transforms a "probability statement" into a "physical law."

The reason you will never see the smoke particles accumulate in one small corner is that there are many, many more ways for the smoke particles to distribute themselves uniformly throughout the room than there are ways for the particles to all be located in just one corner of the room. That said, just as it is not *impossible* for all 10^{23} tosses to land on heads, it is not *impossible* that all the smoke particles will spontaneously move to one corner of the room... just don't hold your breath waiting for it to happen.

Microstates and macrostates

More generally we can say that when the number of atoms or molecules in a system is large, the system will most likely move toward a thermodynamic state for which there are many possible microscopic "arrangements" of the energy. (And they will be very unlikely to move toward a thermodynamic state for which there are very few possible microscopic arrangements.) If this seems mysterious, go back to the discussion of **coin tosses (http://umdberg.pbworks.com/w/page/68375351/Probability%20(2013))** - it's a pretty good analogy. The H/T ratio (say, 5/5) -- which we refer to as a ***macrostate*** of the system -- is analogous to a thermodynamic state of a system, where only the pressure, temperature, and density of the molecules are specified. The different ways in which that H/T ratio can be obtained (say, HTTHTTHHHT) -- which we refer to as a ***microstate*** -- is analogous to the specification of the spatial and energetic arrangement of each of the atoms/molecules that compose a particular thermodynamic state. As we saw in the coin toss discussion, if one only looks at the macrostate description, one is much more likely to get a H/T result that corresponds to a large number of arrangements. Likewise, one is much more likely to get an atom/molecule distribution that corresponds to a large number of arrangements.

The second law

The Second Law of Thermodynamics can now be stated in this qualitative way:

When a system is composed of a large number of particles, the system is exceedingly likely to spontaneously move toward the thermodynamic (macro)state that correspond to the largest possible number of particle arrangements (microstates).

There are a few really important words to make note of in this definition. First, the system must have a LARGE number of particles. If the system has just a few particles, it is not exceedingly likely that the particles will be in one state rather than another. Only when the number of particles is large do the statistics become overwhelming. If one tosses a coin just twice, there is a reasonable chance (namely, 25%) that one will obtain all heads. Secondly, the system is EXCEEDINGLY LIKELY, but not guaranteed, to move toward a state for which there are the most particle arrangements. The larger the number of particles, the more likely it is, but it is never a guarantee. Thirdly, this law does not specify the specific nature of these "arrangements." It may be that we are only interested in spatial location, in which case an arrangement corresponds to the spatial location of each particle in the system. More arrangements would then correspond to more ways of positioning the particles in space. In other contexts we may be interested in energy, and arrangements would then correspond to the set of energies corresponding to the system's constituents. In either case, the most likely thermodynamic state is the one for which there are the most microscopic arrangements.

Biological implications

The Second Law of Thermodynamics is a statistical law of large numbers. But we have to be careful. Although biological systems almost always consist of a huge number of atoms and molecules, in some critical cases there are a *very small* number of molecules that make a big difference. For example, a cell may contain only a single copy of a DNA molecule containing a particular gene. Yet that single molecule may be critical to the production of protein molecules that are critical to the survival of the cell. For some processes a small number of molecules in a cell (fewer than 10!) can make a big difference. On the other hand, a cubic micron of a fluid in an organism typically contains on the order of 10^{14} molecules! The second law of thermodynamics is a law that is indispensable in analyzing biological systems in countless contexts; but it is essential to understand it well -- not to just use it mindlessly.

[Just as our probability that the number of Heads we got in flipping coins got narrower as the the number of flips got larger, the probability that our results are those predicted by statistical mechanics (most probable macrostates) gets sharper and sharper. The variation around that perfect probability (corresponding to an infinite number of flips or particles) is called *fluctuations*. The scale of fluctuations can be estimated crudely as about 1/(square root of the number). So for 10^{14} molecules, our corrections due to fluctuations are about 1 part in 10^7. Whereas, if we only have 100 = 10^2molecules, our fluctuations are expect to be about 1 part in 10^1 or 10%.]

Entropy

Since the number of microstates corresponding to a particular macrostate plays a critical role, we need a way to count them in order to quantify what's going on with the probabilities. The number of arrangements is so large, that it turns out to be convenient to work with a smaller number -- the **log (http://umdberg.pbworks.com/w/page/68375276/Powers%20and%20exponents%20(2013))** of the number of microstates. This is just like counting the powers of 10 in a large number rather than writing out all the zeros. For a very large N, the number 10^N is considerably larger! And it turns out that working with the log of the number of microstates is very much more convenient. Essentially what is happening is that when you put two systems together (imagine combining two boxes of gases into one) the number of microstates of the combination is basically the *product* of the number of microstates in each. (If we flip a coin 10 times, the number of microstates is 2^{10}. If we flip it another 10 times, the new number of microstates is 2^{20} -- the product of 2^{10} with 2^{10}.) If we take the log of the number of microstates, when we add two systems together, the logs of their number of microstates add to get the total number. This turns out to be both easier to work with and to lead to a number of nice ways of expressing things mathematically.

The log of the number of distributions of the energy that correspond to the thermodynamic state of a system is termed the "entropy" of the system, and is given the symbol *S*. Another way of stating the Second Law, therefore, is to say that systems are exceedingly likely to spontaneously move toward the state having the highest entropy *S*. Using the symbol W to represent the number of arrangements of the energy that correspond to a particular thermodynamics state, we can write an expression for entropy as follows:

$S = kB \ln W$

The constant *kB* is called Boltzmann's constant, and its value is $1.38\text{x}10^{-23}$ J/K. (Yes, it's the same constant we ran into in our discussion of kinetic theory of gases -- the gas constant *R* divided by Avogadro's number, N_A.) The important thing to take from this equation is that the entropy *S* is a measure of the number of arrangements *W*. As *W* goes, so goes the entropy.

But of course the number *W* is usually a HUGE number, and counting up arrangements to arrive at its value would usually take you forever. Fortunately, it is very rarely the case that we actually need to do the counting. Rather, we usually need only to compare two thermodynamic states and to decide which one is consistent with the greatest number of microscopic arrangements. That is the state to which the system will evolve.

Systems

When discussing the Second Law of Thermodynamics, it is crucial to be very careful about defining the system that one is considering. While it is always the case that the entropy of the *universe* is overwhelmingly likely to increase in any spontaneous process, it is not necessarily the case that a particular sub-system of the universe will experience an increase in entropy. If the system being studied is isolated, i.e., if no matter or energy is allowed to enter or leave the system, then the system's entropy will increase in any spontaneous process. But, if the system is NOT isolated, it is entirely possible its entropy will decrease. Stated more generally, it is entirely possible that one *part* of the universe will exhibit an entropy decrease during a spontaneous process while the rest of the universe exhibits a *larger increase* in entropy, such that the overall entropy in the universe has increased. All of this is just to say that it is of utmost importance to be clear about the system to which the Second Law of Thermodynamics is being applied.

It is not obvious at this stage that the statement of the Second Law of Thermodynamics presented here will be practically useful in understanding which processes in nature are spontaneous and which ones are not. What, for example, does any of this have to do with the fact that heat spontaneously transfers from hot objects to cold objects and not the other way around? What does this have to do with chairs sliding across a room? And what does it have to do with the electrostatic potential across a biological membrane? As it turns out, the Second Law of Thermodynamics as defined above can in fact explain those examples.

2.4 Example: Arranging energy and entropy

The following is based off of
1.umdberg / Example: Arranging energy and entropy. Available at: **http://umdberg.pbworks.com/w/page/104869513/Example%3A%20Arranging%20energy%20and%20entropy. (http://umdberg.pbworks.com/w/page/104869513/Example%3A%20Arranging%20energy%20and%20entropy.)** (Accessed: 20th July 2017)

The 2nd law of thermodynamics says that energy will tend to spontaneously distribute itself so that it is, on the average (and this phrase is very important -- see **How Energy is distributed: Fluctuations (http://umdberg.pbworks.com/w/page/104491534/How%20energy%20is%20distributed%3A%20Fluctuations)**), spread equally to all degrees of freedom. It is not easy to see what this means, so let's consider a problem and work out a simple example in detail.

The entropy of a particular macrostate is proportional to the logarithm of the number of microstates corresponding to that macrostate. To see what that means and why entropy tells us about how a system will spontaneously tend to redistribute its energy, let's consider a "toy model" -- one that is sufficiently simplified that we can understand clearly the mechanism behind the mathematics.

One of the reasons that it is difficult to understand entropy as about energy distribution is that many of the degrees of freedom we deal with -- kinetic energy in three directions, energy of rotation,... -- are continuous. The energy in them can take any value. This makes it hard to see that entropy is actually about *counting* -- counting the number of ways energy can be distributed. That math showing this involves breaking the continuum up into bits, counting the arrangements of those bits, and then taking a limit as the size of the bits go to zero. This involves more math than we would like to get into at this point.

Fortunately, some degrees of freedom are not continuous: they are discrete -- their energies can only take on specific values. Here are two.

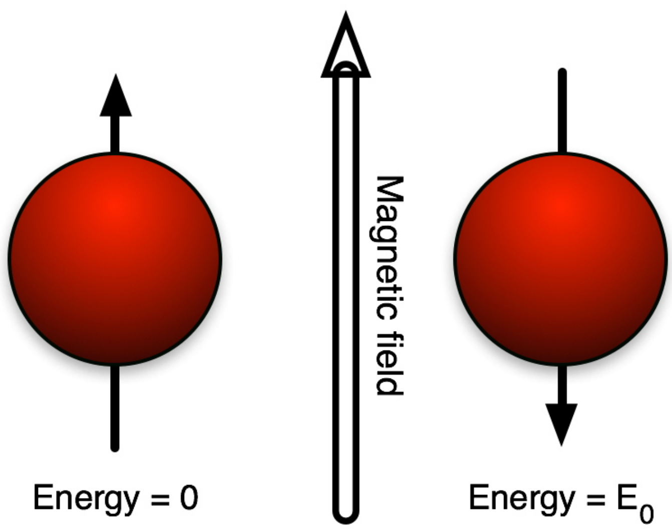


Figure 2.1 The alignment of the magnetic moment of protons in a magnetic field -- The protons that make up the nucleus of hydrogen atoms are little bar magnets. Because of the laws of quantum mechanics, if they are placed in a magnetic field they can only either line up with or against the magnetic field. If it's lined up with the magnetic field it has a lower energy, which, if we are only discussing magnetic energy, we can call 0. (This is the way it "wants" to be.) If it is lined up in the opposite direction from the magnetic field, it has a higher energy, which we will call E0.*

If you start with a bunch of protons in a magnetic field and you have some energy, you can distribute it by flipping some number of protons against the field. In the figure at the left, we have 6 protons, all aligned with the magnetic field, so the (magnetic) energy of the system is 0. In the figure at the right, we have flipped 3 of the protons to be anti-aligned with the magnetic field, so the energy of the system is $3E_0$.

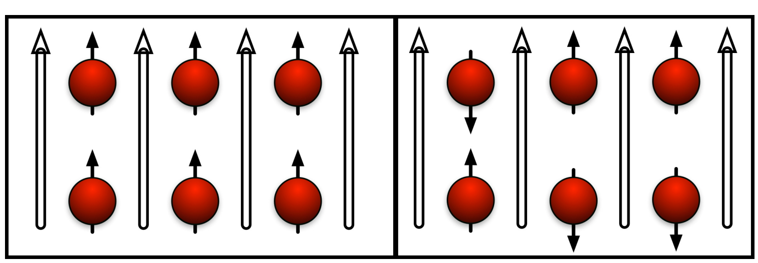


Figure 2.2

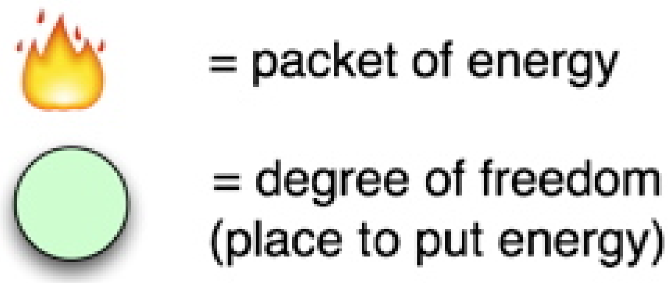
The orientation of the proton in a magnetic field is not only a discrete degree of freedom, it can only hold one "packet" of energy. It either has the energy E_0 or it has none. It can't hold 2 or 3 packets. It's like an on-or-off switch.

This system has relevance to how an MRI works.

Now that we have (we hope!) convinced you that a model with discrete packets of energy is "not just a toy" but also useful in real physical situations, let's solve a typical problem.

1. Consider a system consisting of four discrete degrees of freedom, each of which can only hold 1 packet of energy. Suppose we have 2 packets of energy to distribute. This system of 4 bins with 2 packets is a macrostate -- it's a system with a given amount of energy. How many microstates -- states corresponding to specific ways of distributing those energy packets -- correspond to that macrostate? What is the entropy of the macrostate with 4 bins and 2 energy packets?

Questions



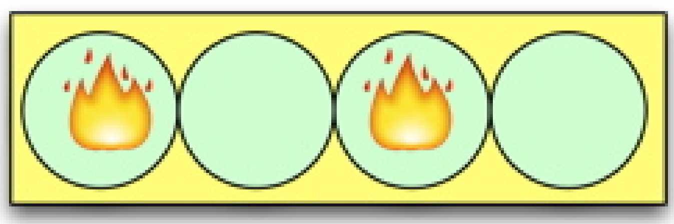


Figure 2.3

2. What if we have two adjacent identical systems, A and B, of 4 bins each. What is the entropy of the state with 4 packets of energy all residing in A? Compare that to the entropy of the state with 2 packets of energy in A and 2 packets in B.

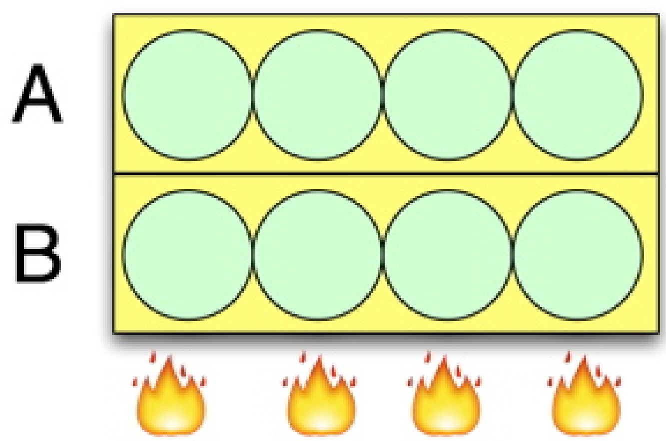


Figure 2.4

Solutions

1. How many ways can we put 2 packets of energy into 4 bins? Let's label the bins 1, 2, 3, and 4. We can only have one packet in a bin or none, so counting is pretty straightforward. We can put the first packet into our bins in any of 4 ways. The second packet can't go in the bin that we have used, so we have only three places to put it. So we have 12 (= 4 x 3) ways in which we can place our 2 bits of energy into 4 places. But energy is energy. If we have one packet in bin 1 and one packet in bin 3, it doesn't matter whether we put a packet into 1 first and 3 second or the other way round. Counting 4 x 3 counts both those ways

separately. So we have calculated each arrangement twice. The real results is half of 4 x 3 or 6. We can easily enumerate them: We can have the bins occupied by an energy packet as: (12), (13), (14), (23), (24), and (34). A total of 6, just as we calculated.

So there are 6 ways (microstates) of making a state with 2 energy packets in 4 bins (a macrostate). Since the entropy is

$S = k_B \ln W$

where W is the number of microstates, we have $S = \ln 6\ k_B = 1.79\ k_B$. Note that entropy has the units of k_B -- energy per degree Kelvin.

2. For the second situation, our macrostate is specified by how many energy packets are in A and how many in B. (This is like specifying the temperature of each object.) Our number of microstates is the number of ways to get that result.

For the first situation, all 4 packets in A, there is only one way we can do it. Since each bin can only hold one packet, we have to put one in each. Since there is only one way, the entropy of this macrostate is $S = k_B \ln W = k_B \ln 1 = 0$.

For the second situation, since the order doesn't matter (all energy packets being equivalent), we can put 2 packets into A first and then 2 packets into B. In part 1 we calculates that there were 6 ways to put 2 packets into 4 bins. So there are 6 ways to put 2 packets into A and 6 ways to put 2 packets into B. You can enumerate them just as we did in part 1. So how many total ways are there? Just the product of 6 x 6 = 36. We can easily see this by looking at what a particular microstate looks like. For A we have a list of 6 possibilities: (12), (13), (14), (23), (24), (34). For B we have a similar list. Any AB microstate is a specification something like:

A(13)B(24). Clearly there are 6 x 6 possibilities. So the entropy of this macrostate is $S = k_B \ln W = k_B \ln 36 = 3.58\ k_B$. (It should be no surprise that this is twice the entropy of the state found in 1.)

This tells us that the transition from all the energy in A (A hot, B cold -- entropy = 0) can spontaneously go to a state where A and B have equal energies (same temperature -- entropy =3.58 k_B.)

* This has the value μB, where μ is the magnetic moment of the proton and B is the strength of the magnetic field.

** The reduced mass of two masses, m_1 and m_2, is equal to $m_1m_2/(m_1+m_2)$. This adjusts the KE for the fact that both masses are moving in coordinated ways. See **Center of Mass (http://umdberg.pbworks.com/w/page/99210514/Center%20of%20Mass%3A%20General%20(technical))** and **Diatomic Vibrations (http://umdberg.pbworks.com/w/page/53869907/Diatomic%20vibrations)** .

Joe Redish 2/4/16

2.5 A Way to Think about Entropy - Sharing

The following is based off of
umdberg / A way to think about entropy -- sharing. Available at: **http://umdberg.pbworks.com/w/page/50323410/A%20way%20to%20think%20about%20entropy%20--%20sharing. (http://umdberg.pbworks.com/w/page/50323410/A%20way%20to%20think%20about%20entropy%20--%20sharing. (Accessed: 20th July 2017))**

We've now read a lot about the second law of thermodynamics and the idea of *entropy.* The basic idea was that we are looking coarsely at a system that has a fine-grained structure that is changing rapidly in a random way. Specifically, we are looking at things *macroscopically* -- and by this we mean at a level at which the structure of matter in terms of molecules and atoms can't be seen. We only care about average properties of the molecules; things like temperature, pressure, and concentration. It is generally useful to ignore the fact that the molecules are actually moving around chaotically, colliding with each other, and chemical reactions are happening (and unhappening).

What we *are* interested in, is the following:

If two parts of a system we are considering are NOT in thermodynamic equilibrium, what will naturally tend to happen?

This is the question that the second law of thermodynamics gives the answer to. It tells us that if one part of a system is hotter than another, the natural spontaneous tendency of the system is for the temperature to even out. If a chemical reaction can occur, the reaction will continue in one direction until the rate of the reverse reaction is equal to the rate of the reaction. When the rates of forward and reverse reactions are equal, the amount of each chemical stays the same and it is called chemical equilibrium.

To understand these situations in general and to figure out which way things will happen under what conditions, we introduced the concept of entropy and the second law of thermodynamics. The core idea is that to each coarse-grained view of a particular system (its pressure, temperature, chemical concentrations, etc. -- its *macrostate*) there are many, many different possible possible arrangements and motions of the individual molecules (its *microstate*). The idea of the second law is:

A system that is not in thermodynamic equilibrium will spontaneously go towards the state with the largest number of microstates.

The reason for this is that we assume that as the system goes through its various chaotic states, each microstate is equally probable. Therefore, the system will most often wind up in the macrostate that corresponds to the largest number of microstates.

Since the *entropy* is defined as (a constant times) the logarithm of the number of microstates, $S = kB \ln(W)$, (see **Why entropy is logarithmic (http://umdberg.pbworks.com/w/page/49691685/Why%20entropy%20is%20logarithmic)**), the second law can be restated as

Systems that are not in thermodynamic equilibrium will spontaneously transform so as to increase the entropy.

Well. This is an impressive sounding statement. But what does it mean? It's pretty plausible to think about flipping coins and deciding whether 5 heads and 5 tails is more or less likely to happen than tossing 10 heads in a row. But how does counting microstates help us see that hot and cold objects placed together will tend to go to a common temperature? You can do it, but it takes a LOT of heavy mathematical lifting -- and doesn't particularly help us conceptually. Another way to think about it that might help, is to think of entropy as a measure of *sharing*.

If energy is uniformly spread, it's useless.

Thermodynamic equilibrium means that the energy in a system is uniformly spread among all the degrees of freedom (i.e. distributed evenly among all places energy can go, for example, for each molecule among both its potential and kinetic energies). In such a state, the energy no longer "flows" from one set of molecules to another or from one kind of energy to another. Thus in thermodynamic equilibrium the energy in a system is useless; no work, either physical or chemical can be done. If we want to think about how useful some energy is, we need to know not just how much energy we have, but how it is distributed. The further from equilibrium it is, the more useful it will be. We are working towards developing the idea of not just energy, but *free energy* -- useful energy.

In some sense, entropy is a measure of how uniformly the energy is distributed in a system. If the system is fully at thermodynamic equilibrium the entropy is a maximum. If the entropy is lower than than maximum, then there is room for the entropy to go up as the system moves towards thermodynamic equilibrium. The system will spontaneously and naturally be redistributing its energy toward the equilibrium state. During such a redistribution, work can get done and an organism can make a living.

Joe Redish 1/29/12

2.6 Why Entropy is Logarithmic

The following is based off of
umdberg / Why entropy is logarithmic. Available at: **http://umdberg.pbworks.com/w/page/49691685/Why%20entropy%20is%20logarithmic. (http://umdberg.pbworks.com/w/page/49691685/Why%20entropy%20is%20logarithmic.)** (Accessed: 20th July 2017)

We defined the entropy (S) of a system as $k_B \ln W$, where W is the number of possible arrangements of the system. But why? Why not just say that entropy **is** the number of arrangements? Let's think through why it has to be defined this way.

We want to define entropy as an **extensive property**, i.e. if I have two systems A and B, the total entropy should be the entropy of A plus the entropy of B. This is like mass (2 kg + 2 kg = 4 kg), and not an **intensive property** like temperature (if you combine two systems that are each at 300 K, you have a system at 300 K, **not** at 600 K!).

What happens to the number of possible arrangements when you combine two systems? If system A can be in 3 different arrangements and system B can be in 5 different arrangements, then there are 3*5 = 15 possible combinations. They multiply! This '80s **music video (http://www.youtube.com/watch?v=w0i_ZFlGTVY)** explains why.

So we can't just define entropy as the number of possible arrangements, because we need the entropy to **add**, not **multiply**, when we combine two systems.

How do you turn multiplication into addition? Just take the logarithm. $3 * 5 = 15$, but $\ln 3 + \ln 5 = \ln 15$.

So that's why entropy is defined as a constant times $\ln W$. W (the number of arrangements) is a dimensionless number, so $\ln W$ is too.

The constant out in front could be any constant, but we use Boltzmann's constant, 1.38×10^{-23} J/K. When we get to Gibbs free energy, we'll see that this constant has the right units, since we need entropy to be in units of energy/temperature.

Ben Dreyfus 1/9/2012